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Shape Limit in Triangular Spiral Tilings

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Phyllotaxis is the study of arrangements of leafs and florets. The topology of triangular spiral (multiple) tilings with opposed parastichy pairs is intimately related to the phyllotaxis theory and continued fractions. It is shown that, if the divergence angle of the genetic spiral is given as a quadratic irrational and fixed, then the limit set of the shape parameters of triangular tiles, as the parastichy numbers tend to infinity, is a finite set. In particular, the limit is the golden section if the divergence angle is 'ultimately golden'.

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1. Introduction

Phyllotaxis [1] is an interdisciplinary subject related to physics, biology, and mathematics [2, 3], where the golden section $\tau = (1+\sqrt{5})/2$, the Fibonacci numbers, and continued fraction expansions play important roles [4, 5]. In [6], a quasi-crystalline structure is observed in a circular defect line in the *parastichy transition* of spiral phyllotaxis. Recently, there are intensive studies on the dynamical models that generate spiral phyllotaxic patterns [7–10], as well as geometric approaches on the spiral phyllotaxis [11–13].

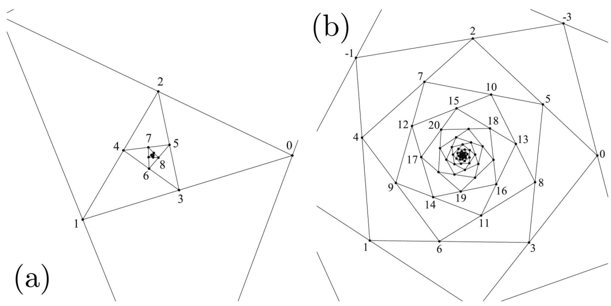

Fig. 1. Triangular spiral tilings with vertex sets $\{z^j\}_{j\in\mathbb{Z}}$, $z = r\,\mathrm{e}^{2\pi\mathrm{i}\tau}$, $\tau = (1+\sqrt{5})/2$. The number j denotes the point $z^j \in \mathbb{C}$. (a) $r = 0.6780$, opposed parastichy pair $\{2, 3\}$. (b) $r = 0.9328$, opposed parastichy pair $\{5, 3\}$.

One of the most simplified mathematical models of spiral phyllotaxis is the triangular spiral tiling that admits transitive action of a similarity transformation group, with no rotational symmetry, Fig. 1. In the previous paper [14], we have shown that the parameter space P_v of the generators $z = r\,\mathrm{e}^{\mathrm{i}\theta}$ of triangular spiral (multiple) tilings of multiplicity $|v| \neq 0$ with *opposed parastichy pairs* is a nowhere dense subset of the unit disk

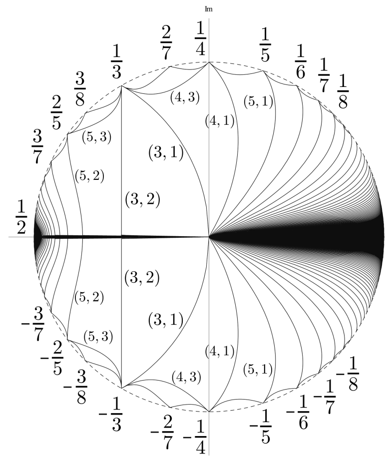

Fig. 2. The set $P_1 \cup P_{-1}$ of generators for triangular spiral tilings with opposed parastichy pairs. The arcs $P_{m,n,1}$ and $P_{m,n,-1}$, defined in [14], are denoted by (m, n).

$\mathbb{D}$, which is a countable family of real algebraic curves parametrized by the *divergence angle* θ, see Fig. 2, whereas the parameter space Q_v for triangular spiral (multiple) tilings with *non-opposed parastichy pairs* is a dense subset of $\mathbb{D}$, being a countable union of real algebraic curves parametrized by the *plastochrone ratio* $1/r$. In P_v, the *opposed parastichy pair* is described by the continued fraction expansion of $\theta/2\pi$. The union $\cup_v P_v$ is a dense subset of $\mathbb{D}$.

