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Route to Chaos in Generalized Logistic Map

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We postulate a generalization of well-known logistic map to open the possibility of optimization the modelling process of the population evolution. For proposed generalized equation we illustrate the character of the transition from regularity to chaos for the whole spectrum of model parameters. As an example we consider specific cases for both periodic and chaotic regime. We focus on the character of the corresponding bifurcation sequence and on the quantitative nature of the resulting attractor as well as its universal attribute (Feigenbaum constant).

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1. Introduction

Since the dawn time the chaos is an indispensable part of human life. One of the most famous models with chaotic behavior is the logistic map [1–3]. Proposed extension of the case in form of generalized difference equation reads [4]:

$$x_{n+1} = r x_n^p (1 - x_n^q), \tag{1}$$

$x \in [0,1]$, $(p,q) > 0$, $n = 0,1,2,...$, where two new parameters p and q may assume any positive values. The standard logistic map thus corresponds to special case of Eq. (1) for $p = q = 1$. Its simple analytic form meant that it was used in many scientific disciplines such as biology, cryptography, communication chemical physics and stock market [5–8]. One of the mathematical models of chaos was discovered by Mitchell Feigenbaum [9–12]. Feigenbaum considered ordinary difference equations used for example in biology to describe the development of population in its dependence on time. He discovered that population oscillates in time between stable powers (fixed points), the number of which doubles according to changes in the power of external parameter. The generalization of the Feigenbaum model contains all first-order difference equations $f(x_n) = x_{n+1}$. The condition for the existence of chaos is the single maximum of a function $f(x_n)$. Feigenbaum also proved that the transition to chaos is described by two universal constans α and δ, later named the Feigenbaum numbers.

We quote here a few definitions which are basic in the development of this exposition.

Definition 1. [cf. [13], p. 62]
Let $I = [a,b] \subset \Re$, $a < b$, $f(I) \subset I$. Discrete iterations of a function $f : I \to I$ are called functions $f^n : I \to I$ defined inductively: $f^0 = id$, $f^{n+1} = f \circ f^n$, $n = 1,2,3,....$

Definition 2. [cf. [13], p. 62]
The sequence $\{x_n\}$, where $x_n = f^n(x_0)$ is called the sequence of iterates of a function f generated by point $x_0 \in I$.

Definition 3. [cf. [13], p. 65]
A point $s \in I$, with $f(s) = s$ is called a first-order fixed point of f. A fixed point s of a function f is locally stable if there exists a neighbourhood U_s such that all sequences of iterates are convergent to s for $x_0 \in U_s$.

Theorem 1. [see [13], p. 68]
If function f is differentiable in fixed point s and $|f'(s)| < 1$, then s is attractive.

Definition 4. [cf. 13], p. 72]
Let $f : I \to I$. We say that s is a periodic point of the function f, or that s generates a cycle of order $k \geq 2$, if $f^k(s) = s$ and $\forall_{i<k} f^{(i)}(s) \neq s$.

Theorem 2. [14]
If function f has a cycle of order different to 2^k, then f is chaotic.

Theorem 3. [cf. [13], p. 79]
For arbitrary continuous function $f : I \to I$ existence cycles of order m involve the existence cycles of order n in the following order: $3 \gg 5 \gg 7 \gg 2\cdot 3 \gg 2\cdot 5 \gg 2\cdot 7... \gg 2^k\cdot 3 \gg 2^k\cdot 7 \gg ... \gg 2^k \gg ... \gg 8 \gg 4 \gg 2 \gg 1$.

