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Numerical Analysis of Two Coupled Kaldor–Kalecki Models with Delay

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This paper is concerned with two coupled Kaldor–Kalecki models of business cycles with delays in both the gross product and the capital stock. We consider two types of investment functions that lead to different behavior of the system. We introduce the model with unidirectional coupling to investigate the influence of a global economy (like the European Union) on a local economy (like Poland). We present detailed results of numerical analysis.

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1. Introduction

Until the late XIX century not much research on business cycles was done. Some of the first contributions were provided by Clement Juglar and Michail Tuhan-Baranowski, but perhaps the most decisive of early investigations, partly stimulated by the great depression, should be attributed to independent studies of John Maynard Keynes, Michał Kalecki, and Nicholas Kaldor (see [1–3] for many interesting historical remarks and observations). The original Kalecki's model of business cycle is an example of a difference-differential model [4]. The most important contribution was incorporation of a new factor into the model: investment decisions B. According to Kalecki, the increase of capital $K(t)$ at the moment t results form investment decisions undertaken at the moment $t-\tau$, where τ denotes the average time of investment completion. The thorough discussion of the model and its solutions can be found in [5, 6]. Although not recognized at first, the Kalecki's model is now regarded to be the first fully endogenous model of business cycles and a starting point for further investigations. Then Kaldor introduced his model of business cycles which is now regarded to be a prototype model of nonlinear dynamics [7, 8]. Major contribution of Kaldor was the assumption that the investment function and the saving function are nonlinear functions of income y:

$$\frac{dy}{dt} = \alpha(I(y,K) - S(y,K)), \tag{1}$$

$$\frac{dK}{dt} = I(y,K) - \delta K, \tag{2}$$

where $I(y,K)$ is an investment function depending on income y and capital K, $S(y,K)$ is a saving function, α is an adjustment coefficient, and $\delta \in (0,1)$ is the depreciation rate of capital stock.

In 1999 Krawiec and Szydłowski applied the Kalecki's idea of investment gestation period to the Kaldor model [9] what resulted in a system of differential equations with delay:

$$\frac{dy}{dt} = \alpha(I(y(t),K(t)) - S(y(t),K(t))), \tag{3}$$

$$\frac{dK}{dt} = I(y(t-\tau),K(t)) - \delta K(t), \tag{4}$$

where τ is the time delay and we keep the previous denotations. They proposed the following form of the investment function:

$$I(y,K) = \eta y + \beta K, \tag{5}$$

where $\eta > 0$ and $\beta < 0$. In the analysis of the above model the authors used the Poincare–Andronov–Hopf theorem to predict the occurrence of a bifurcation to a limit cycle for some value of parameter τ.

In the following years many researchers contributed to such a version of the Kaldor–Kalecki model, which has recently been referred to as a Krawiec–Szydłowski model [10]. The amendments done concerned delay and the form of investment and saving functions [9–16]:

$$\frac{dy}{dt} = \alpha(I(y(t),K(t)) - S(y(t),K(t))), \tag{6}$$

$$\frac{dK}{dt} = I(y((t-\tau),K(t-\tau)) - \delta K(t), \tag{7}$$

where

$$I(y,K) = I(y) + \beta K \tag{8}$$

and

$$S(y,K) = \gamma y. \tag{9}$$

Also investigations of the cyclic behavior of this model for s-shaped investment functions were performed.

2. Construction of the model

The above discussed model needs changes in order to reproduce actual features of an economy: gross product and the capital stock. Nowadays, when local economies are subject to various economic and political dependences, e.g., due to various unions (monetary, economic, etc.), macroeconomic models cannot be treated as isolated systems any more. Especially the influence of large and dominating economies on Gross Domestic Product of local economies cannot be neglected. In this paper we focus on the investigation of properties of the modified two Kaldor–Kalecki equations with feedback and delay. In the model we study the coupling describes

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the situation when the global market influences the local market and modifies the business cycles. In our study we consider two types of investment functions that lead to different behavior of the system.

