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Dynamical Properties of k-Free Lattice Points

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We revisit the visible points of a lattice in Euclidean n-space together with their generalisations, the k-th-power-free points of a lattice, and study the corresponding dynamical system that arises via the closure of the lattice translation orbit. Our analysis extends previous results obtained by Sarnak and by Cellarosi and Sinai for the special case of square-free integers and sheds new light on previous joint work with Peter Pleasants.

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1. Introduction

In Ref. [1], the diffraction properties of the visible points of $\mathbb{Z}^2$ and the k-th-power-free numbers were studied. It was shown that these sets have positive, pure-point, translation-bounded *diffraction spectra* with countable, dense support. This is of interest because these sets fail to be Delone sets: they are uniformly discrete (subsets of lattices, in fact) but not relatively dense. The lack of relative denseness means that these sets have arbitrarily large 'holes'. In Ref. [2], it was shown that the above results remain true for the larger class of k-th-power-free (or k-free for short) points of arbitrary lattices in n-space. Furthermore, it was shown there that these sets have positive *patch counting entropy* but zero *measure-theoretical entropy* with respect to a measure that is defined in terms of the 'tied' frequencies of patches in space. Note that this is not the measure of maximum entropy; compare [3, 4].

Recent independent results by Sarnak [5] and by Cellarosi and Sinai [6] on the natural dynamical system associated with the square-free (resp. k-th-power-free) integers (in particular on the ergodicity of the above frequency measure and the dynamical spectrum, but also on the topological dynamics) go beyond what was covered in [2]. For further generalisations in one dimension, see [7, 8]. The aim of this short note is to generalise some of these results to the setting of k-free lattice points.

2. k-free points

The *k-free points* $V = V(\varLambda, k)$ of a lattice $\varLambda \subset \mathbb{R}^n$ are the points with the property that the greatest common divisor of their coordinates in any lattice basis is not divisible by any non-trivial k-th power of an integer. Without restriction, we shall assume that $\varLambda$ is unimodular, i.e. $|\det(\varLambda)| = 1$. One can see that V is *non-periodic*, i.e. V has no non-zero translational symmetries. As particular cases, we have the visible points (with respect to the origin 0) of $\varLambda$ (with $n \geq 2$ and $k = 1$) and the k-free integers (with $\varLambda = \mathbb{Z}$), both treated in [1] and [9]; see Fig. 1. We exclude the trivial case $n = k = 1$, where V consists of just the two points of $\varLambda$ closest to 0 on either side.

Let $v_n = \mathrm{vol}(B_1(0))$, so that $v_n R^n$ is the volume of the open ball $B_R(0)$ of radius R about 0 in $\mathbb{R}^n$. If $Y \subset \varLambda$, its 'tied' *density* $\mathrm{dens}(Y)$ is defined by

$$\mathrm{dens}(Y) := \lim_{R\to\infty} \frac{|Y \cap B_R(0)|}{v_n R^n},$$

when the limit exists. The following result is well known.

Theorem 1. [2], Cor. 1. *One has* $\mathrm{dens}(V) = 1/\zeta(nk)$, *where* ζ *denotes Riemann's* ζ*-function.* ☐

An application of the Chinese Remainder Theorem immediately gives the following result on the occurrence of 'holes' in V.

Proposition 1. [2], Prop. 1. *V is uniformly discrete, but has arbitrarily large holes. Moreover, for any $r > 0$, the set of centres of holes in V of inradius at least r contains a coset of $m^k\varLambda$ in $\varLambda$ for some $m \in \mathbb{N}$.* ☐

Given a radius $\rho > 0$ and a point $t \in \varLambda$, the *ρ-patch* of V at t is

$$(V - t) \cap B_\rho(0),$$

the translation to the origin of the part of V within a distance ρ of t. We denote by $\mathcal{A}(\rho)$ the (finite) set of