In this paper we describe the relationship between the continued fraction of the divergence angle and the triangular spiral (multiple) tilings with an *opposed parastichy pairs*, and study the limit set $\Omega(\theta)$ of the 'shape parameters' of tiles, as $r \to 1$, of triangular spiral multiple tilings with opposed parastichy pairs. It is shown that if $\theta/2\pi$ is a quadratic irrational, then $\Omega(\theta)$ is a finite set of quadratic irrationals. It is known that most divergence angles θ of the spirals observed in plant phyllotaxis are written in the form $\theta = 2\pi(a\tau+b)/(c\tau+d)$, for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ [5]. If it is the case, then we have $\Omega(\theta) = \{-\tau, -1/\tau\}$.

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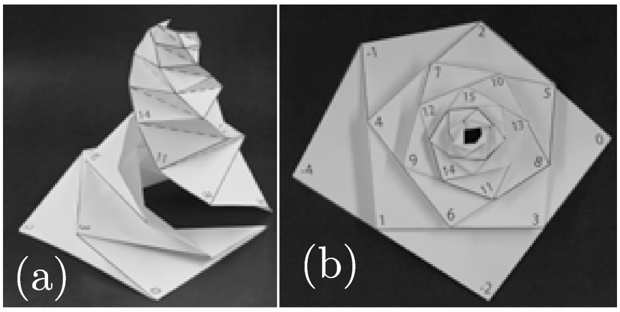

Fig. 3. Paper-folding for the tiling Fig. 1b.

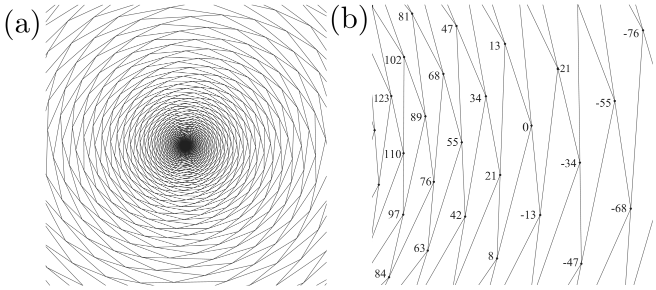

Fig. 4. Triangular spiral tiling generated by $(0.9965)\exp(2\pi \mathrm{i}\cdot\tau)$, $\tau = (1+\sqrt{5})/2$, with an opposed parastichy pair $\{8, 13\}$. (a) Global view around the origin. (b) Local view around the tile T_0.

In the International Conference of Quasicrystals, Krakow, September 2013, we presented paper-folding sheets that build spiral towers whose top-down views are triangular tilings, Fig. 3. See [14, 15] for some backgrounds in figurative arts as applications of phyllotaxis and quasicrystals.

2. Triangular spiral tilings and continued fractions

For $x \in \mathbb{R}$, let

$$x = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \dots}} = [a_0; a_1, a_2, \dots]$$

be its continued fraction expansion [16, 17], where $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}$, $i \geq 1$. Define the sequences $\{p_j\}_{j\geq -1}$ and $\{q_j\}_{j \geq -1}$ by $p_{-1} = 1$, $p_0 = a_0$, $p_1 = a_0 a_1 + 1$, $p_{j+1} = a_{j+1}p_j + p_{j-1}$, $j \geq 1$; $q_{-1} = 0$, $q_0 = 1$, $q_1 = a_1$, $q_{j+1} = a_{j+1}q_j + q_{j-1}$, $j \geq 1$. In this paper we denote by $p_{j,k} = kp_j + p_{j-1}$, and $q_{j,k} = kq_j + q_{j-1}$ for $j \geq 0$, $0 \leq k \leq a_{j+1}$. The fraction $p_j/q_j = [a_0; a_1, \dots, a_j]$, $j \geq 0$, is called a *principal convergent* of x, and $p_{j,k}/q_{j,k} = [a_0; a_1, \dots, a_j, k]$, $j \geq 0$, $0 < k < a_{j+1}$, is called an *intermediate converegent* of x. Note that $p_{j,0} = p_{j-1}$, $q_{j,0} = q_{j-1}$, $p_{j,a_{j+1}} = p_{j+1}$, $q_{j,a_{j+1}} = q_{j+1}$.