In electronic engineering, in which mostly linear four-poles are considered, we can distinguish positive (in which a signal is amplified) and negative feedback loops. In nonlinear dynamics, feedback does not have to behave like that. Nonlinearity has a very serious impact on the behavior of a system (e.g, [17, 18]). Therefore, including the time delay in the coupling, we obtain the final version of our model that reads:

$$\dot{y}_1 = \alpha_1(F_1(t) - \delta_1 y_2(t) - \gamma_1 y_1(t)), \tag{10}$$

$$\dot{y}_2 = F_1(t-\tau) - \delta_1 y_2(t-\tau) - \delta y_2(t), \tag{11}$$

$$\dot{y}_3 = \alpha_2(F_2(t) - \delta_2 y_4(t) - \gamma_2 y_3(t)), \tag{12}$$

$$\dot{y}_4 = F_2(t-\tau) - \delta_2 y_4(t-\tau) - \delta y_4(t) + s(y_1(t) - y_3(t)), \tag{13}$$

where $F_1(t)$ and $F_2(t)$ are investment functions, $s \geq 0$ is a coupling coefficient, α_1 and α_2 are the adjustment coefficients (correction factors), $\delta \in (0,1)$ is the depreciation rate of capital stock, γ_1, γ_2, δ_1, δ_2 are constants, and τ denotes the delay.

In our discussion, we assume that the first two equations of the above model describe the global market, say, the European Union, and the last two equations are responsible for the behavior of the local market, say, Poland. We consider only an unidirectional coupling, although under some very special conditions it is not impossible that "smaller" or "weaker" economy can affect the "larger" or "stronger" one. Another important, and in fact a quite natural, assumption is also the condition $\alpha_1 \geq \alpha_2$, i.e., that the correction factor for the "stronger" market is equal to or greater than analogous coefficient corresponding to the "weaker" market.

In the literature, investment functions are often (not to say typically) given as logistic functions, what means that the investment is increasing for a certain interval, and then takes a constant positive value. However, taking into account various policies and actions present on real markets (e.g., the EU subsidy programs) it could happen that the investment function of a smaller local market is of different form and can, in some cases, exhibit oscillatory behavior. Therefore we consider two particular cases. First, we assume that both investment functions are the logistic functions $F_{1,2}(t) = e^{y_{1,3}(t)}/(1 + e^{y_{1,3}(t)})$. Next, we keep the logistic form for the investment function describing the "larger" economy $F_1(t) = e^{y_1(t)}/(1 + e^{y_1(t)})$ and propose an oscillatory investment function for the "weaker" market $F_2(t) = 0.8\sin(y_3(t))$. The amplitude 0.8 was carefully chosen by the trial and error method to provide clear and useful illustrative results. It is worth noting that in the literature (see, e.g., [15]) one can also find different investment functions, e.g., in the form of hyperbolic tangent.

3. Results

To perform an extensive analysis of the above formulated and discussed system, a numerical model was created using Dynamics Solver and a program in C++ was written. A very successfull Dormand–Prince 8 integration algorithm was used. This method is a modification of the explicit Runge–Kutta formula with a variable integration step. Results were checked, confirmed, and plotted using the Matlab dde23 solver.

We start with the following set of the model parameters: $\alpha_1 = 3.0$, $\alpha_2 = 3.0$, $\gamma_1 = 0.2$, $\gamma_2 = 0.2$, $\delta_1 = 0.2$, $\delta_2 = 0.2$, and $\delta = 0.1$. The time delay is given by $\tau = 3.0$ and will remain constant in all computations performed in this paper. For the basic model system without coupling ($s = 0$), periodic oscilations and the corresponding limit cycle (with a transient state) were obtained for this set of parameters and visualised in Fig. 1.

