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From Colorings to Weavings

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We present a methodology for constructing weavings from 2-colorings of the plane. In particular, we consider tilings $\mathcal{T}$ of the plane by triangles and their corresponding triangle groups G. We derive 2-colorings of $\mathcal{T}$ using the index 2 subgroups of G.

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1. Introduction

In Ref. [1] Ramsden et al. showed that crystalline nets in the Euclidean space can be constructed from reticulations of the hyperbolic plane $\mathbb{H}^2$. The construction of crystalline three-dimensional Euclidean nets is done by projecting two-dimensional hyperbolic tilings onto a family of triply periodic minimal surfaces.

Among the goals of this study is to construct weavings in $\mathbb{H}^2$ which can then be used to construct three-periodic patterns. We define a weaving as an interlacement, determined by a weaving map, of two disjoint geometrically identical nets. In the following, we discuss and illustrate the methodology using tilings in the Euclidean plane $\mathbb{E}^2$ and, consequently, weavings in $\mathbb{E}^2$, but the methodology also works for the hyperbolic plane $\mathbb{H}^2$ with no adjustments.

2. Tilings and their colorings

Let Δ be any triangle in $\mathbb{E}^2$ with interior angles π/p, π/q, π/r, where p, q, r are integers ≥ 2. Repeatedly reflecting Δ on its sides results in a triangle tiling $\mathcal{T} := \mathcal{T}(p,q,r)$ of the appropriate plane by copies of Δ. Let P, Q, R, respectively, denote the reflections on the sides of Δ opposite the angles π/p, π/q, π/r. The group $G := G(p,q,r)$ of isometries generated by P, Q, R is called a *triangle group* and has group presentation $\langle P, Q, R \mid P^2 = Q^2 = R^2 = (QR)^p = (RP)^q = (PQ)^r \rangle$.

The group G acts on $\mathbb{E}^2$, and the *orbit* of a point $x \in \mathbb{E}^2$ is the set of the images of x under the elements of G, denoted by $Gx = \{gx \mid g \in G\}$. The set of orbits of $\mathbb{E}^2$ under G form a partition of $\mathbb{E}^2$. A subset of $\mathbb{E}^2$ which contains exactly one point from each of these orbits is called a *fundamental region for the action of G on $\mathbb{E}^2$* or, in short, a *fundamental region of G*.

It is easy to see that the triangle Δ contains exactly one point from the orbits of G (except on its boundary which may contain duplicates of representative of some orbits). Then Δ is a fundamental region of G.

The tiling $\mathcal{T}$ is the G-orbit of Δ, and G acts transitively on $\mathcal{T}$ and $Stab_G\Delta = \{e\}$. Consequently, there exists a one-to-one correspondence between G and $\mathcal{T}$ given by $g \mapsto g\Delta$ where $g \in G$, and $g' \in G$ acts on $g\Delta \in \mathcal{T}$ by sending it to its image under g'.

Suppose H is a subgroup of G of index n. Let $\{g_1, g_2, \ldots, g_n\}$ be a complete set of left coset representatives of H in G with $g_1 \in H$, and $\{c_1, c_2, \ldots, c_n\}$ a set of n colors. Then the assignment $g_iH\Delta \mapsto c_i$ defines an n-coloring of $\mathcal{T}$ which is G-transitive.

To construct the coloring, assign color c_i to $g_iH\Delta$ for $i = 1, 2, \ldots, n$. The group G acts transitively on $\{g_1H\Delta, g_2H\Delta, \ldots, g_nH\Delta\}$ with $g \in G$ sending $g_iH\Delta$ to $gg_iH\Delta$. The elements of G which fix c_1 constitute the subgroup H, while the elements which fix color c_i constitute the conjugate subgroup $g_iHg_i^{-1}$ of H. Thus, the symmetry group of the coloring is the intersection of all conjugate subgroups of H which is the $Core_GH = \cap g_iHg_i^{-1}$. This connection between subgroups and coloring is described in [2, 3].