A pair of rational numbers $\frac{a}{m}, \frac{b}{n}$ is called a *pair of convergents* of $x \in \mathbb{R}$ if $|bm - an| = 1$ and either $\frac{a}{m} < x < \frac{b}{n}$ or $\frac{b}{n} < x < \frac{a}{m}$. It is known that if $\frac{a}{m}, \frac{b}{n}$ is a pair of convergents of x, then either $a = p_j$, $m = q_j$, $b = p_{j,k}$, $m = q_{j,k}$ with j even, or $a = p_{j,k}$, $m = q_{j,k}$, $b = p_j$, $n = q_j$ with j odd, and $0 < k \leq a_{j+1}$.

Let $I = (-\pi, \pi]$ be a half-open interval, and $\Delta = \Delta_+ \cup \Delta_-$, where $\Delta_+ = \{(\theta_1, \theta_2) \in I^2 : 0 < \theta_1 < \theta_2 + \pi < \pi\}$, $\Delta_- = \{(\theta_1, \theta_2) \in I^2 : 0 < \theta_2 < \theta_1 + \pi < \pi\}$. Denote by $[\![x]\!]$ an integer closest to $x \in \mathbb{R}$ such that $-\frac{1}{2} < \langle x \rangle := x - [\![x]\!] \leq \frac{1}{2}$. Denote a line segment with the endpoints $\zeta_1, \zeta_2 \in \mathbb{C}$ by $\ell(\zeta_1, \zeta_2)$, and a triangle with the vertices $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{C}$ by $\triangle(\zeta_1, \zeta_2, \zeta_3)$.

Proposition 1. *Let $m, n > 0$ be relatively prime integers, and $\theta \in \mathbb{R}$. If $\left(2\pi\langle\frac{m\theta}{2\pi}\rangle, 2\pi\langle\frac{n\theta}{2\pi}\rangle\right) \in \Delta_+$, then there exists a unique $0 < r < 1$ such that for $z = r\mathrm{e}^{\mathrm{i}\theta}$,*

$$\mathcal{T} = \{T_j = \triangle(z^{j+m}, z^{j+n}, z^j)\}_{j\in\mathbb{Z}} \tag{1}$$

is a triangular spiral multiple tiling of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of multiplicity $v = (n\langle\frac{m\theta}{2\pi}\rangle - m\langle\frac{n\theta}{2\pi}\rangle)/2\pi$.

Proof. If $\left(2\pi\langle\frac{m\theta}{2\pi}\rangle, 2\pi\langle\frac{n\theta}{2\pi}\rangle\right) \in \Delta_+$, $\mathcal{T}$ is a (multiple) tiling if and only if z^{m+n} lands on the line segment $\ell(z^m, z^n)$. So r is determined as a root of the equation $f(r) = 0$ where

$$f(r) := r^m \sin n\theta - r^n \sin m\theta + \sin(m-n)\theta, \tag{2}$$

by [14, Lemma 4]. The existence and uniqueness of the root $r \in (0, 1)$ follows from the observation that $f(r)$ is a monotone decreasing function of $0 \leq r \leq 1$, such that $f(0) > 0 > f(1)$. □

The integers $m, n > 0$ in Proposition 1 are called an *opposed parastichy pair* of $\mathcal{T}$. The case $\left(2\pi\langle\frac{m\theta}{2\pi}\rangle, 2\pi\langle\frac{n\theta}{2\pi}\rangle\right) \in \Delta_-$ shall also have a triangular spiral multiple tiling, which we omit here.

The following Proposition follows from [14, Proposition 3]. Denote the principal argument of $z \in \mathbb{C}$ by $-\pi < \mathrm{Arg}(z) \leq \pi$.

Proposition 2. *Let $z = r\mathrm{e}^{\mathrm{i}\theta}$, $0 < r < 1$, $\theta \in \mathbb{R}$, and $m, n > 0$. Suppose that $\mathrm{Arg}(z^n) < 0 < \mathrm{Arg}(z^m)$, and that (1) is a triangular spiral multiple tiling of multiplicity $v > 0$. Then there exist integers $a, b > 0$ such that $a/m < b/n$ is a pair of convergents of $x = \theta/2v\pi$, and we have $v = (n\langle\frac{m\theta}{2\pi}\rangle - m\langle\frac{n\theta}{2\pi}\rangle)/2\pi = (bm - an)/2\pi$.*

