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Effect of Magnetic Field on the Fluctuations of Charged Oscillators in Viscoelastic Fluids

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In the present work the generalized Langevin equation is solved for the motion of a charged Brownian oscillator in a magnetic field, when the thermal random force is exponentially correlated in the time. This model is consistent with the assumption that the medium has weakly viscoelastic properties. The velocity autocorrelation function, time-dependent diffusion coefficient and mean square displacement of the particle have been calculated. Our solutions generalize the previous results from the literature and are obtained in a way applicable to other problems of the Brownian motion with memory.

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1. Introduction

Stochastic motion of charged particles in magnetic fields was first studied half a century ago in connection with the diffusion of electrons and ions in plasma. In the classical works by Taylor and Kurşunoğlu the long-time limits of the mean square displacement (MSD) of the particles have been found [1, 2]. Later Furuse on the basis of the standard Langevin theory with the white-noise force driving the particles, generalized their results for arbitrary times [3]. The currently observed revival of these problems is mainly related to memory effects in the particle diffusion [4, 5]. Such effects appear when more realistic coloured random forces act on the particles from their surroundings. In the present work an exact analytical solution of the generalized Langevin equation (LE) has been found for the motion of the particle trapped in a harmonic potential well and exposed to a constant magnetic field in the case when the thermal force is exponentially correlated in the time. This model is consistent with the assumption that the solvent has weakly viscoelastic properties, which corresponds to the theory, originally proposed by Maxwell and later substantiated coming from first principles [5, 6]. The calculated time correlation functions describing the particle motion are more general than the previous results from the literature and are obtained in a way applicable to many other problems of the Brownian motion with memory.

2. Formulation of the problem

If the random force driving the Brownian particle (BP) is not the delta-correlated white noise but a coloured one, the friction during the particle motion cannot be arbitrary (in particular, it cannot be the Stokes one as in the traditional theories) but must obey the fluctuation-dissipation theorem. Then the equation of motion for the BP has a non-Markovian form of a generalized LE [5] that, for a particle of mass M in a harmonic potential with stiffness k, is

$$M\dot{\boldsymbol{v}}(t) + M\int_0^t \Gamma(t-t')\boldsymbol{v}(t')\,\mathrm{d}t' + M\omega^2\boldsymbol{r}(t) = Q\boldsymbol{v}\times\boldsymbol{B} + \boldsymbol{f}(t), \quad (1)$$

where Q is the charge of the particle of mass M, $\boldsymbol{B}$ is the constant induction of magnetic field along the axis z, and the force $\boldsymbol{f}$ has zero mean and its time correlation function at $t > 0$ is $\langle f_i(t)f_j(0)\rangle = k_BT\delta_{ij}\Gamma(t)$. The memory in the system is described by the kernel $\Gamma(t) = \omega_M\omega_m exp(-\omega_m t)$. Here, $\omega = (k/M)^{1/2}$ is the oscillator frequency and $\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t)$ is the velocity of the BP. Let the force $\boldsymbol{f}(t)$ arises from the standard LE $m\dot{\boldsymbol{u}}(t) + \gamma\boldsymbol{u}(t) = \boldsymbol{\eta}(t)$ with the white-noise force $\eta(t)$ and the Stokes friction force proportional to the velocity $\boldsymbol{u}(t)$ of the surrounding particles. The characteristic relaxation times of the particles of mass m and the BP of mass M, respectively, are $\tau_m = 1/\omega_m = m/\gamma$ and $\tau_M = 1/\omega_M = M/\gamma$.

3. Solving the generalized Langevin equation

The projection of Eq. 1 onto the axis z does not contain the magnetic force so that along the field we have just a Brownian oscillator in a Maxwell fluid. The full solution of this problem (including the case of a moving potential well) can be found in [7]. The motion across the field can be considered as follows. We multiply both the projections of Eq. 1 on the axes x and y by $\upsilon_x(0)$ and statistically average. This way we obtain the equations for the velocity correlation functions $\varphi_x(t) = \langle\upsilon_x(t)\upsilon_x(0)\rangle$ and $\upsilon_{xy}(t) = \langle\upsilon_y(t)\upsilon_x(0)\rangle$. The corresponding equations for the Laplace transform (LT) of these quantities, $\tilde{\phi}_x(s) = \Lambda\{\varphi_x(t)\}$ and $\tilde{\upsilon}_{xy}(s) = \Lambda\{\upsilon_{xy}(t)\}$, are