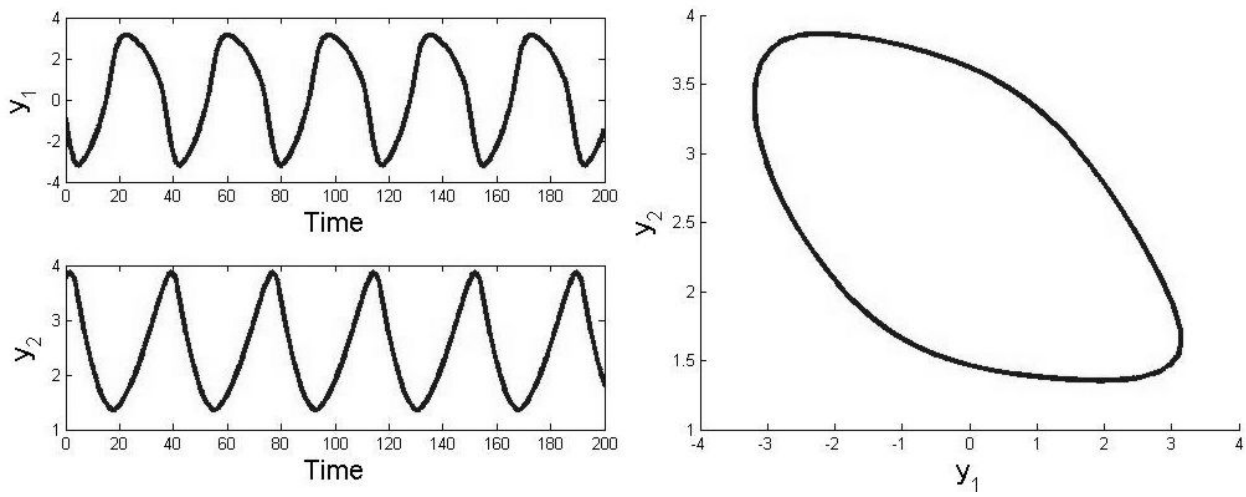

Fig. 1. The basic model with no coupling ($s = 0.0$). Both investment functions are logistic in form. Parameters: $\alpha_1 = 3.0$, $\alpha_2 = 3.0$, $\gamma_1 = 0.2$, $\gamma_2 = 0.2$, $\delta_1 = 0.2$, $\delta_2 = 0.2$, $\delta = 0.1$, and $\tau = 3.0$.