A fundamental region of H denoted by Δ_H consists of n copies of Δ representing each color. Henceforth, we let H be an index two subgroup of G. Then a fundamental region Δ_H may be given by $\Delta \cup g\Delta$ for some $g \notin H$; specifically, Δ_H consists of c_1 = white and c_2 = black copies of Δ. Furthermore, we impose the following condition for the choice of $g \notin H$ in $\Delta_H = \Delta \cup g\Delta$. The reason for imposing this condition will be explained later.

FR Condition. *If H is generated by the set $\{h_1, h_2, \ldots, h_l\}$, we choose g such that the fixed set of each h_j, $\{x \in \mathbb{E}^2 \mid h_jx = x\}$, intersect both Δ and $g\Delta$.*

Let us note that for any index 2 subgroup H of G, it is possible to choose a generating set of H such that the FR Condition is satisfied. However, this is not necessarily true of higher index subgroups of G.

In Fig. 1 (left), we illustrate the tiling $\mathcal{T} := \mathcal{T}(4,4,2)$ which has fundamental region Δ with interior angles $\pi/4$, $\pi/4$, $\pi/2$. It is associated to the triangle group $G := G(4,4,2)$. Then in Fig. 1 (right), we have the 2-coloring of $\mathcal{T}$ corresponding to the subgroup $H_2 = \langle R, Q, PRP \rangle$ of G. The symmetry group of the coloring $Core_GH_2$ is H_2 itself since H_2 is normal in G. A fundamental region of H_2 is $\Delta_{H_2} = \Delta \cup P\Delta$, which consists of a white tile Δ and a black tile $P\Delta$. Let us note that Δ_{H_2} satisfies the FR Condition as the fixed lines of the reflections R, Q, PRP intersect both Δ and $P\Delta$.

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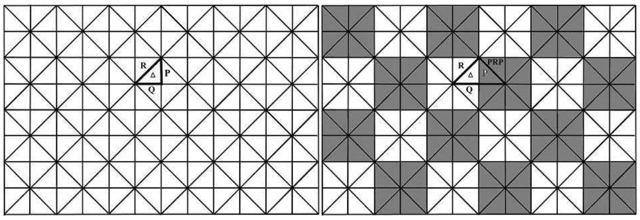


Fig. 1. The tiling $\mathcal{T}(4,4,2)$ (left), and its 2-coloring by $H_2 = \langle R, Q, PRP \rangle$ (right).

3. From colorings to nets

Given a 2-coloring of the tiling $\mathcal{T}$ associated with a subgroup H of G, we simultaneously construct two disjoint nets which are geometrically identical — a black (B-)net and a white (W-)net, which we collectively call *overlapping nets* and denote by $\mathcal{O}_H$. The W-net is represented by a net of dashed lines for obvious reasons.

3.1. Vertex sets of overlapping nets

We define B- and W-vertex sets for $\mathcal{O}_H$ via B- and *W-patches* defined below using ideas from combinatorial tiling theory [4].

Consider the tiles in $\mathcal{T}$ as chambers, and their colors as the chamber classes. We have black and white chamber classes, which are determined by the subgroup H of G used to color $\mathcal{T}$. Furthermore, for each tile $g\Delta$ in $\mathcal{T}$, label the edges opposite the angles π/q, π/r, π/p by $0, 1, 2$, respectively. Then record the neighbor-relations of the chambers which are formally described by maps s_0, s_1, and s_2 from the chamber system onto itself [5]. These maps are defined as follows: the map s_i sends a chamber c to the chamber across the edge i of c.

In our case, the map s_i either fixes (F) the colors or interchanges (I) them. Since the maps s_i cannot fix the colors simultaneously, we only have 7 types of patches defined in Table.

Each patch type corresponds to an index 2 subgroup of $G := G(p,q,r)$. In particular, if all of p, q, r are even, then G has seven index 2 subgroups [3], and each of the seven patch types occurs in a 2-coloring of $T(p,q,r)$.

