Proceedings of the European Conference Physics of Magnetism, Poznań 2014

Point Group Interpretation of Galois Symmetry of Bethe Ansatz Solutions of Magnetic Pentagonal Ring

B. Lulek[a], T. Lulek[b,*], M. Łabuz[c] and J. Milewski[d]

[a]East European State Higher School, T. Terleckiego 6, 37-700 Przemyśl, Poland

[b]Faculty of Physics, Adam Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland

[c]Department of Theoretical Physics, Faculty of Mathematics and Natural Sciences, University of Rzeszow, S. Pigonia 1, 35-310 Rzeszów, Poland

[d]Institute of Mathematics, Poznan University of Technology, Piotrowo 3A, 60-965 Poznań, Poland

Exact solutions of the eigenproblem of the magnetic pentagonal ring exhibit the arithmetic symmetry expressed in terms of a Galois group of a finite extension of the prime field $\mathbb{Q}$ of rationals. We propose here a geometric interpretation of this symmetry in the interior of the Brillouin zone, in terms of point groups. Explicitly, it is a subgroup of the direct product $C_4 \times D_4$. We present also the appropriate irreducible representations of the group.

DOI: 10.12693/APhysPolA.127.336

PACS: 75.10.Jm, 03.65.Fd, 03.67.Lx, 02.10.Ud, 02.10.De

1. Introduction

Bethe Ansatz [1–4] provides unique exact solution of the eigenproblem of the isotropic Heisenberg Hamiltonian for a magnetic ring of N nodes with the spin 1/2. Recently, it has been demonstrated that this solution exhibits a Galois symmetry, stemming from the fact that the eigenproblem in the initial basis of magnetic configurations is expressible in integers, and thus the solutions require only a *finite* extension of the prime field $\mathbb{Q}$ of rationals [5–8]. The Galois group of this field extension has proven to be a useful tool in discovering several algebraic symmetries between exact eigenvalues and eigenstates. Details of this tool are expressed, however, in a somehow hermetic terms of algebraic Galois theory. We intend to interpret them in terms of quantum mechanical notions and calculations. The main aim of the present paper is interpretation of the Galois group for the magnetic pentagon ($N = 5$) [5, 6] in terms of some symmetries of point groups.

2. The eigenproblem of the pentagon and Bethe Ansatz

A detailed description of the diagonalisation procedure for the magnetic pentagon has been given in [5]. Here we focus our attention on the two-magnon sector ($r = 2$ spin deviations from the ferromagnetic saturation), in the interior $B_{\mathrm{int}} = \{k = \pm1, \pm2\}$ of the Brillouin zone for the pentagon, where k is *the quasimomentum*, the exact quantum number responsible for the translational symmetry of the pentagon, given by the cyclic group C_5. The 10×10 block Heisenberg Hamiltonian for the sector $r = 2$, consisting of integers (2 or 4 on the main diagonal, and 1 or 0 outside), decomposes into two-dimensional blocks of the form given in detail in [5]. In principle, an arbitrary quantum mechanical eigenproblem in the finite-dimensional Hilbert space assumes the number field $\mathbb{C}$ of complex numbers, as underlying number field. The field $\mathbb{C}$ is algebraically complete, i.e., any polynomial with coefficients in $\mathbb{C}$ has all its roots in $\mathbb{C}$. However, in many cases, including our case of pentagon, this field is redundant, in spite of the following facts:

(a) the original 10×10 problem in the basis of magnetic configurations requires only the prime field $\mathbb{Q} \subset \mathbb{C}$ (since all matrix elements are integers),

(b) the effective four 2×2 eigenproblems for the interior B_{int} require the cyclotomic extension $\mathbb{Q}(\omega)$ of the prime field $\mathbb{Q}$,

(c) solutions of the eigenproblem are also expressible in $\mathbb{Q}(\omega)$. Due to this fact, $\mathbb{Q}(\omega)$ can be referred to as to *the Heisenberg field* of magnetic pentagon [6, 7].

The solution obtained from 2×2 eigenproblems do not display explicitly the string structure of the Bethe Ansatz solutions. This structure can be derived by solution of the so called inverse Bethe Ansatz [5–9]:

$$a_k b_k = \omega^{-k},$$

$$a_k + a_k^{-1} + b_k + b_k^{-1} = E_k + 4, \tag{1}$$

where a_k and b_k are unknown *portions of phase* related to spectral paramters (λ, μ) and pseudomomenta p_1, p_2, $\omega = \exp(2\pi \mathrm{i}/5)$, and $E_k = -4 + (-1)^k\sqrt{5}$, $k \in B_{\mathrm{int}}$, stands for eigenenergies of the problem being discussed. Equation (1) has a simple physical meaning of conservation of quasimomentum and energy. According to [6], Bethe parameters a_k, b_k can be obtained in the form

$$(a_k, b_k) = \frac{(-1)^k\sqrt{5} \pm \gamma_{\epsilon(k)}}{2(1+\omega^k)}, \quad k \in B_{\mathrm{int}}, \tag{2}$$

where $\epsilon : \mathbb{Z}_5^{\times} \to C_2$ is the homomorphism given by $\epsilon(2) = -1$, and $\gamma_1 = \mathrm{i}\sqrt{1+2\sqrt{5}}$, $\gamma_{-1} = \sqrt{-1+2\sqrt{5}}$, so that γ_1 and γ_{-1} corresponds to scattered ($k = \pm1$)

*corresponding author; e-mail: tadlulek@amu.edu.pl

and bound ($k = \pm 2$) states of pentagon, denoted by rigged string configurations $\begin{array}{|c|}\hline \mp 2 \\ \hline \mp 2 \\ \hline \end{array}$ and $\begin{array}{|c|c|}\hline \pm 2 & \quad \\ \hline \end{array}$, respectively [10–12].