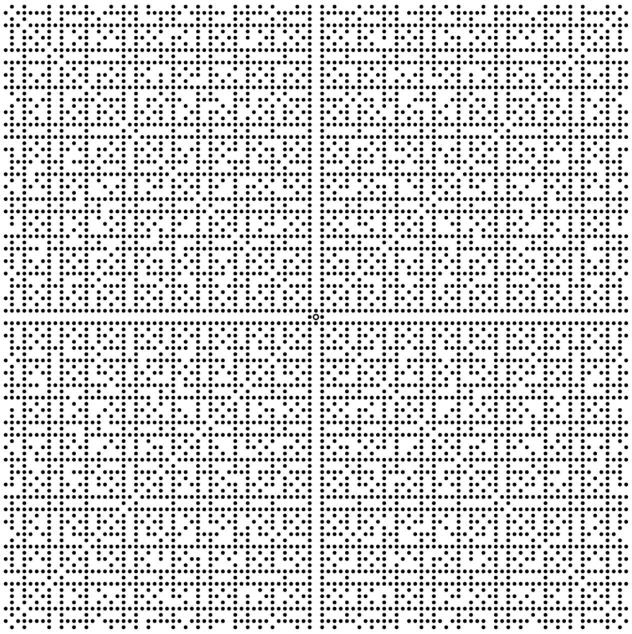

Fig. 1. A central patch of the visible points of the square lattice $\mathbb{Z}^2$. Note the invariance with respect to $\mathrm{GL}(2,\mathbb{Z})$.

all ρ-patches of V, and by $N(\rho) = |\mathcal{A}(\rho)|$ the number of distinct ρ-patches of V. In view of the binary configuration space interpretation, and following [2], the *patch counting entropy* of V is defined as

$$h_{\mathrm{pc}}(V) := \lim_{\rho\to\infty} \log_2 N(\rho)/(v_n \rho^n).$$

It can be shown by a classic subadditivity argument that this limit exists.

Following [1, 2], the 'tied' *frequency* $\nu(\mathcal{P})$ of a ρ-patch $\mathcal{P}$ of V is defined by

$$\nu(\mathcal{P}) := \mathrm{dens}\bigl(\{t \in \Lambda \mid (V-t) \cap B_\rho(0) = \mathcal{P}\}\bigr), \qquad (1)$$

which can indeed be seen to exist. Moreover, one has

Theorem 2. [2], Thms. 1 and 2. *Any ρ-patch $\mathcal{P}$ of V occurs with positive frequency, given by*

$$\nu(\mathcal{P}) = \sum_{\mathcal{F}\subset(B_\rho(0)\cap\Lambda)\setminus\mathcal{P}} (-1)^{|\mathcal{F}|} \prod_p \left(1 - \frac{|(\mathcal{P}\cup\mathcal{F})/p^k\Lambda|}{p^{nk}}\right),$$

where p runs through all primes. □

3. Diffraction

Recall that the *dual* or *reciprocal lattice* Λ^* of Λ is

$$\Lambda^* := \{y \in \mathbb{R}^n \mid y \cdot x \in \mathbb{Z} \text{ for all } x \in \Lambda\}.$$

Further, the *denominator* of a point y in the $\mathbb{Q}$-span $\mathbb{Q}\Lambda^*$ of Λ^* is defined as

$$\mathrm{den}(y) := \min\{m \in \mathbb{N} \mid my \in \Lambda^*\}.$$

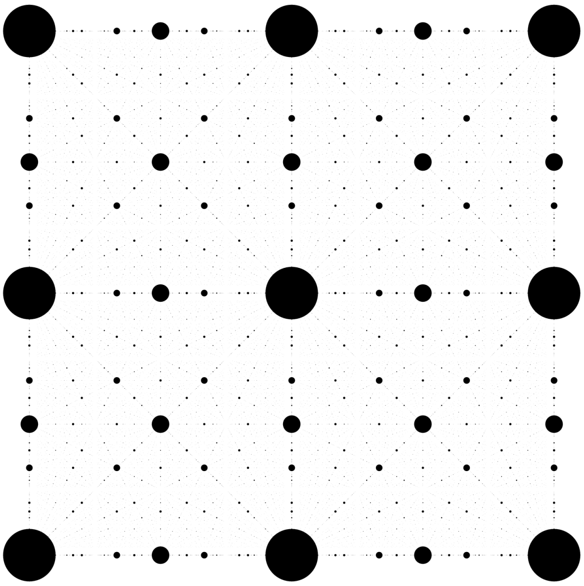

Fig. 2. Diffraction $\widehat{\gamma}$ of the visible points of $\mathbb{Z}^2$. Shown are the intensities with $I(y)/I(0) \geq 10^{-6}$ and $y \in [0,2]^2$. Its lattice of periods is $\mathbb{Z}^2$, and $\widehat{\gamma}$ turns out to be $\mathrm{GL}(2,\mathbb{Z})$-invariant; see [9] for details.