Proposition 3. *Let $\theta \in \mathbb{R}$ and $v \in \mathbb{N}$. Suppose that $a/m < b/n$ is a pair of convergents of $\theta/2v\pi$, and $m, n > 0$ are sufficiently large. Then there exists a unique $0 < r < 1$ such that for $z = r\mathrm{e}^{\mathrm{i}\theta}$, (1) is a triangular spiral multiple tiling of $\mathbb{C}^*$ of multiplicity v.*

Proof. If m, n are large, then a/m, b/n are close to $\theta/2v\pi$, and so we have $\left(2\pi\langle\frac{m\theta}{2\pi}\rangle, 2\pi\langle\frac{n\theta}{2\pi}\rangle\right) \in \Delta_+$. Proposition 1 is applied to obtain the tiling (1). The multiplicity is obtained from [14, Proposition 3]. □

3. Shape limit in triangular spiral tilings

Let $v > 0$, $\theta \in (-v\pi, v\pi]$. In this section we suppose that $\theta/2v\pi$ is a fixed irrational number. In the continued fraction expansion of $x = \theta/2v\pi$, we consider the sequences q_j and $q_{j,k}$, $j>0$, $0 \le k \le a_{j+1}$, as defined in Sect. 2. For each $j>0$ and $0 \le k \le a_{j+1}$, denote by $a_{j,k}/m_{j,k} < b_{j,k}/n_{j,k}$ a pair of convergents of $x=\theta/2v\pi$ such that $\{m_{j,k}, n_{j,k}\} = \{q_j, q_{j,k}\}$. Suppose that j is sufficiently large that $\left(2\pi\langle\frac{m_{j,k}\theta}{2\pi}\rangle, 2\pi\langle\frac{n_{j,k}\theta}{2\pi}\rangle\right) \in \Delta_+$. Let $0 < r = r_{j,k} < 1$ be the root of the Eq. (2), and $z_{j,k} = r_{j,k}\mathrm{e}^{\mathrm{i}\theta}$. Then we obtain a (multiple) tiling (1) with an opposed parastichy pair $\{m,n\} = \{m_{j,k}, n_{j,k}\}$.

Lemma 4. $\mathrm{Arg}(z_{j,k}^{q_{j,k}}) \to 0$ *as* $j \to \infty$.

Proof. It is known that

$$\left|\frac{\theta}{2v\pi} - \frac{p_j}{q_j}\right| \le \frac{C}{q_j^2}$$

where the constant $C>0$ is independent of j. Hence

$$\begin{aligned}\left|\mathrm{Arg}(z_{j,k}^{q_{j,k}})\right| &\le \left|\mathrm{Arg}(z_j^{q_j})\right| = 2\pi\left|\langle\frac{q_j\theta}{2\pi}\rangle\right| \\ &= 2\pi\left|\frac{q_j\theta}{2\pi} - vp_j\right| \le \frac{2Cv\pi}{q_j} \to 0 \qquad (3)\end{aligned}$$

as $j \to \infty$. □

Lemma 5. *Let* $v>0$, $\theta \in (-v\pi, v\pi]$, *and suppose that* $\theta/2v\pi$ *is an irrational number. Then, the angles* $\angle(1, z_{j,k}^{m_{j,k}}, z_{j,k}^{n_{j,k}})$ *and* $\angle(z_{j,k}^{m_{j,k}}, z_{j,k}^{n_{j,k}}, 1)$ *tend to* 0 *as* $j \to \infty$.

Proof. By [14, Lemma 4], the four points $z_{j,k}^{m_{j,k}}$, 0, $z_{j,k}^{n_{j,k}}$, 1 lie on a same circle. Thus we have $\angle(1, z_{j,k}^{m_{j,k}}, z_{j,k}^{n_{j,k}}) = \angle(1,0,z_{j,k}^{n_{j,k}}) = \mathrm{Arg}(z_{j,k}^{n_{j,k}}) \to 0$ as $j\to\infty$, and similarly $\angle(z_{j,k}^{m_{j,k}}, z_{j,k}^{n_{j,k}}, 1) = \angle(z_{j,k}^{m_{j,k}}, 0, 1) = \mathrm{Arg}(z_{j,k}^{m_{j,k}}) \to 0$ as $j\to\infty$. □

Lemma 6. *Suppose that the coefficients* $\{a_j\}_{j\ge 0}$ *in the continued fraction expansion* $\theta/2v\pi = [a_0; a_1, a_2, \dots]$ *are bounded. Then we have*

$$0 < 1 - r_{j,k} \le \frac{C}{m_{j,k}^3},$$

where $C>0$ *is a constant independent of* j,k.

