New Exact Solutions for Boussinesq Type Equations by Using $(G'/G, 1/G)$ and $(1/G')$-Expansion Methods

S. Demiray, Ö. Ünsal and A. Bekir*

Eskişehir Osmangazi University, Art-Science Faculty, Department of Mathematics-Computer, Eskişehir, Turkey

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In this paper, the $(G'/G, 1/G)$ and $(1/G')$-expansion methods with the aid of Maple are used to obtain new exact traveling wave solutions of the Boussinesq equation and the system of variant Boussinesq equations. The travelling wave solutions are expressed by the hyperbolic functions, the trigonometric functions, and the rational functions. It is shown that the proposed method provides a powerful mathematical tool for solving nonlinear wave equations in mathematical physics and engineering.

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1. Introduction

Finding exact solutions of nonlinear evolution equations (NEEs) is a very important part of nonlinear physical phenomena. It is a fact that exact solutions provide much physical information and help one to understand the mechanism that governs some physical models, such as plasma physics, optical fibers, biology, solid state physics, chemical physics, and so on. In recent years, different methods for finding exact solutions of nonlinear evolution equations have been proposed, developed and extended. These are the Jacobi elliptic function method [1], the Hirota bilinear tranformation [2], the Weierstrass function method [3], the Darboux and Backlund transform [4], the Wronskian technique [5], homotopy perturbation method [6], the theta function method [7], symmetry method [8, 9], the homogeneous balance method [10, 11], sine/cosine method [12–14], F-expansion method [15], exp function method [16–19], the Painleve expansion method [20], the transformed rational function method [21], the inverse scattering method [22–24], the mapping method [25], tanh–coth method [26, 27], the collocation method [28], first integral method [29, 30], the rank analysis method [31], the parameter-expansion method [32], the auxiliary equation method [33], (G'/G)-expansion method [34–38]. The key idea of the original (G'/G)-expansion method is that the exact solutions of nonlinear partial differential equations (PDEs) can be expressed by a polynomial in one variable (G'/G) in which $G = G(\xi)$ satisfies the second ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ and μ are constants.

In the present paper, we will use the two-variable $(G'/G, 1/G)$-expansion method and $(1/G')$-expansion method. $(G'/G, 1/G)$-expansion method is considered as a generalization of the original (G'/G)-expansion method. The key idea of the two-variable $(G'/G, 1/G)$-expansion method is that the exact traveling wave solutions of nonlinear PDEs can be expressed by a polynomial in two variables (G'/G) and $(1/G)$, in which $G = G(\xi)$ satisfies a second-order linear ordinary differential equation (ODE) $G''(\xi) + \lambda G(\xi) = \mu$. Similarly, the main idea of $(1/G')$-expansion method is that our solutions can be expressed by a polynomial $(1/G')$ and $G = G(\xi)$ satisfies a second-order linear ODE $G''(\xi) + \lambda G'(\xi) + \mu = 0$ where λ and μ are constants. For both methods, the degree of the polynomials can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms in the given nonlinear PDEs. Besides, the coefficients of this polynomial can be obtained by solving a set of algebraic equations resulting from the process of using the method. As a pioneer work, Li et al. [39] have applied the two-variable $(G'/G, 1/G)$-expansion method and found the exact solutions of the Zakharov equations. Then Zayed and Abdelaziz [40, 41] determined exact solutions of some nonlinear evolution equations. $(1/G')$-expansion method has first been introduced by Yokuş [42].

The present paper investigates the applicability and effectiveness of the $(G'/G, 1/G)$-expansion method and $(1/G')$-expansion method on nonlinear evolution equations and systems of NEEs. In Sect. 2 and Sect. 3 we describe these methods for finding exact travelling wave solutions of nonlinear evolution equations. In Sect. 4, we illustrate these methods in detail with the Boussinesq equation and the system of variant Boussinesq equations. Finally, some conclusions are given.

2. The $(G'/G, 1/G)$ expansion method

In this section, we describe the main steps of the $(G'/G, 1/G)$-expansion method to find travelling wave solutions of nonlinear evolution equations. Li et al. [39] has summarized the $(G'/G, 1/G)$-expansion method as follows.