Theorem 3. [1], Thms. 3 and 5; [2], Thm. 8; [9]. *The natural diffraction measure $\widehat{\gamma}$ of the autocorrelation γ of V exists and is a positive, translation-bounded, pure-point measure which is concentrated on the set of points in $\mathbb{Q}\Lambda^*$ with $(k{+}1)$-free denominator, the Fourier–Bohr spectrum of γ, and whose intensity is*

$$\Bigl(\zeta(nk)^{-1} \prod_{p|q} (p^{nk}-1)^{-1}\Bigr)^2$$

at any point with such a denominator q (Fig. 2). □

4. The hull of V

Endowing the power set $\{0,1\}^\Lambda$ of the lattice Λ with the product topology of the discrete topology on $\{0,1\}$, it becomes a compact topological space (by Tychonov's theorem). This topology is in fact generated by the metric d defined by

$$d(X,Y) := \min\Bigl\{1, \inf_{\epsilon>0}\{X \cap B_{1/\epsilon}(0) = Y \cap B_{1/\epsilon}(0)\}\Bigr\}$$

for subsets X, Y of Λ; cf. [10]. Then, $(\{0,1\}^\Lambda, \Lambda)$ is a *topological dynamical system*, i.e. the natural translational action of the group Λ on $\{0,1\}^\Lambda$ is continuous.

Let X now be a subset of Λ. The closure

$$\mathbb{X}(X) := \overline{\{t + X \mid t \in \Lambda\}}$$

of the set of lattice translations $t + X$ of X in $\{0,1\}^\Lambda$ is the (*discrete*) *hull* of X and gives rise to the topological dynamical system $(\mathbb{X}(X), \Lambda)$, i.e. $\mathbb{X}(X)$ is a compact topological space on which the action of Λ is continuous.

By construction of the hull, Proposition 1 implies

Lemma 1. *For any $r > 0$ and any element $X \in \mathbb{X}(V)$, the set of centres of holes in X of inradius at least r contains a coset of $m^k\Lambda$ in Λ for some $m \in \mathbb{N}$.* □

For a ρ-patch $\mathcal{P}$ of V, denote by $C_{\mathcal{P}}$ the set of elements of $\mathbb{X}(V)$ whose ρ-patch at 0 is $\mathcal{P}$, the so-called *cylinder set* defined by the ρ-patch $\mathcal{P}$. Note that these cylinder sets form a basis of the topology of $\mathbb{X}(V)$.

It is clear from the existence of holes of unbounded inradius in V that $\mathbb{X}(V)$ contains the empty set (the configuration of 0 on every lattice point). Denote by $\mathbb{A}$ the set of *admissible* subsets A of Λ, i.e. subsets A of Λ having the property that, for every prime p, A does *not* contain a full set of representatives modulo $p^k\Lambda$. In other words, A is admissible if and only if $|A/p^k\Lambda| < p^{nk}$ for any prime p, where $A/p^k\Lambda$ denotes the set of cosets of $p^k\Lambda$ in Λ that are represented in A. Since $V \in \mathbb{A}$ (otherwise some point of V is in $p^k\Lambda$ for some prime p, a contradiction) and since $\mathbb{A}$ is a Λ-invariant and closed subset of $\{0,1\}^\Lambda$, it is clear that $\mathbb{X}(V)$ is a subset of $\mathbb{A}$. By [2], Thm. 2, the other inclusion is also true. One thus obtains the following characterisation of the hull of V, which was first shown by Sarnak [5] for the special case of the square-free integers.