1.1 The conventional logistic map

The Logistic map [1–3],

$$f(x_n) = x_{n+1} = r x_n (1 - x_n),$$

$$x \in [0,1], \quad n = 0,1,2,... \tag{2}$$

is one of the most simple forms of a chaotic process. Basically, this map, like any one-dimensional map, is a rule for getting a number from a number. The parameter r is fixed, but if one studies the map for different values of r (up to 4, else the unit interval is no longer invariant) it is found that r is the catalyst for chaos. After many iterations x reaches some values independent of its starting value 3 regimes: $r < 1 : x = 0$ for large n, $1 < r < 3 : x =$ constant for large n, $3 < r < r_\infty$: cyclic behavior, where $r_\infty \approx 3.56$, $r_\infty < r \leq 4$: mostly chaotic. An interesting thing happens if a value of r greater than 3 is chosen. The map becomes unstable and we get a pitchfork bifurcation with two stable orbits of period two corresponding to the two stable fixed points of the second iteration of f. With r slightly bigger than 3.54, the population will oscillate between 8 values, then 16, 32, etc. The lengths of the parameter intervals which yield the same number of oscillations decrease rapidly; the ratio between the lengths of two successive such bifurcation intervals approaches the *Feigenbaum constant* $\delta = 4.669...$. The period doubling bifurcations come faster and faster (8, 16, 32, ...), then suddenly break off. Beyond a certain point, known as the accumulation point r_∞, periodicity gives way to chaos. In the middle of the complexity, a window suddenly appears with a regular period like 3 or 7 as a result of mode locking. The 3-period bifurcation occurs at $r = 1 + 2\sqrt{2}$, and period doubling then begin again with cycles of 6, 12,... and 7, 14, 28,... and then once again break off to chaos.

2. Generalized logistic map

One-dimensional representations are the simplest arrangements capable to producing a chaotic movement. Let us consider $p, q > 0$. We postulate non-linear function:

$$f_r(x) = rx^p(1 - x^q), \quad x \in [0,1] \tag{3}$$

set by difference equation:

$$x_{n+1} = rx_n^p(1 - x_n^q), \quad x \in [0,1], \quad n = 0,1,2,.... \tag{4}$$

For the first time the function (3) has been presented in [4]. The maximum of the function (3) is reached for $x = \left(\frac{p}{p+q}\right)^{1/q}$, because $f_r^{'}(x) = r(px^{p-1} - (p+q)x^{p+q-1})$ and $f_r^{'}(x) = 0 \Leftrightarrow x = \left(\frac{p}{p+q}\right)^{1/q}$, $\Re \ni (p,q) > 0$. It results that $\frac{f_r\left(\left(\frac{p}{p+q}\right)^{\frac{1}{q}}\right)=rqp^{\frac{p}{q}}}{(p+q)^{\frac{p+q}{q}}}$ and for $r \in \left(0, \frac{\left(\frac{p+q}{p}\right)^{\frac{p}{q}}(p+q)}{q}\right)$ where $\left(0, \frac{\left(\frac{p+q}{p}\right)^{\frac{p}{q}}(p+q)}{q}\right) = r_{\max}$ function f_r is $f([0,1]) \subset [0,1]$.

Parameter $r_{\max}$ depending on p and q accepts the following limit values (Fig. 1):

$$\lim_{q\to\infty} r_{\max} = 1, \quad \lim_{q\to 0^+} r_{\max} = \infty,$$

$$\lim_{p\to\infty} r_{\max} = \infty, \quad \lim_{p\to 0^+} r_{\max} = 1.$$

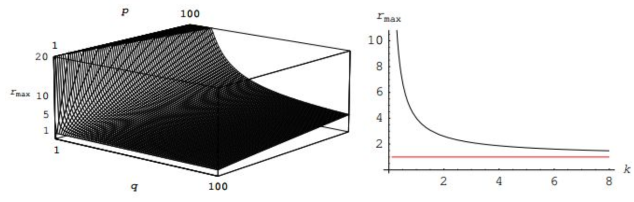

Fig. 1. Values of $r_{\max}$ for generalized logistic map depending on p and q (left) and depending on k (right).

For the sake of parameters p and q the function (3) generates a lot of new functions which can be analyzed using mathematical methods, just like for a logistic map, thus obtaining very interesting results.

a) Let us assume that $k = \frac{q}{p}$ and $k > 0$. Then r is depend on the value k and $r \in \left[0, \frac{(k+1)^{\frac{k+1}{k}}}{k}\right]$.

b) If $k = 1$, then $p = q$ and $r_{\max}$ is independent on p, q and $r_{\max} = 4$ always — for logistic map, we have $p = q = 1$ and $r_{\max} = 4$.

c) If $k \to \infty$, then $r_{\max} \to 1$, because $\lim_{k\to\infty} \frac{(k+1)^{\frac{k+1}{k}}}{k} = 1$.

d) If $k \to 0^+$, then $r_{\max} \to \infty$, because $\lim_{k\to\infty} \frac{(k+1)^{\frac{k+1}{k}}}{k} = +\infty$.

e) Now let us consider a case, when $p = 1$ and $q > 0$. Then we have

$$f_r(x) = rx(1 - x^q), \quad x \in [0,1], \quad n = 0,1,2,.... \tag{5}$$

and $r \in \left[0, \frac{(q+1)^{\frac{q+1}{q}}}{q}\right]$.