The Bethe Ansatz solution of the eigenproblem is expressible within the extension of the cyclotomic field $\mathbb{Q}(\omega)$ by γ_1 and γ_{-1}. This extension can be denoted by $\mathcal{B} = \mathbb{Q}(\omega, \gamma_1, \gamma_{-1})$, and referred to as the *Bethe field.*

3. Galois symmetries

The Bethe field $\mathcal{B}$ is a 16-dimensional linear space over $\mathbb{Q}$, with the basis (cf. [6]):

$$[k; l_1, l_2] := \gamma_1^{l_1} \gamma_{-1}^{l_{-1}} \omega^k, \quad l_1, l_2 \in \mathbb{Z}_2, k \in \mathbb{Z}_5^\times. \tag{3}$$

Within this context, the interior B_{int} of the Brillouin zone for pentagon can be identified with the multiplicative group $\mathbb{Z}_5^\times$ of the finite number field $\mathbb{Z}_5$, and the ranges for l_1 and l_{-1} with $\mathbb{Z}_2$. The Galois group of the Bethe field $\mathcal{B}$, i.e. the group $G = \text{Aut}(\mathcal{B}/\mathbb{Q})$ of all automorphisms of $\mathcal{B}$ acts on the basis (3) as

$$(l; \eta_1, \eta_{-1})[k; l_1, l_{-1}] = \eta_1^{l_1} \eta_{-1}^{l_{-1}} [lk; l_{\epsilon(k)}, l_{-\epsilon(k)}], \tag{4}$$

such that

$$G = \{(l; \eta_1, \eta_{-1}) |\, \eta_1, \eta_{-1} \in C_2, l \in \mathbb{Z}_5^\times\}, \tag{5}$$

where $C_2 \equiv \{1, -1\}$ can be interpreted as the cyclic group C_2 with the multiplicative group composition (to be distinguished from the additive group $(\mathbb{Z}_2, +)$). Equation (4) yields the multiplication law

$$(l'; \eta'_1, \eta'_{-1})(l; \eta_1, \eta_{-1}) = \left(l'l; \eta'_{\epsilon(l)} \eta_1, \eta'_{-\epsilon(l)} \eta_{-1}\right). \tag{6}$$

Equation (6) shows that the group G is a semidirect product

$$G = \mathbb{Z}_5^\times \times_\psi (C_2 \times C_2), \tag{7}$$

with $\psi : \mathbb{Z}_5^\times \to \text{Aut}(C_2 \times C_2)$ given by $\psi(l)(\eta_1, \eta_{-1}) = (\eta_{\epsilon(l)}, \eta_{-\epsilon(l)})$, being the action of the active group $\mathbb{Z}_5^\times$ on the passive group $C_2 \times C_2$, which permutes only the order of elements in the semidirect product $C_2 \times C_2$, so that the group (7) is the wreath product of the group C_2 and C_4. It determines the action of the Galois group G in the Bethe field $\mathcal{B}$, in the basis (3) adapted to exact Bethe Ansatz eigenstates. It thus allows one to generate all eigenstates within the interior B_{int} of the Brillouin zone from a single one. We describe the group G as the Bethe–Galois group for the interior of the Brillouin zone for the Heisenberg pentagon.

4. Point group extensions

The chain of subfields $\mathbb{Q} \subset \mathbb{Q}(\omega) \subset \mathcal{B}$ implies that the Bethe–Galois group G is the extension of the (Abelian) group

$$D_2 = \text{Aut}(\mathcal{B}/\mathbb{Q}(\omega)) = \{(l_1, l_{-1}) |\, l_1, l_{-1} \in \mathbb{Z}_2\}, \tag{8}$$

isomorphic to the dihedral point group D_2, by the group

$$C_4 = \text{Aut}(\mathbb{Q}(\omega)/\mathbb{Q}) = \mathbb{Z}_5^\times \cong B_{\text{int}}, \tag{9}$$

isomorphic to the cyclic point group C_4, in accordance with the short exact sequence of groups and homomorphisms

$$1 \to D_2 \to G \to C_4 \to 1. \tag{10}$$

Roughly speaking, the passive group changes at most only the ring of numbers γ_1 and γ_{-1}: $\gamma_1 \longmapsto \pm\gamma_1$, $\gamma_{-1} \longmapsto \pm\gamma_{-1}$, whereas the active group C_4 permutes quasimomenta in the interior of the Brillouin zone. The combined action of these two groups, described precisely by Eq. (6), admits transformations of bound and scattered two-magnon states, accompanied by a gauge by ± 1.