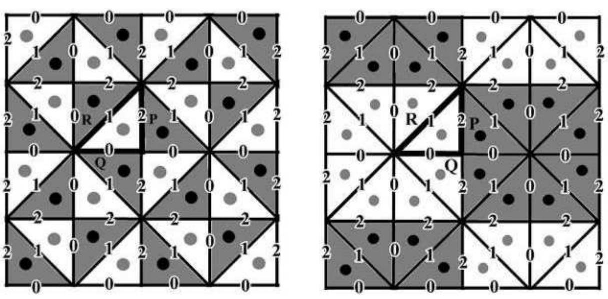


Fig. 2. Patches by: $H_1 = \langle QR, RP \rangle$ (left) and $H_2 = \langle R, Q, PRP \rangle$ (right).

In Fig. 2, we illustrate 2 types of B- and W-patches using two of the seven index 2 subgroups of $G(4,4,2)$. The black and grey dots correspond to B- and W-vertices, respectively, which are defined below.

Vertex Sets Construction. *We define the B-vertex set as the collection of the centroids of the B-patches. The W-vertex set is similarly defined.*

TABLE

The definition of the 7 types of B- and W-patches.

s_0	s_1	s_2	Definition of W- and B-patches
I	I	I	B-patches are the black tiles; W-patches are the white tiles
I	F	F	B-patches are the union of black tiles about the common vertex of 1- and 2-edges; similarly for W-patches
F	I	F	B-patches are the union of black tiles about the common vertex of 0- and 2-edges; similarly for W-patches
F	F	I	B-patches are the union of black tiles about the common vertex of 0- and 1-edges; similarly for W-patches
I	I	F	B-patches are the union of adjacent black tiles along 2-edges; similarly for W-patches
I	F	I	B-patches are the union of adjacent black tiles along 1-edges; similarly for W-patches
F	I	I	B-patches are the union of adjacent black tiles along 0-edges; similarly for W-patches

3.2. Edges of overlapping nets

To construct the edges of $\mathcal{O}_H$ corresponding to the 2-coloring of the tiling $\mathcal{T}$ by H, impose a *motif* on the fundamental region $\Delta_H = \Delta \cup g\Delta$ of H, where g is chosen such that the FR Condition is satisfied.

Motif Construction. *Suppose $\{h_1, h_2, \ldots, h_l\}$ is a generating set of H. Consider the B- and W-vertices on Δ_H, and denote them by b and w, respectively. For each generator h_i, connect the vertex b to the vertices $h_i b$ and ${h_i}^{-1} b$ using straight line segments. Similarly, connect w to $h_i w$ and ${h_i}^{-1} w$. The segment of the edges $(b, h_i b)$, $(b, {h_i}^{-1} b)$, $(w, h_i w)$, $(w, {h_i}^{-1} w)$ within Δ_H together with the vertices w and b is the desired* motif Δ_{mH} *for $\mathcal{O}_H$. The edges $(w, h_i w)$ and $(w, {h_i}^{-1} w)$ are represented by dashed line segments.*

Because of the FR Condition imposed on the choice of the fundamental region Δ_H of H, we are assured that the tiles containing the images of vertices w and b intersect Δ_H and the resulting edges are (possibly) minimized. We place the B-edges above the W-edges whenever they intersect in Δ_{mH} as in Fig. 3.