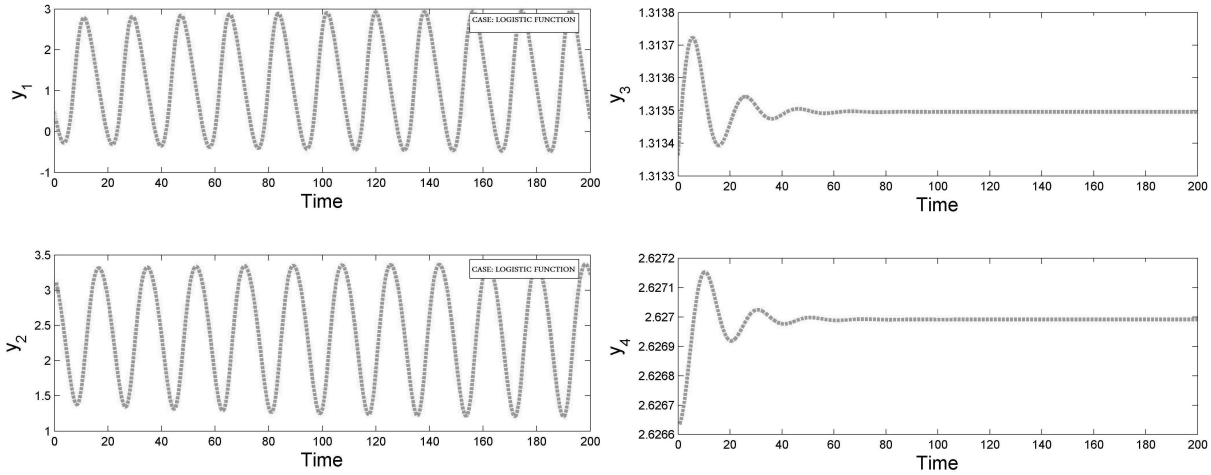

Fig. 2. Time series for the model without coupling ($s = 0.0$). Both investment functions are logistic in form. Parameters: $\alpha_1 = 3.0$, $\alpha_2 = 1.5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.2$, $\delta_1 = 0.2$, $\delta_2 = 0.2$, $\delta = 0.1$, and $\tau = 3.0$.

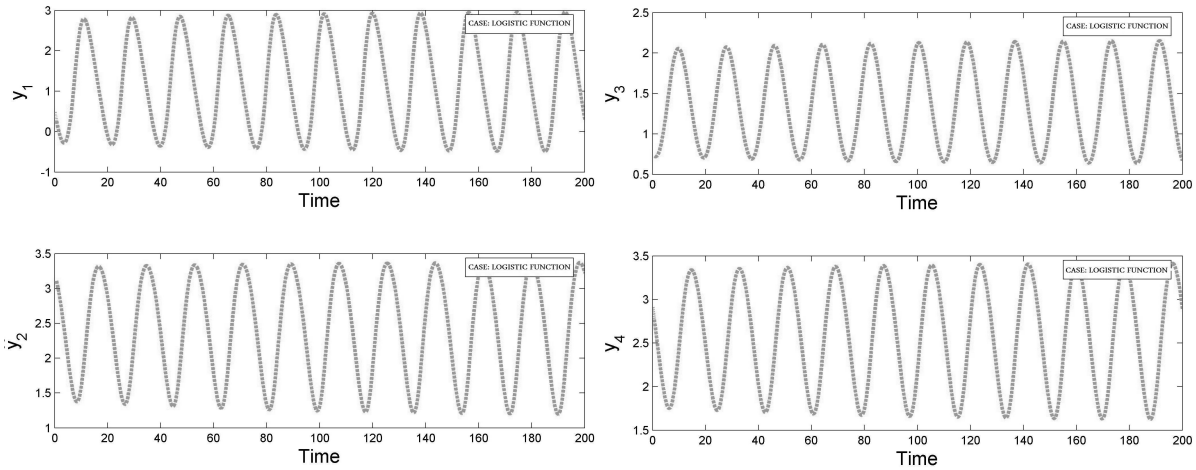

Fig. 3. Time series for the model with coupling ($s = 0.1$). Both investment functions are logistic in form. Parameters: $\alpha_1 = 3.0$, $\alpha_2 = 1.5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.2$, $\delta_1 = 0.2$, $\delta_2 = 0.2$, $\delta = 0.1$, and $\tau = 3.0$.