Consider the second order linear ODE (LODE)

$$G''(\xi) + \lambda G(\xi) = \mu \tag{2.1}$$

and we take

*corresponding author; e-mail: abekir@ogu.edu.tr

$$\phi = G'/G, \quad \psi = 1/G \tag{2.2}$$

for simplicity here and after. Using (2.1) and (2.2), we get

$$\phi' = -\phi^2 + \mu\psi - \lambda, \quad \psi' = -\phi\psi. \tag{2.3}$$

From the three cases of the general solutions of the LODE (2.1), we have:

Case 1. When $\lambda < 0$, the general solutions of the LODE (2.1) is given as

$$G(\xi) = A_1 \sinh(\sqrt{-\lambda}\xi) + A_2 \cosh(\sqrt{-\lambda}\xi) + \mu/\lambda$$

and we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{2.4}$$

where A_1 and A_2 are two arbitrary constants and $\sigma = A_1^2 - A_2^2$.

Case 2. When $\lambda > 0$, the general solutions of the LODE (2.1) is clearly

$$G(\xi) = A_1 \sin(\sqrt{\lambda}\xi) + A_2 \cos(\sqrt{\lambda}\xi) + \mu/\lambda$$

and we have

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma - \mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{2.5}$$

where A_1 and A_2 are two arbitrary constants and $\sigma = A_1^2 + A_2^2$.

Case 3. When $\lambda = 0$, the general solutions of the LODE (2.1) is

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2$$

and we have

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}(\phi^2 - 2\mu\psi), \tag{2.6}$$

where A_1 and A_2 are two arbitrary constants.

Now consider a nonlinear evolution equation with time and spatial variable one each,

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0. \tag{2.7}$$

In general, the left-hand side of Eq. (2.7) is a polynomial in u and its various partial derivatives. The main steps of the $(G'/G, 1/G)$ expansion method are:

Step 1. By coordinates transformation $\xi = x - ct$ and with $u(x,t) = u(\xi)$, Eq. (2.7) can be reduced to an ODE on $u(\xi)$ with

$$P(u, -cu', u', c^2u'', -cu', u'', \ldots) = 0. \tag{2.8}$$

Step 2. Suppose that the solution of ODE (2.8) can be expressed by a polynomial in ϕ and ψ as

$$u(\xi) = \sum_{i=0}^{N} a_i\phi^i + \sum_{i=1}^{N} b_i\phi^{i-1}\psi, \tag{2.9}$$

where $G = G(\xi)$ satisfies the second order LODE (2.1), a_i $(i = 0, \ldots, N)$, b_i $(i = 1, \ldots, N)$, c, λ and μ are constants to be determined later, and the positive integer N can be determined by using homogeneous balance between the highest order derivatives and the nonlinear terms appearing in ODE (2.8).

Step 3. Substituting (2.9) into Eq. (2.8), using (2.3) and (2.4) (here case 1 is taken as example) the left-hand side of (2.8) can be converted into a polynomial in terms of ϕ and ψ, in which the degree of ψ is not larger than 1. Equating each coefficient of the polynomial to zero yields a system of algebraic equations in a_i $(i = 0, \ldots, N)$, b_i $(i = 1, \ldots, N)$, c, λ $(\lambda < 0)$, μ, A_1, and A_2.

Step 4. Solve the algebraic solutions in the Step 3 with the aid of maple. Substituting the values of a_i $(i = 0, \ldots, N)$, b_i $(i = 1, \ldots, N)$, c, λ, μ, A_1, and A_2 obtained into (2.9), one can obtain the travelling wave solutions expressed by the hyperbolic functions of Eq. (2.8).

Step 5. Similar to Step 3 and Step 4, substituting (2.9) into Eq. (2.8), using (2.3) and (2.5) (or (2.3) and (2.6)), we obtain the travelling wave solutions of Eq. (2.8) expressed by trigonometric functions (or expressed by rational functions).

3. The $(1/G')$ expansion method

Step 1. This step is the same as the other method, that means, a PDE (2.7) can be converted to ODE (2.8).