Taking into account the above relationship we can observe change maximum of the function (5) depending on q (Fig. 2).

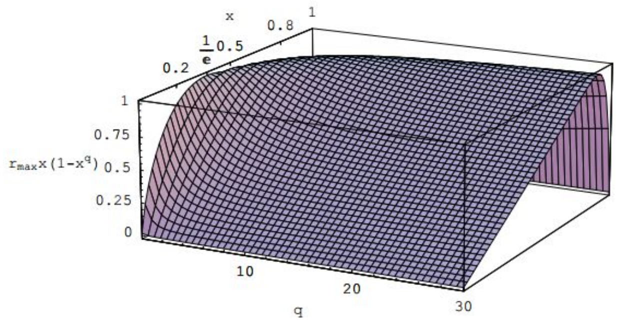

Fig. 2. $f_r(x) = rx(1-x^q)$ for $r = r_{\max}$ depending on q.

2.1. The property of $x_{n+1} = rx_n^p(1 - x_n^q)$ for $p = 1$, $q = 2$ in the periodic region

In the following part of this paper we will show that the Feigenbaum model of transition to chaos is correct for representation (4) for given positive values of parameters p and q. Therefore, we will study the dynamics of this map in the case of $p = 1$ and $q = 2$. Then the representation (5) assumes the following form:

$$f_r(x) = rx(1 - x^2), \quad x \in [0,1], \quad n = 0,1,2,.... \tag{6}$$

A function f_r in the point $x = \frac{\sqrt{3}}{3}$ has a local maximum and the value of function in this point is equal to $f_r(\frac{\sqrt{3}}{3}) = \frac{2r\sqrt{3}}{9}$. From this relationship, we have $f([0,1]) \subset [0,1]$ for $r \in (0, \frac{3\sqrt{3}}{2}]$. The trajectories behavior generated by f_r depends on r and, therefore, we have to distinguish several intervals of this value:

1) For $r \in (0,1]$ all the sequences of iterates $x_1, x_2, x_3, ..., x_n$ depend on the initial value x_0 tend to $s = 0$. This point is the fixed point of the map for $r \in (0,1]$ and according to *Definition 3* it is also the stable point of this map.

2) Let $r = r_1 > 1$. If $x = rx(1 - x^2)$ then, $x_1 = 0$, $x_2 = \sqrt{\frac{r-1}{r}}$, so for $r > 1$ function (6) has two fixed points. Using *Theorem 1* we get: $f_r^{'}(x) = r - 3rx^2$, $|f_r^{'}(0)| = r$ and $|f_r^{'}(\sqrt{\frac{r-1}{r}})| = |3 - 2r|$.

Therefore, we have that $x_1 = 0$ is an unstable fixed point, whereas $x_2 = \sqrt{\frac{r-1}{r}}$ is an attractive fixed point for $r \in (1,2)$.

Change r from 1 to 2 has caused the migration of the attractive point from 0 to $\frac{\sqrt{2}}{2}$.

Conclusion 1.

For $1 < r < 2$ there exists a stable fixed point attracting all points. When $r = r_1 = 2$, then $|f_r^{'}(\sqrt{\frac{r-1}{r}})| = |3 - 2r| = 1$ and this point stops being an attractive point.

3) For $r = r_2 \geq 2$ we will look into the stability of fixed points f_r and $f_r^{(2)}$ of the map (5) as a function of parameter r, because the Feigenbaum model is generated by bifurcations connected with iterations of a function.

Let us consider the second iteration of f_r, namely $f_r^{(2)}(x) = r^2x - r^2x^3 - r^4x^3 + 3r^4x^5 - 3r^4x^7 + r^4x^9$, with fixed points: $x_1 = 0$, $x_2 = \sqrt{\frac{r-1}{r}}$, $x_3 = \sqrt{\frac{1}{2} - \frac{\sqrt{r^2-4}}{2r}}$, $x_4 = \sqrt{\frac{1}{2} + \frac{\sqrt{r^2-4}}{2r}}$. Because $f_r^{'(2)}(x) = r^2 - 3r^2x^2 - 3r^4x^2 + 15r^4x^4 - 21r^4x^6 + 9r^4x^8$ we have $f_r^{'(2)}(x_1) = r^2$, $f_r^{'(2)}(x_2) = (3 - 2r)^2$, $f_r^{'(2)}(x_3) = f_r^{'(2)}(x_4) = 9 - 2r^2$ whereas for $2 < r_2 < \sqrt{5}$ fixed points x_3 and x_4 are attractive points according to condition in *Theorem 1.*

Conclusion 2.