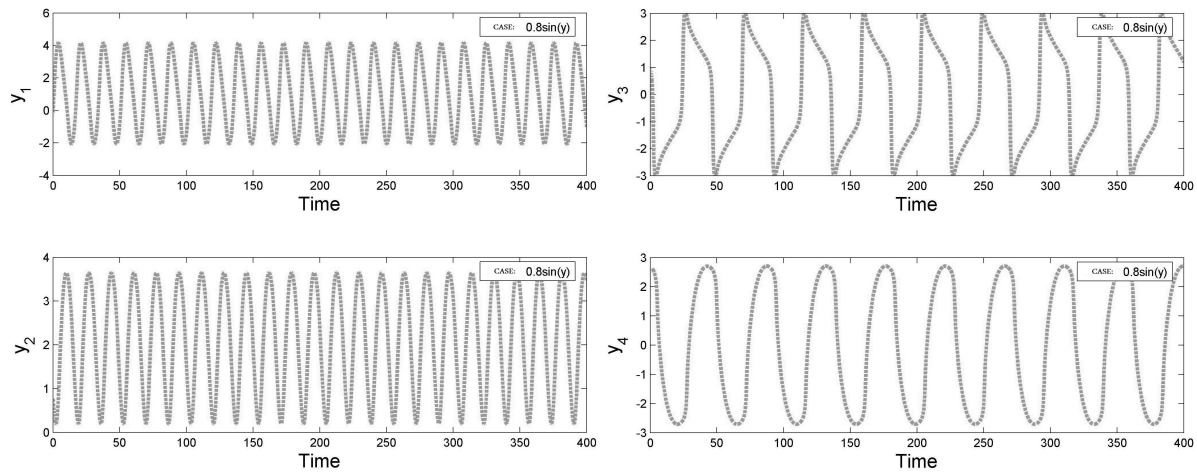

Fig. 4. Time series for the model without coupling ($s = 0.0$). The first investment function is of logistic form and the second is oscillatory. Parameters: $\alpha_1 = 4.0$, $\alpha_2 = 3.0$, $\gamma_1 = 0.2$, $\gamma_2 = 0.2$, $\delta_1 = 0.2$, $\delta_2 = 0.2$, $\delta = 0.1$, and $\tau = 3.0$.

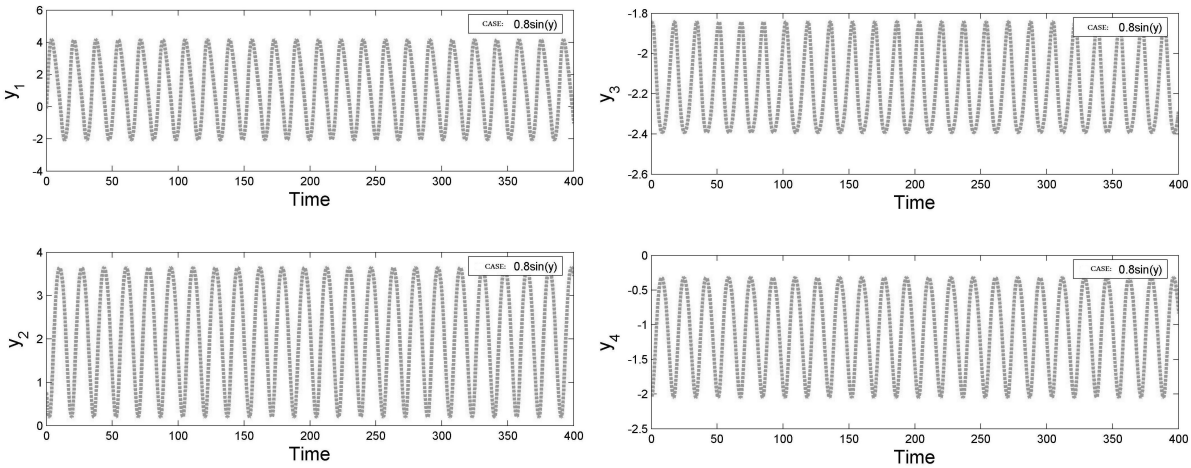

Fig. 5. Time series for case with coupling $s = 0.1$. The first investment function is of logistic form and the second is oscillatory. Parameters: $\alpha_1 = 4.0$, $\alpha_2 = 3.0$, $\gamma_1 = 0.2$, $\gamma_2 = 0.2$, $\delta_1 = 0.2$, $\delta_2 = 0.2$, $\delta = 0.1$, and $\tau = 3.0$.

