Proc. of the 11 Int. School on Theoretical Physics Symmetry and Structural Properties of Condensed Matter, Rzeszów 2014

Some Exact Results in the One-Dimensional Attractive Hubbard Model

D. Jakubczyk*

Department of Physics, Rzeszów University of Technology, al. Powstańców Warszawy 6, 35-959 Rzeszów, Poland

The one-dimensional attractive Hubbard model ($U \ll 0$) is discussed for the chains of N nodes and the same number of electrons, where $N-1$ of them have the same spin projection, assuming periodic boundary conditions and the half-filling case. Based on the analysis of the eigenvalue problem we provided the general analytical expression for the eigenvalues, for any number N. This formula implies the existence of two elementary particles with mutually dependent momenta on the ring with N sites the same number of electrons including $N-1$ of the same spin projection.

DOI: 10.12693/APhysPolA.128.170

PACS: 75.10.Pq, 03.65.Aa, 03.65.Fd

1. Introduction

The Hubbard model [1–3] has become increasingly important in condensed matter physics since it is related to topics such as ferromagnetism, antiferromagnetism, the Mott transition, high-temperature superconductivity or Bose–Einstein condensate in cold optical lattice [4–7]. The exact solution of the Hubbard Hamiltonian exists only for one spatial dimension, published by Lieb and Wu [8] in year 1968. They used the method called nested Bethe Ansatz [9], and found the large set of normalized and mutually orthogonal eigenfunctions. The results are not easy to analyze and the completeness of the obtained set of eigenfunctions was not considered [10]. We continue the work with one spatial dimension [11–14], which may become more important due to its possible applications in intensively studied carbon nanotubes, one-dimensional organic superconductors, or one-dimensional organic ferromagnets [15]. There is also possibility of modelling the one-dimensional Hubbard model of fermionic quantum gas immersed into an optical lattice [16], which is a promising candidate for quantum information processing.

We consider the one-dimensional attractive Hubbard model ($U \ll 0$) [13, 14] assuming periodic boundary conditions and the half-filling case. The considered chains have N nodes, the same number of electrons including $N-1$ of the same spin projection.

2. The symmetry of the system

The dynamics of the finite set of electrons, occupying the one-dimensional chain consisting of N atoms, can be described by the Hubbard Hamiltonian in the following form:

$$\hat{H} = -t \sum_{i\in\tilde{2}} \sum_{j\in\tilde{N}} (\hat{a}^{\dagger}_{ji}\hat{a}_{j+1i} + \hat{a}^{\dagger}_{j+1i}\hat{a}_{ji}) + U \sum_{j\in\tilde{N}} \hat{n}_{j+}\hat{n}_{j-}, \tag{1}$$

where $\tilde{N} = \{j = 1, 2, \ldots, N\}$ denotes the set of atoms of the chain, $\tilde{2} = \{i = +, -\}$, $\hat{n}_{ji} = \hat{a}^{\dagger}_{ji}\hat{a}_{ji}$, and finally $\hat{a}^{\dagger}_{ji}$, $\hat{a}_{ji}$ are the canonical Fermi operators, that is creation and anihilation operators of electron of the spin i, on the site j. In general U can be any value, with $U < 0$ ($U \ll 0$ — the case presented in this article) and $U > 0$ ($U \gg 0$ [17]) are responsible for attraction and repulsion, respectively, but $U = 0$ stands for no effect or plain gas of fermions.

Since the periodic boundary condition are assumed, the Hamiltonian (1) has the obvious translational symmetry ($\hat{a}_{N+1i} = \hat{a}_{1i}$), this means that one-particle Hamiltonian of the form (1) is completely diagonalized by a Fourier transformation [18] of the form

$$a^{\dagger}_{k} = \frac{1}{\sqrt{N}} \sum_{j\in\tilde{N}} \mathrm{e}^{\mathrm{i}2\pi kj/N} a^{\dagger}_{j}, \quad k \in B, \tag{2}$$

where

$$B = \{0, \pm 1, \pm 2, \ldots, \begin{cases} \pm(N/2-1), N/2 & \text{for } N \text{ even} \\ \pm(N-1)/2, & \text{for } N \text{ odd} \end{cases}\}. \tag{3}$$

The symmetry $\mathrm{SU}(2) \times \mathrm{SU}(2)$ [19] provides two sets of generators, $\{\hat{S}_z, \hat{S}^+, \hat{S}^-\}$ and $\{\hat{J}_z, \hat{J}^+, \hat{J}^-\}$, for spin and charge, respectively. These generators can be written in the following forms:

$$\hat{S}_z = \frac{1}{2} \sum_{j\in\tilde{N}} (\hat{a}^{\dagger}_{j+}\hat{a}_{j+} - \hat{a}^{\dagger}_{j-}\hat{a}_{j-}),$$
$$\hat{S}_+ = \hat{S}^{\dagger}_- = \sum_{j\in\tilde{N}} \hat{a}^{\dagger}_{j+}\hat{a}_{j-}, \tag{4}$$

$$\hat{J}_z = \frac{1}{2} \sum_{j\in\tilde{N}} (\hat{a}^{\dagger}_{j+}\hat{a}_{j+} + \hat{a}^{\dagger}_{j-}\hat{a}_{j-} - 1),$$
$$\hat{J}_+ = \sum_{j\in\tilde{N}} (-1)^j \hat{a}^{\dagger}_{j+}\hat{a}^{\dagger}_{j-}, \quad \hat{J}_- = \sum_{j\in\tilde{N}} (-1)^j \hat{a}_{j+}\hat{a}_{j-} \tag{5}$$