We remark that the motif Δ_{mH} is dependent on H and on the set of generators chosen for H. For example, a subgroup H_1 of $G(4,4,2)$ may be generated by $H_{1a} = \langle QR, RP \rangle$, $H_{1b} = \langle QR, PQ \rangle$, or $H_{1c} = \langle RP, PQ \rangle$. These generating sets result in the following fundamental

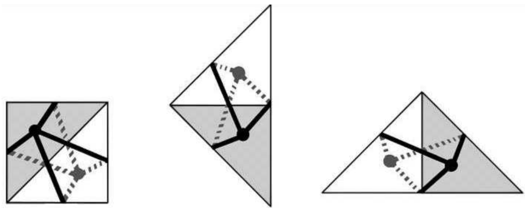


Fig. 3. The motifs derived from three fundamental regions (background) of H_1: $\Delta_{mH_{1a}}$ (left), $\Delta_{mH_{1b}}$ (middle), $\Delta_{mH_{1c}}$ (right).

regions satisfying the FR Condition: $\Delta_{H_{1a}} = \Delta \cup R\Delta$, $\Delta_{H_{1b}} = \Delta \cup Q\Delta$, and $\Delta_{H_{1c}} = \Delta \cup P\Delta$, respectively. They yield three distinct motifs shown in Fig. 3.

Overlapping Nets Construction. *To get the overlapping nets $\mathcal{O}_H$, we let H act on Δ_{mH}. When black edges abut, they are joined continuously to form the B-lines of the B-net; similarly, when dashed edges abut, they are joined continuously to form the W-lines of the W-net.*

We denote by $\mathcal{L}_b$ and $\mathcal{L}_w$ the sets of B-lines and W-lines (or line segments), respectively, of $\mathcal{O}_H$. Now, consider a subset $\mathcal{I}$ of the set $\mathcal{L}_b \times \mathcal{L}_w$ defined as:

$$\mathcal{I} = \{(\ell_b, \ell_w) \in \mathcal{L}_b \times \mathcal{L}_w \mid \ell_b \text{ and } \ell_w \text{ are intersecting lines}\}.$$

Moreover, we interpret (ℓ_b, ℓ_w) as the intersection point of the lines ℓ_b and ℓ_w. This means that the set $\mathcal{I}$ is precisely the set of all intersection points of the B- and W-lines in $\mathcal{O}_H$. We now define a *weaving* using these notations.

Definition 1. *A* weaving *is a triple $(\mathcal{O}_H, \mathcal{I}, \omega)$ where ω is a map from $\mathcal{I}$ to the two-point set $\{\oplus, \ominus\}$, called a* weaving map.

Let $(\ell_b, \ell_w) \in \mathcal{I}$. If $\omega(\ell_b, \ell_w) = \oplus$, then we say that ℓ_b is above *ℓ_w; while if $\omega(\ell_b, \ell_w) = \ominus$, we say that ℓ_b is* below *ℓ_w.*

In other words, a *weaving* is merely an interlacement of the B- and W-nets in $\mathcal{O}_H$. Denote a weaving derived from the subgroup H with respect to a set of generators of H by $(\mathcal{O}_H, \mathcal{I}, \omega)$ or, simply, $\mathcal{W}_H$.

In the next section, we construct the final component of a weaving $\mathcal{W}_H$, which is the weaving map ω.

4. From nets to weavings

Consider a motif Δ_{mH} of overlapping nets $\mathcal{O}_H$. The intersection points of the B- and W-edges within Δ_{mH} are referred to as IPs, and IPs in the same orbit of H are said to be identifiable. Let us note that the elements of the set $\mathcal{I}$ are just images of the IPs under the elements of H. Hence, if we define the weaving map ω on the IPs, then we can naturally extend the mapping to $\mathcal{I}$.

Weaving Construction. *Define ω on the IPs of Δ_{mH} by assigning $\oplus$ and $\ominus$ to them. If there are identifiable IPs, assign an identical symbol to each one of them. Then, let H act on the decorated Δ_{mH}, and assign values ($\oplus$ or $\ominus$) to the rest of the intersection points in $\mathcal{I}$ as follows: if an IP is given a certain symbol, then the elements of its orbit under H will be given the same symbol.*

Clearly, the following scenarios will not yield a (proper) weaving.

No Weaving Scenarios (NWS)

1. Δ_{mH} has only one IP.

2. All the IPs of Δ_{mH} are identifiable.

We now outline the methodology for constructing weavings from 2-colorings.