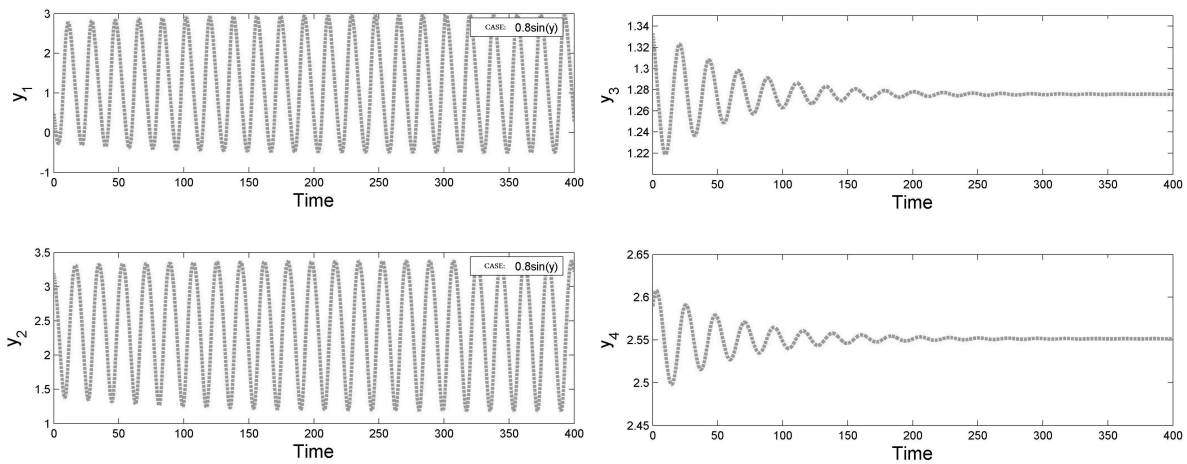

Fig. 6. Time series for the model without coupling ($s = 0.0$). The first investment function is of logistic form and the second is oscillatory. Parameters: $\alpha_1 = 3.0$, $\alpha_2 = 1.5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.2$, $\delta_1 = 0.2$, $\delta_2 = 0.2$, $\delta = 0.1$, and $\tau = 3.0$.

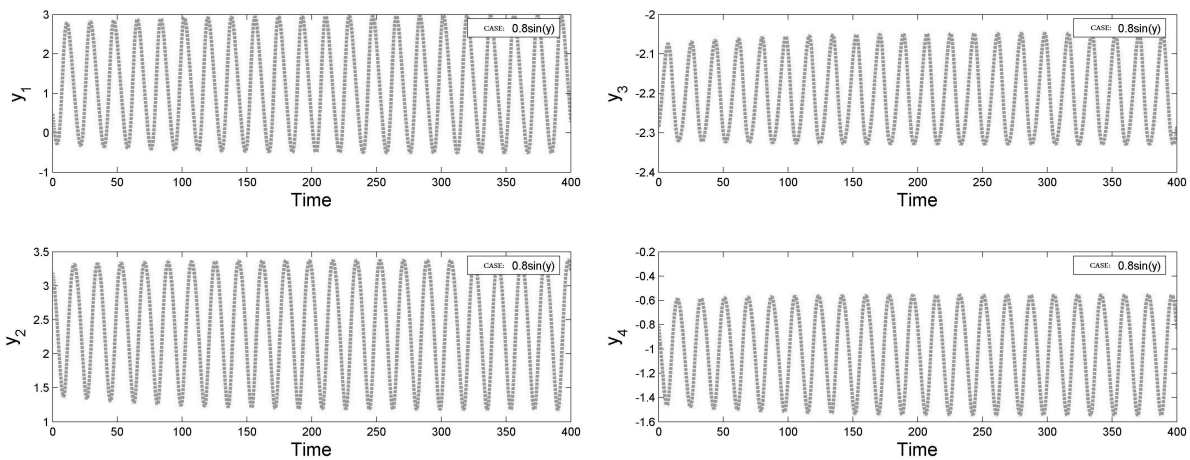

Fig. 7. Time series for the model with coupling ($s = 0.1$). The first investment function is of logistic form and the second is oscillatory. Parameters: $\alpha_1 = 3.0$, $\alpha_2 = 1.5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.2$, $\delta_1 = 0.2$, $\delta_2 = 0.2$, $\delta = 0.1$, and $\tau = 3.0$.