Proof. We shall adopt a notation $\varphi = O(m^{-s})$ when there exists a constant C independent of j,k such that $|\varphi| \le C/m_{j,k}^s$. Since the coefficients a_j are bounded, the ratios $n_{j,k}/m_{j,k}$ are also bounded, so we may write $O(m^{-s}) = O(n^{-s})$. We have

$$\begin{aligned}\sin m\theta &= 2\pi\langle\frac{m\theta}{2\pi}\rangle - \frac{(2\pi)^3}{6}\langle\frac{m\theta}{2\pi}\rangle^3 + O(m^{-5}) \\ &= O(m^{-1})\end{aligned}$$

by (3), and

$$\begin{aligned}&\sin n\theta - \sin m\theta + \sin(m-n)\theta \\ &= 4\pi^3\langle\frac{m\theta}{2\pi}\rangle\langle\frac{n\theta}{2\pi}\rangle\langle\frac{(m-n)\theta}{2\pi}\rangle + O(m^{-5}) \\ &= O(m^{-3}),\end{aligned}$$

so

$$\frac{\sin(m-n)\theta}{\sin m\theta - \sin n\theta} = 1 + O(m^{-2}),$$

where we denote by $m = m_{j,k}$, $n = n_{j,k}$ for the sake of simplicity. Since $0 < r = r_{j,k} < 1$ is a root of (2), we have $\lim_{j\to\infty} r_{j,k}^{m_{j,k}} = \lim_{j\to\infty} r_{j,k}^{n_{j,k}} = 1$, and

$$r_{j,k}^{m_{j,k}} = 1 + O(m^{-2}), \qquad r_{j,k}^{n_{j,k}} = 1 + O(m^{-2}). \qquad (4)$$

Let $t_{j,k} := 1 - r_{j,k}$. Then we have $t_{j,k} = O(m^{-3})$ by (4), which completes the proof. □

Suppose that $\theta/2v\pi$ is a quadratic irrational. Then it has a periodic continued fraction expansion

$$\begin{aligned}\frac{\theta}{2v\pi} &= [a_0; a_1, a_2, \dots] \\ &= [a_0; a_1, \dots, a_{j_0}, \overline{b_1, \dots, b_d}] \\ &= [a_0; a_1, \dots, a_{j_0}, b_1, \dots, b_d, b_1, \dots, b_d, \dots].\end{aligned}$$

We may assume that j_0, d are even, by choosing larger ones if necessary. For each $1 \le h \le d$, let

$$\omega_h = [b_h; \overline{b_{h+1}, \dots, b_d, b_1, \dots, b_h}]$$

be a purely periodic continued fraction.

Let $R(\theta, v)$ be the set of ratios $(z_{j,k}^{n_{j,k}} - 1)/(z_{j,k}^{m_{j,k}} - 1)$ for $j>0$ and $0 < k \le a_j$. Let

$$\Omega(\theta, v) := \Omega(R(\theta, v))$$

be the limit set, i.e., the set of the accumulation points, of $R(\theta, v)$.

Theorem 7. *Suppose that* $\theta/2v\pi$ *is a quadratic irrational. Then we have*

$$\Omega(\theta, v) = \{(\omega_j - k)^{(-1)^j} : 0 < j \le d, 0 < k \le b_j\}. \qquad (5)$$

In particular, it is a finite set of quadratic irrationals.