From Colorings to Weavings: The Methodology

1. Consider a tiling $\mathcal{T} := \mathcal{T}(p,q,r)$ by triangles with interior angles π/p, π/q, π/r, together with its corresponding triangle group $G := G(p,q,r)$.

2. Get a 2-coloring of $\mathcal{T}$ using an index 2 subgroup H of G. The symmetry group of such coloring is $Core_G H = H$.

3. Choose the fundamental region of H to be $\Delta_H = \Delta \cup g\Delta$ where $g \notin H$ and g is chosen such that *FR Condition* is satisfied.

4. Identify the B- and W-patches using the chamber system, and then perform *Vertex Set Construction*.

5. Execute *Motif Construction* to get the motif Δ_{mH}. Then, construct the overlapping nets $\mathcal{O}_H$ via *Overlapping Nets Construction*.

6. If none of the *NWS* occurs, carry out *Weaving Construction*. Otherwise, consider an index 2 subgroup N of H.

The motif Δ_{mN} is composed of two copies of Δ_{mH}, effectively yielding more IPs. The process is repeated if any of the *NWS* occurs on Δ_{mN}, either by considering another index 2 subgroup of H or an index 2 subgroup of N.

4.1. Equivalent weavings

We now define when two weavings are equivalent. The following is adapted from the definition given in [6].

Definition 2. *Let H and H' be subgroups of G which, respectively, yield the weavings $\mathcal{W}_H = (\mathcal{O}_H, \mathcal{I}, \omega)$ and $\mathcal{W}_{H'} = (\mathcal{O}_{H'}, \mathcal{I}', \omega')$. Let $\mathcal{L}_b$ and $\mathcal{L}_w$ be the sets of B- and W-lines in $\mathcal{O}_H$, and $\mathcal{L}'_b$ and $\mathcal{L}'_w$ be the sets of B- and W-lines in $\mathcal{O}_{H'}$.*

We say that $\mathcal{W}_H$ and $\mathcal{W}_{H'}$ are equivalent weavings *if there is a one-to-one correspondence ϕ between the elements of $\mathcal{L}_b$ and $\mathcal{L}'_b$, and between the elements of $\mathcal{L}_w$ and $\mathcal{L}'_w$ under which*

1. *$\mathcal{O}_H$ and $\mathcal{O}_{H'}$ are combinatorially equivalent, and*

2. *$\omega(\ell_b, \ell_w) = \omega'(\phi(\ell_b, \ell_w))$ $\forall (\ell_b, \ell_w)_H \in \mathcal{I}$, or $\omega(\ell_b, \ell_w) = \overline{\omega'(\phi(\ell_b, \ell_w))}$ $\forall (\ell_b, \ell_w)_H \in \mathcal{I}$, where $\overline{\oplus} = \ominus$ and $\overline{\ominus} = \oplus$.*

The equivalence classes of weavings with respect to this equivalence relation are called weaving patterns.

It is straightforward to show that if a weaving $\mathcal{W}$ can be obtained from a weaving $\mathcal{W}'$ by some isometry of the plane, then they are equivalent weavings. For example, consider the weavings generated by the subgroup H_1 using its two sets of generators $H_{1b} = \langle QR, PQ \rangle$ and $H_{1c} = \langle RP, PQ \rangle$ shown in Fig. 4. Observe that if we rotate $\mathcal{W}_{H_{1b}}$ about the centroid of ⊞ by 90° clockwise and then reflect along the vertical line passing through that point, we get $\mathcal{W}_{H_{1c}}$. Thus, $\mathcal{W}_{H_{1b}}$ and $\mathcal{W}_{H_{1c}}$ are equivalent.