In Fig. 2 we set $\alpha_2 = 1.5$ and consider no coupling. The system with small values of the α is not able to function properly; such a small adjustment coefficient prevents us from obtaining the business cycle, which confirms the experience that countries with very high debt may go bankrupt without outside help. The next step is to add a unidirectional coupling to our system and look at how the global system oscillatory behavior will affect for extinguished oscillations of the local system. We assume that $s = 0.1$. We note that the oscillations of y_3 and y_4 are raised but phase portraits of individual systems are of similar nature, as shown in the following Fig. 3. It follows that a strong economic system has a big impact on smaller markets, and it is worth noting that the coupling coefficient adopted here a small value.

In the next point we consider the case in which we change the value of $\alpha_1 = 4$ and we have no coupling. Other parameters are like in the first reference case. In this case we do not have any oscillation extinction, but we note that the phase portrait of the first equation is stable and describes a fully periodic behavior, where $T_1 \approx 16.9$, $T_2 \approx 16.8$, $T_3 = 18.14$, $T_4 \approx 18.07$. Adding to our system an unidirectional coupling s, causes that (y_1, y_2) directs (y_3, y_4). An attempt to fit the y_3 and y_4 to the y_1 and y_2 resulted in disorder oscillations as well as changing the length of the local amplitude of the system. Now we consider results for logistic function for global economy and the sinus function for local economy. First we consider the case without coupling, in which $\alpha_1 = 4$. Other parameters are the same as in the reference model. The slight increase of α_1 results in considerable decrease of periods of the oscillation. The system still oscillates while the reference investment function in the second equation extends the period of oscillation ($T_1 \approx 16.9$, $T_2 \approx 16.8$, $T_3 \approx 44.3$, $T_4 \approx 44.6$), what is presented in Fig. 4.

Next, we consider the system as described above but with unidirectionally coupling, with coupling coefficient $s = 0.1$. We note that under the influence of the feedback oscillation periods for both y_3 and y_4 are significantly reduced, while still they are characterized by periodic behavior. It is worth emphasizing that the system (y_1, y_2) controls the (y_3, y_4). The period of oscillation for y_1 and y_3 is the same: $(T_1 = T_3)$, what we can see in Fig. 5. In order to verify the influence of the feedback on the first step, we start from the model without feedback, where $\alpha_2 = 1.5$ and other parameters are the same as in the reference case. We can observe, that the behavior of the income and capital for the economic system is very unfavorable for small correction factor (α), what is presented in Fig. 6 for y_3 and y_4. Subsequently, we add unidirectional coupling, where $s = 0.1$. We can observe that the weaker market influenced by the powerful global system is able to restore its oscillatory character, see Fig. 7. This is true even for a relatively small value of the coupling coefficient.

4. Concluding remarks

The Kaldor–Kalecki model, although created many years ago, can be adopted to describe today's business cycle working in larger structures by modifying and taking into account coupling between the described structures. It is worth noting that most of the research of this model done so far is based on a very narrow range of parameters. Nowadays, when local economies are subject to various economic and political dependences, e.g., due to various unions (monetary, economic, and others), macroeconomic models cannot be treated as isolated systems any more. Especially the influence of dominating economies on Gross Domestic Product of local economies cannot be neglected. In our paper was proposed an innovation idea: Kaldor–Kalecki model has not been used earlier as a system of two coupled Kaldor–Kalecki submodels. In our study we have modelled the interaction of two economic systems. We have also demonstrated that the stronger global system directs the interdependent local system. In some circumstances the local system is able to restore its oscillatory behavior, even if it has very adverse economic parameters and has been blanked system. We believe that our results will contribute to a better understanding of the mechanisms of business models and economic problems.

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