5. Geometric interpetation

The Bethe–Galois group G, presented in previous sections as the extension of D_2 by C_4 with the operator action ψ, can be also seen as a subgroup of the index 2, embedded into the direct product $D_4 \times C_4$ of standard point groups. Within this setting, G acts on two squares, presented in Fig. 1. Vertices of the first square represent the regular orbit of the defining action of C_4 as the Galois group of the cyclotomic field along Eq. (9), whereas those of the second — a transitive representation of the point group D_4, having D_2 of Eq. (8) as its subgroup.

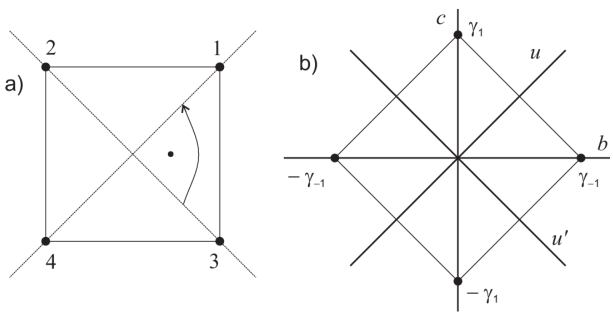

Fig. 1. The action of Galois group on two squares: (a) passive group, (b) active group.

On the first square there acts the group $C_4 = \langle c_4 \rangle$, which is generated by the element c_4, corresponding to the rotation by the $\pi/2$ angle. It is compatible with a multiplication of an index of vertices by $2 \bmod 5$. On the second square there acts the group D_4, which, except of rotations by a multiple of the right angle, contains also reflections versus b, c, u and u' axes. For both groups, C_4 and D_4, one can distinguish adequate subgroups of the rank 2: $C_4^2 := \langle {c_4}^2 \rangle$ in C_4, and $D_2 := \langle b, c \rangle$ in D_4. The Bethe–Galois group is isomorphic with the group G:

$$G = (C_4^2 \times D_2) \cup (c_4 C_4^2 \times c_4 D_2). \tag{11}$$

Now we see that groups described by Eqs. (5) and (11) are isomorphic. Indeed, the isomorphism is given by generators of these groups [6]:

$$\tilde{a} = (c_4, u) \cong (2; 1, 1), \quad \tilde{b} = (\mathrm{e}, b) \cong (1; -1, 1). \tag{12}$$

The form (11) is convenient to make a split of the group G into classes of conjugated elements. As all the classes from the group C_4 are single-element (like in every Abelian group), and taking into account classes in the group D_4 of the form $\{\mathrm{e}\}, \{c_4, {c_4}^{-1}\}, \{{c_4}^2\}, \{a, b\}, \{u, u'\}$, one gets the split of Bethe–Galois group into 4 one-element classes

$$\{(\mathrm{e},\mathrm{e})\},\quad \{(\mathrm{e},{c_4}^2)\},\quad \{({c_4}^2,\mathrm{e})\},\quad \{({c_4}^2,{c_4}^2)\},\tag{13}$$

and 6 two-element classes

$$\begin{aligned}&\{({c_4}^l,c_4),({c_4}^l,{c_4}^{-1})\}\,;\quad l=\pm1,\\ &\{({c_4}^l,b),({c_4}^l,c)\}\,;\quad l=0,2,\\ &\{({c_4}^l,u),({c_4}^l,u')\}\,;\quad l=\pm1.\end{aligned}\tag{14}$$

Furthermore, the formula (11) gives possibility of construction of irreducible representations of the group G via representation of the group C_4 and D_4. Appropriate matrices of irreducible representations are of the form

$$\tau_{n,m}(\tilde{b})=(-1)^n,\ \tau_{n,m}(\tilde{a})=\mathrm{i}^m,\qquad n\in\mathbb{Z}_2,\ m\in\mathbb{Z}_4,\tag{15}$$

for one-dimensional representations, and

$$E_0(\tilde{b})=E_1(\tilde{b})=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad E_0(\tilde{a})=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad E_1(\tilde{a})=\begin{pmatrix}0&\mathrm{i}\\\mathrm{i}&0\end{pmatrix}\tag{16}$$

for two-dimensional representations. One can distinguish eight one-dimensional representations, and two two-dimensional representations, which is concordant with the Burnside theorem $8\cdot1^2+2\cdot2^2=16$.

6. Conclusions

We have proposed an interpretation of the Galois symmetries of the Bethe Ansatz solutions for magnetic pentagonal ring inside its Brillouin zone, expressed by so-called Bethe group, in terms of point groups C_4, D_2, and D_4. Namely, the semidirect product of D_2 by C_4 can be embedded as a subgroup of index 2 in the direct product $C_4\times D_4$, which admits a geometric interpretation of actions of the relevant group on two deformed squares in the complex plane. It makes transparent the structure of conjugacy classes and irreducible representations of the Bethe group. In particular, bound and scattered two-magnon states are interchanged by the action of the Bethe group.

Acknowledgments

The research was partially supported by the grant: NCN 2013/09/B/ST1/04416.

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