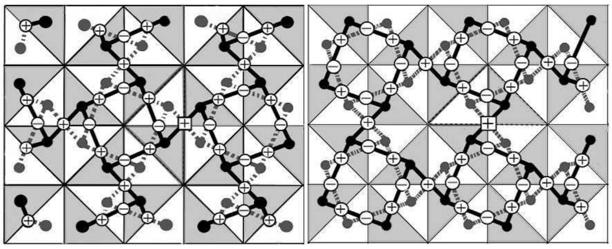


Fig. 4. Equivalent weavings generated by H_1: $\mathcal{W}_{H_{1b}}$ (left), $\mathcal{W}_{H_{1c}}$ (right).

Suppose that N and K are conjugate subgroups in G; that is, $K = gNg^{-1}$ for some $g \in G$. It can be shown that if Δ_N is a fundamental region of N, then $g\Delta_N$ is a fundamental region of K. Consequently, given a weaving $\mathcal{W}_N$ from N, we can derive a weaving $\mathcal{W}_K$ of K which is equivalent to $\mathcal{W}_N$. In particular, applying the isometry g on $\mathcal{W}_N$ results in the weaving $\mathcal{W}_K$.

Now suppose a subgroup H of G, with fundamental region Δ_H, yields a weaving $\mathcal{W}_H$ generated by the decorated motif Δ_{mH}. A fundamental region of an index 2 subgroup N of H is a union of two copies of Δ_H. Then one can get a weaving $\mathcal{W}_N$ by using the union of two copies of the decorated Δ_{mH}. In fact, the weavings $\mathcal{W}_H$ and $\mathcal{W}_N$ are identical.

We summarize the discussion above in the following proposition.

Proposition 1. *Let H, K, N be subgroups of the triangle group.*

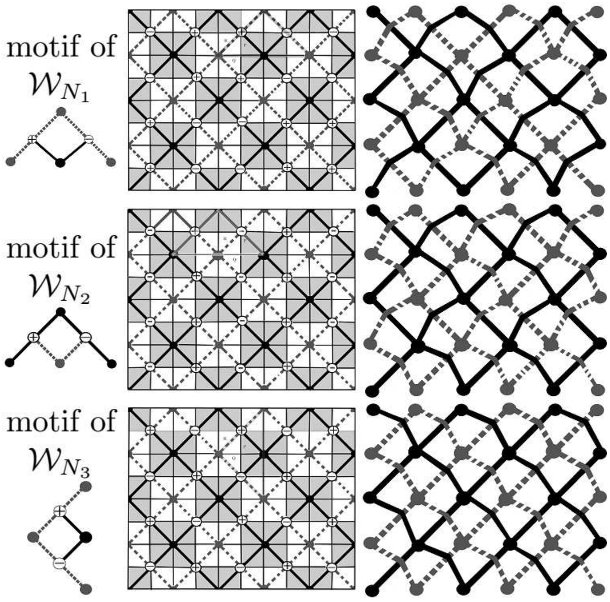
1. *If $\mathcal{W}$ can be obtained from the weaving $\mathcal{W}'$ by some isometry of the plane, then they are equivalent weavings.*

2. *Suppose K and N are conjugate subgroups and $\mathcal{W}_N$ is a weaving generated by N, then K generates a weaving $\mathcal{W}_K$ which is equivalent to $\mathcal{W}_N$.*

3. *Suppose H yields a weaving $\mathcal{W}_H$ and N is a subgroup of H of even index, then N yields a weaving $\mathcal{W}_N$ which is equivalent (identical) to $\mathcal{W}_H$.*

4.2. Example

The following are some of the weaving patterns generated from $H_2 = \langle R, Q, PRP \rangle$ and its subgroups.



The motif Δ_{mH_2} has only one IP, so we consider the index 2 subgroups of H_2, and subsequently their index 2 subgroups when *NWS* occur. We obtain the above non-equivalent weaving patterns from index 4 subgroups of H_2: $N_1 = \langle Q, R, PRPQRQPRP \rangle$, $N_2 = \langle Q, RPRP, RQRPRQRP \rangle$, and $N_3 = \langle R, QRQ, PRQRQP \rangle$. They are all index 2 subgroups of some index 2 subgroups of H_2.

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