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$$s\tilde{\varphi}_x(s)+\tilde{\varphi}_x(s)\tilde{\Gamma}(s)+\omega^2 s^{-1}\tilde{\varphi}_x(s)-\omega_c\tilde{v}_{xy}(s) = k_BT/M, \quad (2)$$

$$s\tilde{v}_{xy}(s)+\tilde{v}_{xy}(s)\tilde{\Gamma}(s)+\omega^2 s^{-1}\tilde{v}_{xy}(s)+\omega_c\tilde{\varphi}_x(s) = 0, \quad (3)$$

where $\tilde{\Gamma}(s)=\omega_M\omega_m(\omega_m+s)^{-1}$ and $\omega_c=QB/M$ is the cyclotron frequency. We have used the equipartition theorem, $\varphi_i(0)=k_BT/M$, and the fact that at $t=0$ different projections of the velocity are uncorrelated. Analogous equations are obtained for $\tilde{\phi}_y(s)$ and $\tilde{v}_{yx}(s)$. These sets of equations have the following solutions for $i=x$, y:

$$\tilde{\phi}_i(s)=\frac{k_BT}{2M}\frac{1}{\psi(s)+i\omega_c}+c.c. \quad (4)$$

Here, "c.c." stands for "complex conjugate" and $\psi(s)=s+\tilde{\Gamma}(s)+\omega^2 s^{-1}$. Using the solutions of Eq. 4, the time-dependent diffusion coefficients $D_i(t)$ of the BP and its mean square displacements (MSD) $\xi_i(t)$ can be calculated according to the formulae [7] $D_i(t)=\int_0^t\phi_i(t')dt'$ and $\xi_i(t)=2\int_0^t D_i(t')dt'$. Expanding the denominator in Eq. 4 in simple fractions, we get

$$\tilde{D}_i(s)=\frac{k_BT}{2M}(s+\omega_m)\sum_{i=1}^{3}\frac{A_i}{s-s_i}+c.c., \quad (5)$$

with s_i being the roots of the cubic equation $s^3+s^2(i\omega_c+\omega_m)+s(i\omega_c\omega_m+\omega_M\omega_m+\omega^2)+\omega^2\omega_m=0$. For the coefficients A_i we have $1/A_1=(s_1-s_2)(s_1-s_3)$, and $A_{2,3}$ are obtained by the cyclic change of the indexes $1\to 2\to 3\to 1$. In Eq. 5, we have used the relation $\sum_{i=1}^3 A_i=0$. It also holds $\sum_{i=1}^3 A_i s_i=0$. One thus sees that the inverse LT of Eq. 5,

$$D_i(t)=\frac{k_BT}{2M}\sum_{i=1}^{3}A_i(\omega_m+s_i)\exp(s_i t)+c.c., \quad (6)$$

describes at $t\to 0$ the physically expected ballistic motion, $D_i(t)\approx k_BTt/M$. As $t\to\infty$, $D_i(t)$ at nonzero ω disappears.

4. Limiting cases

The analysis of the general result (6) for the diffusion function $D_{xy}(t)=D_x(t)+D_y(t)$ or the MSD across the field, $\xi_{xy}(t)=\xi_x(t)+\xi_y(t)$, can be done analytically or numerically. Here we show the limiting cases covered by the obtained solution. First, at the limit of zero correlation time of the random force ($\omega_m\to\infty$, which corresponds to white noise) the solution is described by only two roots $s_{1,2}$, expressed as $2s_{1,2}=-\omega_M-i\omega_c\pm[(\omega_M+i\omega_c)^2-4\omega^2]^{1/2}$. In the plane perpendicular to the field we then have

$$\xi_{xy}(t)=\frac{k_BT}{Ms_1s_2}\left(1+\frac{s_1\exp(s_2t)-s_2\exp(s_1t)}{s_2-s_1}\right)+c.c. \quad (7)$$

with the ballistic behaviour $\xi_{xy}(t)\approx 2k_BTt^2/M$ at $t\to 0$, independent on the external forces and the properties of the random force. Using $s_1s_2=\omega^2$, we obtain $\xi_{xy}(t)\approx 4k_BT/M\omega^2$ as the main approximation for long times. These results are the same as for the situation with zero magnetic field. The full solution in the absence of the field, but when the oscillator is driven by the correlated noise, has been found in [7]. For $\boldsymbol{B}\neq 0$, but neglecting the harmonic force, the MSD across the field (normalized to $(4k_BT/\gamma\omega_M)[1+(\omega_c/\omega_M)^2]^{-1}$), is at long times expressed in a dimensionless form [8]

$$\bar{\xi}_{xy}(t)\approx\omega_M t+\left(1-\frac{\omega_M}{\omega_m}\right)\frac{(\omega_c/\omega_M)^2-1}{(\omega_c/\omega_M)^2+1}+\ldots \quad (8)$$

Next to the 'Einstein' term proportional to t and the constant term there are exponentially decreasing contributions. At $t\to 0$ we again have $\xi_{xy}(t)\approx 2k_BTt^2/M$. These results correct the previous solution [4]. The limits $\omega\to 0$ and $\omega_m\to\infty$ fully correspond to the classical result for particles driven by the white noise in an external magnetic field [3, 8].

5. Conclusions

In a number of recent papers the classical works on the motion of particles under the influence of external forces in a fluctuating environment have been developed. The aim of the present paper was to consider the Brownian motion of a charged oscillator in a constant magnetic field. The inertial and memory effects on the particle motion across the field have been analysed within the generalized Langevin theory. The memory in the system corresponds to that of Maxwell's viscoelastic fluids. Exact solutions for the time correlations functions describing the oscillator fluctuations have been obtained. The general solution is new and its limit in the absence of the potential well corrects the previous attempts from the literature. Our findings could be tested in experiments on trapped particles, similar to those carried out in [6], where memory effects in the Brownian motion were directly probed.

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