Theorem 4. [2], Thm. 6. *One has $\mathbb{X}(V) = \mathbb{A}$.* □

In particular, $\mathbb{X}(V)$ contains *all* subsets of V (and their translates). In other words, V is an interpolating set for $\mathbb{X}(V)$ in the sense of [11], i.e.

$$\mathbb{X}(V)|_V := \{X \cap V \mid X \in \mathbb{X}(V)\} = \{0,1\}^V.$$

It follows that V has patch counting entropy at least $\mathrm{dens}(V) = 1/\zeta(nk)$. In fact, one has more.

Theorem 5. [2], Thm. 3; [12], Thm. 1. *One has* $h_{\mathrm{pc}}(V) = 1/\zeta(nk)$. *Moreover,* $h_{\mathrm{pc}}(V)$ *is the topological entropy of the dynamical system* $(\mathbb{X}(V), \varLambda)$. $\square$

5. Topological dynamics

By construction, $(\mathbb{X}(V), \varLambda)$ is topologically transitive [11, 13, 14], as it is the orbit closure of one of its elements (namely V). Equivalently, for any two non-empty open subsets U and W of $\mathbb{X}(V)$, there is an element $t \in \varLambda$ such that

$$U \cap (W + t) \neq \varnothing.$$

In accordance with Sarnak's findings [5] for square-free integers, one has the following results.

Theorem 6. *The topological dynamical system* $(\mathbb{X}(V), \varLambda)$ *has the following properties.*

- (a) $(\mathbb{X}(V), \varLambda)$ *is topologically ergodic with positive topological entropy equal to* $1/\zeta(nk)$.
- (b) $(\mathbb{X}(V), \varLambda)$ *is proximal (i.e. for any* $X, Y \in \mathbb{X}(V)$ *one has* $\inf_{t\in\varLambda} d(X+t, Y+t) = 0$*) and* $\{\varnothing\}$ *is the unique* $\varLambda$*-minimal subset of* $\mathbb{X}(V)$.
- (c) $(\mathbb{X}(V), \varLambda)$ *has no non-trivial topological Kronecker factor (i.e. minimal equicontinuous factor). In particular,* $(\mathbb{X}(V), \varLambda)$ *has trivial topological point spectrum.*
- (d) $(\mathbb{X}(V), \varLambda)$ *has a non-trivial joining with the Kronecker system* $K = (G, \varLambda)$*, where* G *is the compact Abelian group* $\prod_p (\varLambda/p^k\varLambda)$ *and* $\varLambda$ *acts on* G *via addition on the diagonal,* $g \mapsto g + (\bar{x}, \bar{x}, \dots)$*, with* $g \in G$ *and* $x \in \varLambda$*. In particular,* $(\mathbb{X}(V), \varLambda)$ *fails to be topologically weakly mixing.*

Proof. The positivity of the topological entropy follows from Theorem 5 since $1/\zeta(nk) > 0$. For the topological ergodicity [13], one has to show that, for any two non-empty open subsets U and W of $\mathbb{X}(V)$, one has

$$\limsup_{R\to\infty} \frac{\sum_{t\in\varLambda\cap B_R(0)} \theta\big(U \cap (W+t)\big)}{v_n R^n} > 0, \tag{2}$$

where $\theta(\varnothing) = 0$ and $\theta(A) = 1$ for non-empty subsets A of $\mathbb{X}(V)$. It certainly suffices to verify (2) for cylinder sets. To this end, let $\mathcal{P}$ and $\mathcal{Q}$ be patches of V. Then, a suitable translate $V + s$ is an element of $C_{\mathcal{P}}$. Since

$$\begin{aligned}
&\limsup_{R\to\infty} \frac{\sum_{t\in\varLambda\cap B_R(0)} \theta\big(C_{\mathcal{P}} \cap (C_{\mathcal{Q}}+t)\big)}{v_n R^n} \\
&\quad\geq \limsup_{R\to\infty} \frac{\sum_{t\in\varLambda\cap B_R(0)} \theta\big(\{V+s\} \cap (C_{\mathcal{Q}}+t)\big)}{v_n R^n} \\
&\quad= \limsup_{R\to\infty} \frac{\sum_{t\in\varLambda\cap B_R(0)} \theta\big(\{V\} \cap (C_{\mathcal{Q}}+t)\big)}{v_n R^n} = \nu(\mathcal{Q}),
\end{aligned}$$

the assertion follows from Theorem 2. This proves (a).