*e-mail: djak@prz.edu.pl

and the transfer between these two sets is known as the Shiba transformation [8, 19, 20]. The eigenvalues of the operators $\hat{S}_z$ and $\hat{J}_z$ are labeled by M and J_z, whereas the eigenvalues of the operators

$$\hat{S}^2 = \frac{1}{2}(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+) + \hat{S}_z^2 \tag{6}$$

and

$$\hat{J}^2 = \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) + \hat{J}_z^2, \tag{7}$$

are labeled by $S(S+1)$ and $J(J+1)$, respectively, due to the quantum algebra of the angular momentum. In the case under consideration this symmetry is reduced to the SU(2) $\times$ I symmetry active only in the pseudo-spin space, where I denotes the identity element of the group SU(2), since the singly occupied atoms have the same spin projection.

3. Energies

The *electron configuration* can be defined in the form of the following mapping:

$$f : \tilde{N} \longrightarrow \tilde{4}, \tag{8}$$

where

$$\tilde{4} = \{\pm, \emptyset, +, -\}, \tag{9}$$

$\emptyset$ denotes the empty node, $+$ and $-$ stand for one-node spin projection equal to $\frac{1}{2}$ and $-\frac{1}{2}$, respectively, $\pm$ denotes the double occupation of the one node by two electrons with different spin projections. The set of all linearly independent vectors given by the expression (8) provides the initial, orthonormal basis of the Hilbert space $\mathcal{H}$ given as $lc_{\mathbb{C}}\tilde{4}^{\tilde{N}}$, where $lc_{\mathbb{C}}A$ stands for the linear closure of a set A over the complex field $\mathbb{C}$.

For example the set of all electron configurations for the case of $U \ll 0$, the number of electrons equal to $N = 4$, where 3 electrons have the same spin projection, has the following elements:
$\{|\pm\emptyset++>, |+\pm\emptyset+>, |++\pm\emptyset>, |\emptyset++\pm>,$
$|\emptyset\pm++>, |+\emptyset\pm+>, |+\,+\emptyset\pm>, |\pm++\emptyset>,$
$|\pm+\emptyset+>, |+\pm+\emptyset>, |\emptyset+\pm+>, |+\emptyset+\pm>\}$.

In order to correctly identify the Hilbert space $\mathcal{H}$ for the case with $U \ll 0$ let us introduce some notions of the theory of the symmetric group. The action

$$A : \Sigma_N \times \tilde{4}^{\tilde{N}} \longrightarrow \tilde{4}^{\tilde{N}} \tag{10}$$

of the symmetric group Σ_N on the set $\tilde{4}^{\tilde{N}}$ provides the orbits $\mathcal{O}_\mu$ of the group Σ_N labeled by the *weight* μ, given as the sequence of nonnegative integers $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$, where the consecutive μ_i denote the number of occurrences of $\pm$, $\emptyset$, $+$ and $-$ within the electron configuration, respectively, with relation $\sum_{i\in\tilde{4}} \mu_i = N$.

Since we consider only the case of $U \ll 0$, and the half-filling magnetic rings with N nodes occupied by $N_e = N$ electrons, including $N-1$ electrons with the same spin projection the set of allowing weights consists only of two elements, that is

$$\{(1, 1, N-2, 0), (1, 1, 0, N-2)\}, \tag{11}$$

and the dimension of the appropriate Hilbert space is

$$\dim\mathcal{H} = 4\begin{pmatrix} N \\ 2 \end{pmatrix}, \tag{12}$$

with two subspaces corresponding to two elements of the set (11) of equal dimensions

$$\dim\mathcal{H}_1 = \dim\mathcal{H}_2 = 2\begin{pmatrix} N \\ 2 \end{pmatrix}, \tag{13}$$

where

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2. \tag{14}$$

The set of electron configurations for $U \ll 0$ do not contain the elements with two atoms singly occupied by opposite spin projection (unpaired spins).

The representation of the system Hamiltonian takes the form dependent on the parity of the number of electrons N (or nodes in the considered half-filling case), and provides the quasidiagonal structure in the so-called pseudo-spin basis — each block depends on the parity of the value $k + J$, where $J \in \{0, 1\}$. The energies obtained as the eigenvalues of the appropriate blocks of the Hamiltonian are given by the formula

$$E = U + 2\cos\left(\frac{l\pi}{N}\right)a, \quad l \in 1, 2, ..., N-1, \tag{15}$$

where $a = 2t\sin\left(\frac{k\pi}{N}\right)$ or $a = 2t\cos\left(\frac{k\pi}{N}\right)$ for odd and even N, respectively.