The fixed point $x = \sqrt{\frac{r-1}{r}}$ of $f_r^{(1)}$ is also the fixed point of $f_r^{(2)}$ as well as of all higher iterations.

Conclusion 3.

If the fixed point of $f_r^{(1)}$ becomes unstable, then it is also the unstable fixed point of $f_r^{(2)}$ and of all next iterations. From the inequality $|f^{'}(s)| > 1$ we have $|f^{'(2)}(s)| = |f^{'}[f(s)]f^{'}(s)| = |f^{'}(s)| > 1$.

If $\sqrt{5} < r_3 < r_4$, then the fixed points of $f_r^{(2)}$ become repulsive at the same time. Following this instability, the fourth iteration shows two new pitchfork bifurcations giving cycle 2^2 order for four attractive fixed points, which is called period duplication.

To generalize the above examples we get:

a) For $r_{n-1} < r < r_n$ there exists a stable cycle 2^{n-1}, whose elements $\overline{x}_0, \overline{x}_1, \overline{x}_2, \overline{x}_3, ..., \overline{x}_{2^{n-1}-1}$ are defined by the following connections:

$$f_r(\overline{x}_i) = \overline{x}_{i+1}, \quad f_r^{2^{n-1}}(\overline{x}_i) = \overline{x}_i, \quad |\prod_i f_r^{'}(\overline{x}_i)| < 1.$$

b) For r_n all points of the cycle of 2^{n-1} become unstable simultaneously and the bifurcation creates a new stable cycle of 2^n order for $f_r^{2^n} = f_r^{2^{n-1}}(f_r^{2^{n-1}})$ existing for $r_{n-1} < r < r_n$.

c) For a map (6) the value of parameter r_{n+1} we can describe the following connection: $r_{n+1} \approx \sqrt{3 + r_n}$, where $r_0 = 1, r_1 = 2, r_2 = \sqrt{5}, ..., r_\infty$.

d) There exists a point of accumulation of an infinite number of the bifurcation of period duplication for the finite value r which we appointed as r_∞: $r_\infty = \lim_{n\to\infty} r_n = \frac{1+\sqrt{13}}{2} \approx 2.303$.

Taking into account the above, $r_{n+1} \approx r_n$, $\sqrt{3 + r} = r$, $r^2 - r - 3 = 0$, thus $r = \frac{1+\sqrt{13}}{2}$.

e) For the representation (6) there exists a constant, whose value corresponds to the Feigenbaum constant:

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \approx 4.61.$$

Proof. Let us assume that $r_{n-1} = a$. Then $r_n = \sqrt{3 + r_{n-1}} = \sqrt{3 + a}$, $r_{n+1} = \sqrt{3 + r_n} = \sqrt{3 + \sqrt{3 + a}}$ and $\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = \sqrt{14 + 2\sqrt{13}} \approx 4.61$.

A Table of 2^n type cycles and values of r_n is given below. It is clearly visible that doubling bifurcations come faster and faster (8, 16, 32, ...) and suddenly break off beyond a certain point known as the accumulation point and periodicity gives way to chaos.

TABLE

The algebraic orders of the values of r_n for $n = 1, 2, ...$ are given by $1, 2, 4, 8, ...$

n	1	2	3	4	5	...	11	12	...	∞
2^n cycle	2	4	8	16	32	...	2048	4096	...	accum. point
r^n	2	2.23606...	2.28824...	2.29961...	2.30209...	...	2.3027755...	2.30277562...	...	2.30277563...