For part (b), one can easily derive from Lemma 1 that, for any $\rho > 0$ and any two elements $X, Y \in \mathbb{X}(V)$, there is a translation $t \in \varLambda$ such that

$$(X + t) \cap B_\rho(0) = (Y + t) \cap B_\rho(0) = \varnothing,$$

i.e. both X and Y have the empty ρ-patch at $-t$. It follows that $\mathrm{d}(X+t, Y+t) \leq 1/\rho$ and thus the proximality of the system follows. Similarly, the assertion on the unique $\varLambda$-minimal subset $\{\varnothing\}$ follows from the fact that any element of $\mathbb{X}(V)$ contains arbitrarily large 'holes' and thus any non-empty subsystem contains $\varnothing$.

Since the Kronecker systems are distal, the first assertion of part (c) is an immediate consequence of the proximality of $(\mathbb{X}(V), \varLambda)$. Although this immediately implies that $(\mathbb{X}(V), \varLambda)$ has trivial topological point spectrum, we add the following independent argument. Let $f\colon \mathbb{X}(V) \longrightarrow \mathbb{C}$ be a continuous eigenfunction, in particular $f \not\equiv 0$. Let $\lambda_t \in \mathbb{C}$ be the eigenvalue with respect to $t \in \varLambda$, i.e. $f(X - t) = \lambda_t f(X)$ for any $X \in \mathbb{X}(V)$, in particular

$$f(\varnothing) = \lambda_t f(\varnothing). \tag{3}$$

Since $\varLambda$ acts by homeomorphisms on the compact space $\mathbb{X}(V)$ and since $(\mathbb{X}(V), \varLambda)$ is topologically transitive, it is clear that $|\lambda_t| = 1$ and that $|f|$ is a non-zero constant. We shall now show that even $\lambda_t = 1$ for any t and that f itself is a non-zero constant. By Lemma 1, for any $X \in \mathbb{X}(V)$, one can choose a sequence $(t_n)_{n\in\mathbb{N}}$ in $\varLambda$ such that $\lim_{n\to\infty}(X - t_n) = \varnothing$. Since f is continuous, we have

$$f(\varnothing) = \lim_{n\to\infty} f(X - t_n) = \lim_{n\to\infty} \lambda_{t_n} f(X). \tag{4}$$

Assuming that $f(\varnothing) = 0$ thus implies $f \equiv 0$, a contradiction. Hence $f(\varnothing) \neq 0$ and $\lambda_t = 1$ for any $t \in \varLambda$ by (3). Further, by (4), one has $f(X) = f(\varnothing)$ for any $X \in \mathbb{X}(V)$.

For part (d), one can verify that a non-trivial joining [13] of $(\mathbb{X}(V), \varLambda)$ with the Kronecker system K is given by

$$W := \bigcup_{X\in\mathbb{X}(V)} \Big(\{X\} \times \prod_p (\varLambda \setminus X)/p^k\varLambda\Big).$$

Since the Kronecker system K is minimal and distal, a well known disjointness theorem by Furstenberg [15], Thm. II.3, implies that $(\mathbb{X}(V), \varLambda)$ fails to be topologically weakly mixing. $\square$

6. Measure-theoretic dynamics

The frequency function ν from (1), regarded as a function on the cylinder sets by setting $\nu(C_{\mathcal{P}}) := \nu(\mathcal{P})$, is finitely additive on the cylinder sets with

$$\nu(\mathbb{X}(V)) = \sum_{\mathcal{P}\in\mathcal{A}(\rho)} \nu(C_{\mathcal{P}}) = \frac{1}{|\det(\varLambda)|} = 1.$$

Since the family of cylinder sets is a (countable) semi-algebra that generates the Borel σ-algebra on $\mathbb{X}(V)$ (i.e. the smallest σ-algebra on $\mathbb{X}(V)$ which contains the open subsets of $\mathbb{X}(V)$), it extends uniquely to a probability measure on $\mathbb{X}(V)$; cf. [16], §0.2. Moreover, this probability measure is $\varLambda$-invariant by construction. For part (b) of the following claim, note that, in the case of