Step 2. Suppose that the solution of ODE (2.8) can be expressed by a polynomial $(1/G')$ as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i \left(\frac{1}{G'}\right)^i, \tag{3.1}$$

where $G = G(\xi)$ satisfies the second order LODE

$$G''(\xi) + \lambda G'(\xi) + \mu = 0, \tag{3.2}$$

where a_i $(i = 0, \ldots, N)$, λ, μ are constants to be determined later and the positive integer N is homogeneous balance number.

Step 3. The solution of the differential Eq. (3.2) is

$$G(\xi) = -(\mu\xi)/\lambda + c_1 \mathrm{e}^{-\lambda\xi} + c_2. \tag{3.3}$$

Then

$$\frac{1}{G'(\xi)} = \frac{\lambda}{-\mu + \lambda c_1 \left[\cosh(\xi\lambda) - \sinh(\xi\lambda)\right]} \tag{3.4}$$

can be written.

Step 4. By substituting (3.1) into (2.8) and using second order LODE (3.2), the left-hand side of (2.8) can be converted into a polynomial in terms of $(1/G')$. Equating each coefficient of the polynomial to zero yields a system of algebraic equations and solving the algebraic equations by Maple we obtain a_i, c, λ, and μ constants.

4. Applications of the $(G'/G, 1/G)$ and $(1/G')$ expansion method

4.1. The Boussinesq equation

4.1.1. Using the $(G'/G, 1/G)$ expansion method

We consider the Boussinesq equation

$$u_{tt} - u_{xx} - \left(u^2\right)_{xx} + u_{xxxx} = 0, \tag{4.1.1}$$

which describe the surface water waves whose horizontal scale is much larger than the depth of the water. Previously, some authors found exact solutions of Eq. (4.1.1) [43–45]. Using the transformation $\xi = (x - ct)$, Eq. (4.1.1) is carried to an ODE

$$(c^2 - 1)u'' - (u^2)'' + u'''' = 0, \tag{4.1.2}$$

where the prime denotes the derivation with respect to ξ. Integrating Eq. (4.1.2) twice and setting the constants of integration equal to zero we obtain

$$(c^2-1)u - u^2 + u'' = 0. \tag{4.1.3}$$

By balancing u'' and u^2 we get

$$N+2 = 2N, \quad N = 2. \tag{4.1.4}$$

Consequently from (2.9) we get

$$u(\xi) = a_0 + a_1\phi + a_2\phi^2 + b_1\psi + b_2\phi\psi, \tag{4.1.5}$$

where a_0, a_1, a_2, b_1, and b_2 are constants to be determined later. As we mentioned above there are three cases to be disscused.

Case 1. When $\lambda < 0$ (hyperbolic function solutions)

Substituting (4.1.5) into Eq. (4.1.3), we use (2.3) and (2.4). The left-hand side of (4.1.3) becomes a polynomial in ϕ and ψ. Setting the each coefficient of equation to zero yields a system of algebraic equations in a_0, a_1, a_2, b_1, b_2, c, σ, μ, and λ:

$$\begin{aligned}
\phi^3 :\ & 2b_1\lambda b_2 - 2a_1a_2\lambda^2\sigma + 6b_2\lambda\mu \\
& - 2a_1a_2\mu^2 + 2a_1\lambda^2\sigma + 2a_1\mu^2, \\
\phi^2 :\ & b_1\lambda\mu - a_1^2\mu^2 - 2a_0a_2\mu^2 + c^2a_2\mu^2 - 2a_0a_2\lambda^2\sigma \\
& + b_1^2\lambda + 6a_2\mu^2\lambda + c^2a_2\lambda^2\sigma - a_2\mu^2 + 8a_2\lambda^3\sigma \\
& - a_2\lambda^2\sigma - a_1^2\lambda^2\sigma + b_2^2\lambda^2, \\
\psi\phi^2 :\ & - 2a_2b_1\mu^2 - 2a_1b_2\lambda^2\sigma + 2b_1\lambda^2\sigma - 2a_1b_2\mu^2 \\
& - 2a_2b_1\lambda^2\sigma + 2b_1\mu^2 - 10a_2\mu\lambda^2\sigma - 2b_2^2\lambda\mu - 10a_2\mu^3, \\
\phi :\ & - 2a_0a_1\mu^2 - 2a_0a_1\lambda^2\sigma + c^2a_1\mu^2 + 6b_2\lambda^2\mu - a_1\mu^2 \\
& + 2b_1b_2\lambda^2 + 2a_1\mu^2\lambda + c^2a_1\lambda^2\sigma + 2a_1\lambda^3\sigma - a_1\lambda^2\sigma, \\
\psi\phi :\ & - 4b_1\lambda b_2\mu - 2a_1b_1\mu^2 - 2a_0b_2\lambda^2\sigma - 2a_0b_2\mu^2 \\
& - 3a_1\mu\lambda^2\sigma + c^2b_2\mu^2 + 5b_2\lambda^3\sigma + c^2b_2\lambda^2\sigma - b_2\lambda^2\sigma \\
& - 7b_2\lambda\mu^2 - b_2\mu^2 - 2a_1b_1\lambda^2\sigma - 3a_1\mu^3, \\
\phi^0 :\ & b_1\lambda^2\mu + c^2a_0\lambda^2\sigma - a_0\lambda^2\sigma + 2a_2\lambda^4\sigma - a_0\mu^2 \\
& + c^2a_0\mu^2 - a_0^2\lambda^2\sigma + b_1^2\lambda^2 - a_0^2\mu^2, \\
\psi\phi^0 :\ & - 2a_0b_1\mu^2 - b_1\mu^2 - b_1\lambda\mu^2 + c^2b_1\lambda^2\sigma \\
& - 2a_0b_1\lambda^2\sigma - 4a_2\mu\lambda^3\sigma + c^2b_1\mu^2 + b_1\lambda^3\sigma \\
& - 2b_1^2\lambda\mu - b_1\lambda^2\sigma.
\end{aligned} \tag{4.1.6}$$

Solving the algebraic equations by Maple we get,

$$a_0 = 3\lambda, \quad a_1 = 0, \quad a_2 = 3, \quad b_1 = -3\mu,$$
$$b_2 = \sqrt{\frac{9\lambda^2\sigma + 9\mu^2}{-\lambda}}, \quad c = \sqrt{1+\lambda}. \tag{4.1.7}$$

Substituting these solutions into (4.1.5), using (2.2) and (2.4) we obtain travelling wave solution of (4.1.1) as follows:

$$\begin{aligned}
u(\xi) &= 3\lambda - \frac{3\mu}{A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \mu/\lambda} \\
&+ \frac{3[A_1\cosh(\xi\sqrt{-\lambda})\sqrt{-\lambda} + A_2\sinh(\xi\sqrt{-\lambda})\sqrt{-\lambda}]^2}{[A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \mu/\lambda]^2} \\
&+ \frac{[A_1\cosh(\xi\sqrt{-\lambda})\sqrt{-\lambda} + A_2\sinh(\xi\sqrt{-\lambda})\sqrt{-\lambda}]}{[A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \mu/\lambda]^2} \\
&\times \sqrt{\frac{9\lambda^2\sigma + 9\mu^2}{-\lambda}},
\end{aligned} \tag{4.1.8}$$

where

$$\xi = x - (\sqrt{1+\lambda})t \quad \text{and} \quad \sigma = A_1^2 - A_2^2. \tag{4.1.9}$$

In particular if we set $A_1 = 0$, $A_2 > 0$ and $\mu = 0$ in (4.1.8) then we have solitary solution

$$\begin{aligned}
u(\xi) &= 3\lambda\big[1 - \tanh^2(\xi\sqrt{-\lambda}) \\
&\quad + \mathrm{i}\tanh(\xi\sqrt{-\lambda})\mathrm{sech}(\xi\sqrt{-\lambda})\big],
\end{aligned} \tag{4.1.10}$$

but if we set $A_2 = 0$, $A_1 > 0$ and $\mu = 0$ then we have solitary solution

$$\begin{aligned}
u(\xi) &= 3\lambda\big[1 - \coth^2(\xi\sqrt{-\lambda}) \\
&\quad + \coth(\xi\sqrt{-\lambda})\mathrm{csch}(\xi\sqrt{-\lambda})\big].
\end{aligned} \tag{4.1.11}$$