2.2. The property of $x_{n+1} = rx_n^p(1 - x_n^q)$ for $p = 1$, $q = 2$ in chaotic region

a) For r_∞ the bifurcation sequence ends with a set of the infinitely numerous points, which is called the set of Feigenbaum attraction.

b) If $r_\infty < r \leq \frac{3\sqrt{3}}{2}$ then we observe the irregular dynamics of map (6) with narrow ranges in which the set of attraction has a periodical character.

c) Numeric calculations for the representation (6) show that the largest range is for $r = r_c \approx 2.451$. Then there exists 3-cycle (Fig. 3), which according to th. 2 and th. 3, implicates the chaos as well as the existence of cycles of different orders.

d) For the map (6) at $r = r_{\max} = \frac{3\sqrt{3}}{2}$,

$$x_{n+1} = \frac{3\sqrt{3}}{2} x_n(1 - x_n^2) \equiv f_{r_{\max}}(x_n), \tag{7}$$

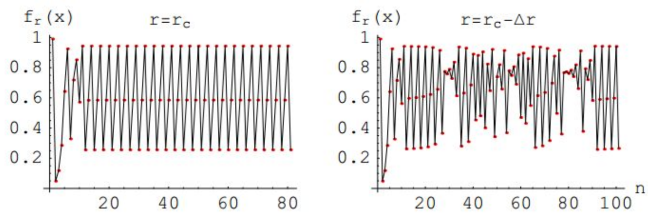

Fig. 3. Iterates of map (6) starting from $x_0 = 0.99$. Left: in the stable 3-cycle region for $r = r_c$; Right: in the chaotic region for $r = r_c - \Delta r$, $\Delta r = 0.001$.

can be solved by the simple change of variables:

$$x_n = \cos(y_n) \equiv g(y_n). \tag{8}$$

Then (7) can be converted into

$$\cos(y_{n+1}) = \frac{3\sqrt{3}}{2} \cos(y_n)[1 - \cos^2(y_n)] = \frac{3\sqrt{3}}{2} \cos(y_n) \sin^2(y_n),$$

which has one solution $y_{n+1} = \arccos\left[\frac{3\sqrt{3}}{2} \cos(y_n) \times \sin^2(y_n)\right]$ and

$$x_n = \arccos\Bigg(\frac{1}{\sqrt{3}\left(-y_0+\sqrt{-1+{y_0}^2}\right)^{\frac{1}{3}}} + \frac{\left(-y_0+\sqrt{-1+{y_0}^2}\right)^{\frac{1}{3}}}{\sqrt{3}}\Bigg).$$

The invariant density function $\rho_{\frac{3\sqrt{3}}{2}}(x)$ of $f_{\frac{3\sqrt{3}}{2}}(x)$ can be calculated using the following definition:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\delta\left(x-x_n\right) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\delta\left(x-g(y_n)\right). \quad (9)$$

Using $\rho(y) = 1$, (9) becomes: $\rho_{\frac{3\sqrt{3}}{2}}(x) = \int_0^1 dy\rho(y)\delta \times[x-h(y)]$ i.e.

$$\rho_{\frac{3\sqrt{3}}{2}}(x) = \quad (10)$$

$$\frac{\left(\sqrt{x^2-1}-x\right)^{2/3}-1}{3\sqrt{x^2-1}\sqrt[3]{\sqrt{x^2-1}-x}\sqrt{-\left(\sqrt{x^2-1}-x\right)^{2/3}-\frac{1}{\left(\sqrt{x^2-1}-x\right)^{2/3}}+1}}.$$

The above function is shown in Fig. 4.

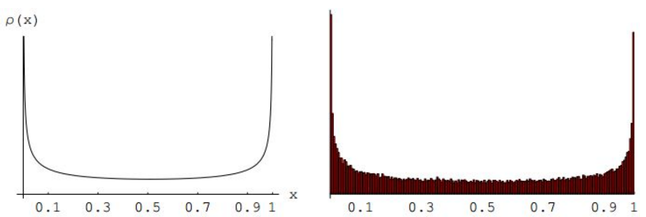

Fig. 4. The invariant density function of map (6) for $r = r_{\max} = \frac{3\sqrt{3}}{2}$. Left: an analytical form represented by the equation (10). Right: numerical simulation — the histogram for 50000 iterations.

The invariant density function of the logistic map for $r = r_{\max} = 4$ is given by $\frac{1}{\pi\sqrt{(1-x)x}}$.