V, the Fourier–Bohr spectrum is itself a group and compare [17], Prop. 17. Turning to the measure-theoretic dynamical system $(\mathbb{X}(V), \varLambda, \nu)$, one has

Theorem 7. *$(\mathbb{X}(V), \varLambda, \nu)$ has the following properties.*

(a) *The $\varLambda$-orbit of V in $\mathbb{X}(V)$ is ν-equidistributed, i.e. for any function $f \in C(\mathbb{X}(V))$, one has*

$$\lim_{R\to\infty} \frac{1}{v_n R^n} \sum_{x\in \varLambda\cap B_R(0)} f(V+x) = \int_{\mathbb{X}(V)} f(X)\, \mathrm{d}\nu(X).$$

In other words, V is ν-generic.

(b) *$(\mathbb{X}(V), \varLambda, \nu)$ is ergodic, deterministic (i.e. it is of zero measure entropy) and has pure-point dynamical spectrum given by the Fourier–Bohr spectrum of the autocorrelation γ, as described in Theorem 3.*

(c) *The Kronecker system $K_\mu = (X_K, \varLambda, \mu)$, where X_K is the compact Abelian group $\prod_p (\varLambda/p^k \varLambda)$, $\varLambda$ acts on X_K via addition on the diagonal (cf. Theorem 6(d)) and μ is the normalised Haar measure on X_K, is metrically isomorphic to $(\mathbb{X}(V), \varLambda, \nu)$.*

Proof. For part (a), it suffices to show this for the characteristic functions of cylinder sets of finite patches, as their span is dense in $C(\mathbb{X}(V))$. But for such functions, the claim is clear as the left hand side is the patch frequency as used for the definition of the measure ν.

For the ergodicity of $(\mathbb{X}(V), \varLambda, \nu)$, one has to show that

$$\lim_{R\to\infty} \frac{1}{v_n R^n} \sum_{x\in \varLambda\cap B_R(0)} \nu\big((C_{\mathcal{P}}+x)\cap C_{\mathcal{Q}}\big) = \nu(C_{\mathcal{P}})\nu(C_{\mathcal{Q}})$$

for arbitrary cylinder sets $C_{\mathcal{P}}$ and $C_{\mathcal{Q}}$; compare [16], Thm. 1.17. The latter in turn follows from a straightforward calculation using Theorem 2 and the definition of the measure ν together with the Chinese Remainder Theorem. In fact, for technical reasons, it is better to work with a different semi-algebra that also generates the Borel σ-algebra on $\mathbb{X}(V)$ [18].

Vanishing measure-theoretical entropy (relative to ν) was shown in [2], Thm. 4, which is in line with the results of [12]. As a consequence of part (a), the individual diffraction measure of V according to Theorem 3 coincides with the diffraction measure of the system $(\mathbb{X}(V), \varLambda, \nu)$ in the sense of [19]. Then, pure point diffraction means pure point dynamical spectrum [19], Thm. 7, and the latter is the group generated by the Fourier–Bohr spectrum; compare [19], Thm. 8 and [17], Prop. 17. Since the intensity formula of Theorem 3 shows that there are no extinctions, the Fourier–Bohr spectrum here is itself a group, which completes part (b).

The Kronecker system can now be read off from the model set description, which provides the compact Abelian group. For the cases $k = 1$ and $d \geq 2$ as well as $k \geq 2$ and $d = 1$, the construction is given in [1]; see also [20], Ch. 5a for an alternative description. The general formalism is developed in [21], though the torus parametrisation does not immediately apply. Some extra work is required here to establish the precise properties of the homomorphism onto the compact Abelian group. □

Let us mention that our approach is complementary to that in [6, 7]. There, ergodicity and pure point spectrum are consequences of determining all eigenfunctions, then concluding via 1 being a simple eigenvalue and via the basis property of the eigenfunctions. Here, we establish ergodicity of the measure ν and afterwards use the equivalence theorem between pure point dynamical and diffraction spectrum [19], Thm. 7, hence employing the diffraction measure of V calculated in [1, 2].

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