Case 2. When $\lambda > 0$ (trigonometric function solutions)

Substituting (4.1.5) into Eq. (4.1.3), we use (2.3) and (2.5). The left-hand side of (4.1.3) becomes a polynomial in ϕ and ψ. Vanishing each coefficient of this polynomial, we get the system of algebraic equations which can be solved by Maple to find the following results:

$$a_0 = 3\lambda, \quad a_1 = 0, \quad a_2 = 3, \quad b_1 = -3\mu,$$
$$b_2 = \sqrt{\frac{-9\lambda^2\sigma + 9\mu^2}{-\lambda}}, \quad c = \sqrt{1+\lambda}. \tag{4.1.12}$$

Substituting these solutions into (4.1.5), using (2.2) and (2.5) we obtain travelling wave solution of (4.1.1) as follows:

$$\begin{aligned}
u(\xi) &= 3\lambda + \frac{3[A_1\cos(\xi\sqrt{\lambda})\sqrt{\lambda} - A_2\sin(\xi\sqrt{\lambda})\sqrt{\lambda}]^2}{[A_1\sin(\xi\sqrt{\lambda}) + A_2\cos(\xi\sqrt{\lambda}) + \mu/\lambda]^2} \\
&- \frac{3\mu}{A_1\sin(\xi\sqrt{\lambda}) + A_2\cos(\xi\sqrt{\lambda}) + \mu/\lambda} \\
&+ \sqrt{\frac{9\mu^2 - 9\lambda^2\sigma}{-\lambda}} \\
&\times \frac{[A_1\cos(\xi\sqrt{\lambda})\sqrt{\lambda} - A_2\sin(\xi\sqrt{\lambda})\sqrt{\lambda}]}{[A_1\sin(\xi\sqrt{\lambda}) + A_2\cos(\xi\sqrt{\lambda}) + \mu/\lambda]^2},
\end{aligned} \tag{4.1.13}$$

where

$$\xi = x - (\sqrt{1+\lambda})t \quad \text{and} \quad \sigma = A_1^2 + A_2^2. \tag{4.1.14}$$

In particular, if we set $A_1 = 0$, $A_2 > 0$ and $\mu = 0$ in (4.1.13) then we have periodic solution

$$u(\xi) = 3\lambda[1 + \tan^2(\xi\sqrt{\lambda}) - \tan(\xi\sqrt{\lambda})\sec(\xi\sqrt{\lambda})], \tag{4.1.15}$$

while we set $A_2 = 0$, $A_1 > 0$ and $\mu = 0$ then we have

$$u(\xi) = 3\lambda[1 + \cot^2(\xi\sqrt{\lambda}) + \cot(\xi\sqrt{\lambda})\csc(\xi\sqrt{\lambda})]. \tag{4.1.16}$$

Case 3. $\lambda = 0$ (rational function solutions)

Substituting (4.1.5) into Eq. (4.1.3), using (2.3) and (2.6) the left-hand side of (4.1.3) becomes a polynomial in ϕ and ψ. Vanishing each coefficient of this polynomial, we get the system of algebraic equations which can be solved by Maple to find the following results:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 3, \quad b_1 = -3\mu,$$

$$b_2 = \sqrt{-18\mu A_2 + 9A_1^2}, \quad c = -1. \tag{4.1.17}$$

Substituting these solutions into (4.1.5), using (2.2) and (2.6) we obtain travelling wave solution of (4.1.1) as follows:

$$u(\xi) = \frac{3(\mu\xi + A_1)^2}{\left(\frac{\mu\xi^2}{2} + A_1\xi + A_2\right)^2} - \frac{3\mu}{\frac{\mu\xi^2}{2} + A_1\xi + A_2} + \frac{\sqrt{-18\mu A_2 + 9A_1^2}(\mu\xi + A_1)}{\left(\frac{\mu\xi^2}{2} + A_1\xi + A_2\right)^2}, \tag{4.1.18}$$

where $\xi = x + t$.