2.3. *Ordered chaos — construction of an attractor for $x_{n+1} = rx_n(1-x_n^2)$ and $r{=}r_{\max}{=}\frac{3\sqrt{3}}{2}$*

At the end of this considerations we will present a method of the *ordering of chaos*. A lot of numeric experiments have led to the situation where it is possible to rank chaos in such a way that the successive iterations generated by chaotic dynamic system would reach an attractor of a regular shape. This method also has it, that from the sequence of several thousand iterations (for representation $x_{n+1} = rx_n(1-x_n^2)$ where $r = r_{\max} = \frac{3\sqrt{3}}{2}$ defined recursively) we draw up the following points $(x_{n+1}, x_n x_{n+1})$ on a plane — we construct an attractor. In this way an attractor comes into being, which is visited irregularly by points of successive iterations (Fig. 5).

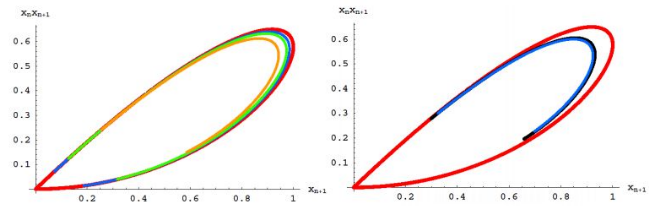

Fig. 5. The set of attraction for representation (6) [60000 iterations]: (a) for $r = r_{\max}$ (red), $r = 2.56$ (blue), $r = 2.53$ (green), $r = 2.45$ (orange); (b) for $r = r_{\max}$ (red), $r = 2.42$ (black), $r = 2.4$ (blue).

2.4. *Bifurcation diagram and Lyapunov exponent*

Nowadays the Lyapunov exponent is applicable in various scientific disciplines such as biology, engineering, bioengineering and informatics [15–20]. A quantitative measure of the sensitive dependence to initial conditions is the Lyapunov exponent, which is a measure of the exponential separation of nearby orbits. For discrete time system $x_{n+1} = f(x_n)$, for an orbit starting with x_0 the Lyapunov exponent $\lambda(x_0)$ can be calculated by

$$\lambda(x_0) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|, \quad (11)$$

A positive Lyapunov exponent can be considered as an indicator of chaos, whereas negative exponents are associated with regular and periodic behavior of the system. Figure 6 shows the bifurcation diagram and Lyapunov exponent for the equation (6). It is clear that even in chaotic areas ($r_\infty < r \le \frac{3\sqrt{3}}{2}$) there are many periodic intervals ($\lambda < 0$).

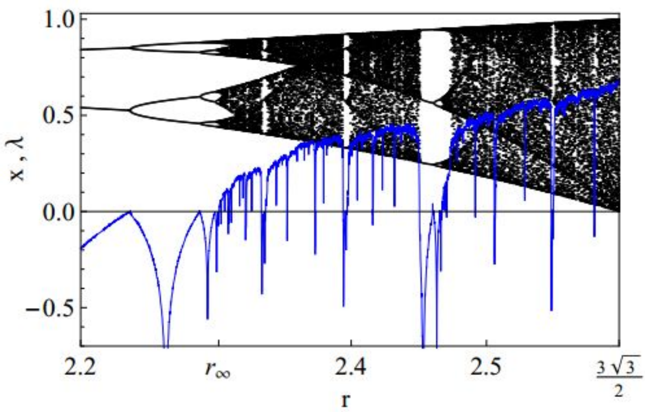

Fig. 6. The bifurcation diagram and the Lyapunov exponent for the map (6) and $2.2 < r \le \frac{3\sqrt{3}}{2}$.

3. Summary

Currently, modeling real-world systems is very popular. A well-known model that takes into account the deterministic chaos is the logistic map. However, this one-dimensional map can be controlled only by one parameter. In the present contribution, we postulate a generalization of the classical logistic map which is generally controlled by three parameters. This could allow to adapt to the conditions prevailing in the modeled system.

We focus on the $r_{\max}$ parameter (depending on p and q) which is responsible for the dynamics of the system — among others shown that if $p = q$ then $r_{\max} = 4$.

As an example, we carry out analytical and quantitative analysis in the case where $p = 1$ and $q = 2$ both in the periodic and chaotic regime where the Lyapunov exponent has positive values. It turns out that the dynamics of this equation is faster than in the case of the logistic map — the value of the parameter $r_{\max} \approx 2.3$ is less than the value of $r_{\max} = 4$ for the logistic map. In the periodic regime we identified Feigenbaum constant δ which is typical for all dissipative systems (also for the logistic map). For chaotic area we have found an analytical form of the invariant density function. For both regimes we have proposed specific form of data representation which allowed to obtain a non-trivial structure of an attractor set.

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