Proof. Since $\theta/2v\pi$ is a quadratic irrational, there exists a constant $C_1, C_2 > 0$, independent of $j > 0$, $0 < k \le a_{j+1}$, such that

$$\frac{C_1}{q_{j,k}^2} < \left|\frac{p_{j,k}}{q_{j,k}} - \frac{\theta}{2v\pi}\right| < \frac{C_2}{q_{j,k}^2}.$$

This implies that

$$\frac{C_1}{m_{j,k}} < \left|\langle\frac{m_{j,k}\theta}{2v\pi}\rangle\right| < \frac{C_2}{m_{j,k}}.$$

We have

$$\begin{aligned}\frac{z_{j,k}^{n_{j,k}} - 1}{z_{j,k}^{m_{j,k}} - 1} &= \frac{-1 + r^n\cos n\theta + \mathrm{i}r^n\sin n\theta}{-1 + r^m\cos m\theta + \mathrm{i}r^m\sin m\theta} \\ &= \frac{-1 + 1 + O(m^{-2}) + \mathrm{i}(2\pi\langle\frac{n\theta}{2\pi}\rangle + O(m^{-2}))}{-1 + 1 + O(m^{-2}) + \mathrm{i}(2\pi\langle\frac{m\theta}{2\pi}\rangle + O(m^{-2}))} \\ &= \frac{2\pi\mathrm{i}\langle\frac{n\theta}{2\pi}\rangle + O(m^{-2})}{2\pi\mathrm{i}\langle\frac{m\theta}{2\pi}\rangle + O(m^{-2})} \\ &= \frac{2\pi\mathrm{i}\langle\frac{n\theta}{2\pi}\rangle(1 + O(m^{-1}))}{2\pi\mathrm{i}\langle\frac{m\theta}{2\pi}\rangle(1 + O(m^{-1}))} \\ &= \frac{\langle\frac{n\theta}{2\pi}\rangle}{\langle\frac{m\theta}{2\pi}\rangle}(1 + O(m^{-1})),\end{aligned}$$

where we denote by $m = m_{j,k}$, $n = n_{j,k}$. Thus it is written as

$$\frac{z_{j,k}^{n_{j,k}} - 1}{z_{j,k}^{m_{j,k}} - 1} = \left(\frac{\langle \frac{q_{j,k}\theta}{2v\pi} \rangle}{\langle \frac{q_j \theta}{2v\pi} \rangle} \right)^{(-1)^j} (1 + O(q^{-1})).$$

By using the continued fractions, we have

$$\langle \frac{q_{j,k}\theta}{2v\pi} \rangle / \langle \frac{q_j\theta}{2v\pi} \rangle = -[a_{j+1} - k, a_{j+2}, a_{j+3}, \dots] \\ = -(\omega_j - k)$$

for j sufficiently large, and $0 \le k \le a_{j+1}$. Thus we obtain Eq. (5). □

It is known that most spirals in plant phyllotaxis have divergence angles θ belonging to a class that might be called 'ultimately golden', or written as $\theta/2\pi = [a_0; a_1, a_2, \dots]$ where $a_j = 1$ for sufficiently large j [5]. This is equivalent to the existence of $a, b, c, d \in \mathbb{Z}$, such that $\theta/2\pi = (a\tau + b)/(c\tau + d)$ and $ad - bc = 1$.

Corollary 8. *Suppose that $\theta/2v\pi$ has a continued fraction expansion $\theta/2v\pi = [a_0; a_1, a_2, \dots]$ such that $a_j = 1$ for sufficiently large j. Then $\Omega(\theta, v) = \{-\tau, -1/\tau\}$.*

Proof. The golden section has a purely periodic continued fraction expansion $\tau = [1; 1, \dots] = [1; \overline{1, 1}]$, and

$$\langle \frac{q_{j,1}\theta}{2v\pi} \rangle / \langle \frac{q_j\theta}{2v\pi} \rangle = -[0; 1, 1, \dots] = -\frac{1}{\tau}$$

for any j. □

Figure 4 shows a triangular spiral tiling generated by $z = r\mathrm{e}^{2\pi \mathrm{i} \tau}$, $r = 0.9965$, $\tau = (1 + \sqrt{5})/2$, with an opposed parastichy pair $\{8, 13\}$, and the ratio $(z^8 - 1)/(z^{13} - 1) = -1.348 + 0.857\mathrm{i}$. If we fix the divergence angle $2\pi\tau$ and consider larger Fibonacci numbers as an opposed parastichy pair, for example $\{55, 89\}$, then we have $r = 0.999989$, and so the ratio $(z^{55} - 1)/(z^{89} - 1) = -1.61208 + 0.13355\mathrm{i}$ gets closer to $-\tau = -1.618$.

Acknowledgments

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