Let us note that our solutions are different from the given ones in [43–45].

4.1.2. Using the $(1/G')$ expansion method

From (4.1.4) we know the balance number is $N = 2$, so if we substitute this number into (3.1), our solution can be expressed as follows:

$$u(\xi) = a_0 + a_1\left(\frac{1}{G'}\right) + a_2\left(\frac{1}{G'}\right)^2. \tag{4.1.19}$$

If we substitute (4.1.19) into ODE (4.1.3) using (3.2) the left hand side of ODE becomes a polynomial in $(1/G')$. Setting the each coefficient of equation to zero yields a system of algebraic equations which can be solved by Maple to find following results:

$$a_0 = 0, \; a_1 = 6\mu\lambda, \; a_2 = 6\mu^2, \; c = \sqrt{1-\lambda^2}. \tag{4.1.20}$$

If we solve our second order LODE (3.2):

$$G''(\xi) + \lambda G'(\xi) + \mu = 0,$$

we find the solution

$$G(\xi) = c_1\frac{-\mathrm{e}^{-\lambda\xi}}{\lambda} - \mu/\lambda\xi + c_2. \tag{4.1.21}$$

By substituting (4.1.20) into (4.1.19) using (4.1.21) we obtain

$$u(\xi) = \frac{6\mu\lambda^2}{-\mu + c_1\lambda[\cosh(\xi\lambda) - \sinh(\xi\lambda)]} + \frac{6\mu^2\lambda^2}{-\mu + c_1\lambda[\cosh(\xi\lambda) - \sinh(\xi\lambda)]^2}, \tag{4.1.22}$$

where $\xi = x - \sqrt{1-\lambda^2}t$.

4.2. The system of variant Boussinesq equations

4.2.1. Using the $(G'/G, 1/G)$ expansion method

The system of variant Boussinesq equations is given as

$$u_t + v_x + u_x u = 0, \tag{4.2.1}$$

$$v_t + (uv)_x + u_{xxx} = 0, \tag{4.2.2}$$

where $u = u(x,t)$ represents the velocity and $v = v(x,t)$ total depth. Some exact solutions of Eq. (4.2.1) have been given by several authors [46, 34]. Using the wave variable

$$u(x,t) = u(\xi), \quad v(x,t) = v(\xi), \quad \xi = x - ct, \tag{4.2.3}$$

the system is carried to a system of ODEs

$$-cu' + v' + u'u = 0, \tag{4.2.4}$$

$$-cv' + (uv)' + u''' = 0. \tag{4.2.5}$$

Integrating Eqs. (4.2.4) and (4.2.5) once we find

$$v = cu - \frac{u^2}{2} + \alpha, \tag{4.2.6}$$

$$-cv + uv + u'' = \beta, \tag{4.2.7}$$

where α, β are the constants of integral. Substituting (4.2.6) into (4.2.7) we obtain

$$u'' = (\beta + \alpha c) + (c^2 - \alpha)u - \frac{3c}{2}u^2 + \frac{1}{2}u^3 = 0. \tag{4.2.8}$$

Balancing u'' with u^3 gives

$$N = 1. \tag{4.2.9}$$

From (2.9) we get

$$u(\xi) = a_0 + a_1\phi + b_1\psi, \tag{4.2.10}$$

where a_0, a_1 and b_1 are constants determined later.

Case 1. When $\lambda < 0$

Substituting (4.2.10) into (4.2.8) using (2.3) and (2.4). The left hand side of equation becomes a polynomial in ϕ and ψ, setting each coefficient of equation to zero to get the system of equations which can be solved by Maple to find the following results:

$$a_0 = c, \quad a_1 = -1, \quad b_1 = \sqrt{-(\lambda^2\sigma + \mu^2)/\lambda},$$
$$c = c, \quad \alpha = -\lambda/2 - c^2/2, \quad \beta = 0. \tag{4.2.11}$$

Substituting these results into (4.2.10) using (2.2) and (2.4) we obtain

$$u(\xi) = c - \frac{A_1\cosh(\xi\sqrt{-\lambda})\sqrt{-\lambda} + A_2\sinh(\xi\sqrt{-\lambda})\sqrt{-\lambda}}{A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \mu/\lambda} + \frac{\sqrt{-(\lambda^2\sigma - \mu^2/\lambda)}}{A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \mu/\lambda}, \tag{4.2.12}$$

where $\xi = x - ct$ and $\sigma = A_1^2 - A_2^2$.