After some simple calculations Eq. (15) can be rewritten in the following forms:

$$E = U + 2t\sin\left(\frac{(k+l)\pi}{N}\right) + 2t\sin\left(\frac{(k-l)\pi}{N}\right), \quad \text{for odd } N, \tag{16}$$

$$E = U + 2t\cos\left(\frac{(k+l)\pi}{N}\right) + 2t\cos\left(\frac{(k-l)\pi}{N}\right), \quad \text{for even } N, \tag{17}$$

or

$$E = U + 2t\sin(p_1) + 2t\sin(p_2), \text{ for odd } N, \tag{18}$$

$$E = U + 2t\cos(p_1) + 2t\cos(p_2), \text{ for even } N, \tag{19}$$

with the relation

$$p_1 + p_2 = 2\pi k/N. \tag{20}$$

Equations (16)–(20) allow to consider the case of the one-dimensional attractive Hubbard model for the chains with N atoms, the same number of electrons, and $N-1$ of them with the same spin projection as the movement of two elementary particles on the ring with N sites. The first particle has the momentum p_1 and the second — p_2 and the momentum of each particle is separately conserved. One pair from $(N-1)|B|$ mutually dependent values of p_1 and p_2, where $|B|$ denotes the cardinality of the set B given by Eq. (3), constitutes one eigenstate of the energy given by one of Eqs. (18) or (19) — ensuring the completeness of the set of eigenvectors, thus

$$\dim\mathcal{H}_1 = \dim\mathcal{H}_2 = (N-1)|B| = 2\begin{pmatrix} N \\ 2 \end{pmatrix}, \tag{21}$$

due to Eq. (13).

4. Conclusions

The set $\{k, J, l\}$ of quantum numbers provides all exact solutions for the case of the one-dimensional attractive Hubbard model for the chains with N atoms, the same number of electrons, and $N-1$ of them with the same spin projection. The translational symmetry provides the quasimomentum k and rotational symmetry within the spinless part of the magnetic ring provides the quantum number J whereas J_z, the total spin S and the total magnetization M are fixed. The formula $E = U + 2t\sin(p_1) + 2t\sin(p_2)$ for the energies of the considered system of electrons implies existence of two elementary particles on the ring of the separately conserved momentum p_1 and p_2, respectively. Each element of the set of pairs of mutually dependent values p_1 and p_2 constitutes the one to one correspondence with the set of eigenvectors of the considered problem, that is with the basis of the appropriate Hilbert space.

References

[1] J. Hubbard, *Proc. R. Soc. A* **276**, 238 (1963).

[2] M.C. Gutzwiller, *Phys. Rev. Lett.* **10**, 159 (1963).

[3] M.C. Gutzwiller, *Phys. Rev. A* **137**, 1726 (1965).

[4] M. Vojta, *Rep. Prog. Phys.* **66**, 2069 (2003).

[5] *The Theory of Superconductivity in the High-T_c Cuprates*, Eds. P.W. Anderson, S.B. Treiman, Princeton Series in Physics, Princeton Univ. Press 1997.

[6] M. Aizenman, E.H. Lieb, R. Seiringer, J.P. Solovej, J. Yngvason, *Phys. Rev. A* **70**, 023612 (2004).

[7] K. Kucab, G. Górski, J. Mizia, *Physica C* **490**, 10 (2013).

[8] E.H. Lieb, F.Y. Wu, *Phys. Rev. Lett.* **20**, 1445 (1968).

[9] C.N. Yang, *Phys. Rev. Lett.* **19**, 1315 (1967).

[10] F.H.L. Essler, V.E. Korepin, K. Schoutens, *Phys. Rev. Lett.* **67**, 3848 (1991).

[11] T. Lulek, B. Lulek, D. Jakubczyk, P. Jakubczyk, *Physica B: Condens. Matter* **382**, 162 (2006).

[12] J. Milewski, B. Lulek, T. Lulek, *Phys. Status Solidi B* **244**, 2497 (2007).

[13] D. Jakubczyk, *Acta Phys. Pol. B* **44**, 1921 (2013).

[14] D. Jakubczyk, *Acta Phys. Pol. B* **45**, 1759 (2014).

[15] D. Jerome, C.R. Pasquier, *Frontiers in Superconducting Materials*, Ed. A.V. Narlikar, Springer, Berlin 2005.

[16] S.R.J.F. Clark, *Strongly Correlated One-Dimensional Systems of Cold Atoms in Optical Lattices*, Trinity College, Oxford 2007.

[17] D. Jakubczyk, *Mod. Phys. Lett. B* **26**, 1150005 (2012).

[18] B. Lulek, T. Lulek, A. Wal, P. Jakubczyk, *Physica B* **337**, 375 (2003).

[19] F.H.L. Essler, H. Frahm, F. Göhmann, A. Klümper, V.E. Korepin, *The One-Dimensional Hubbard Model*, Cambridge University Press, Cambridge 2005.

[20] A.B. Eriksson, T. Einarsson, S. Ostlund, *Phys. Rev. B* **52**, 3662 (1995).