In particular if we set $A_1 = 0$, $A_2 > 0$ and $\mu = 0$ then we have solitary solution

$$u(\xi) = c - \sqrt{\lambda}[\mathrm{i}\tanh(\xi\sqrt{-\lambda}) - \mathrm{sech}\,(\xi\sqrt{-\lambda})], \tag{4.2.13}$$

while we set $A_2 = 0$, $A_1 > 0$ and $\mu = 0$; then we have solitary solution

$$u(\xi) = c - \sqrt{-\lambda}[\coth(\xi\sqrt{-\lambda}) - \mathrm{csch}\,(\xi\sqrt{-\lambda})]. \tag{4.2.14}$$

Case 2. When $\lambda > 0$

Substituting (4.2.10) into (4.2.8) using (2.3) and (2.5). The left hand side of equation becomes a polynomial in ϕ and ψ, vanishing each coefficient of equation to get the system of equations which can be solved by Maple to find following results:

$$a_0 = c, \quad a_1 = -1, \quad b_1 = \sqrt{(\lambda^2\sigma - \mu^2)/\lambda},$$
$$c = c, \quad \alpha = -\lambda/2 - c^2/2, \quad \beta = 0. \tag{4.2.15}$$

Substituting these results into (4.2.10) using (2.2) and (2.4) we obtain travelling wave solution

$$u(\xi) = c - \frac{A_1\cos(\xi\sqrt{\lambda})\sqrt{\lambda} - A_2\sin(\xi\sqrt{\lambda})\sqrt{\lambda}}{A_1\sin(\xi\sqrt{\lambda}) + A_2\cos(\xi\sqrt{\lambda}) + \mu/\lambda} + \frac{\sqrt{(\lambda^2\sigma - \mu^2)/\lambda}}{A_1\sin(\xi\sqrt{\lambda}) + A_2\cos(\xi\sqrt{\lambda}) + \mu/\lambda}, \tag{4.2.16}$$

where $\xi = x - ct$ and $\sigma = A_1^2 + A_2^2$.

In particular if we set $A_1 = 0$, $A_2 > 0$ and $\mu = 0$ then we have periodic solution

$$u(\xi) = c + \sqrt{\lambda}[\tan(\xi\sqrt{\lambda}) + \sec(\xi\sqrt{\lambda})], \tag{4.2.17}$$

while we set $A_2 = 0$, $A_1 > 0$ and $\mu = 0$ then we have periodic solution

$$u(\xi) = c - \sqrt{\lambda}[\cot(\xi\sqrt{\lambda}) - \csc(\xi\sqrt{\lambda})]. \tag{4.2.18}$$

Case 3. When $\lambda = 0$

Substituting (4.2.10) into (4.2.8) using (2.3) and (2.6). The left hand side of equation becomes a polynomial in ϕ and ψ, vanishing each coefficient of equation to get the system of equations which can be solved by Maple to find the following results:

$$a_0 = c, \quad a_1 = -1, \quad b_1 = \sqrt{A_1^2 - 2\mu A_2},$$
$$c = c, \quad \alpha = -\frac{c^2}{2}, \quad \beta = 0. \tag{4.2.19}$$

Substituting these results into (4.2.10) using (2.2) and (2.6) we obtain travelling wave solution

$$u(\xi) = c - \frac{\mu\xi + A_1}{\frac{\mu\xi^2}{2} + A_1\xi + A_2} + \frac{\sqrt{A_1^2 - 2\mu A_2}}{\frac{\mu\xi^2}{2} + A_1\xi + A_2}, \tag{4.2.20}$$

where $\xi = x - ct$.

These are new exact solutions and different from the solutions found in [46, 34].

4.2.2. Using the $(1/G')$ expansion method

From (4.2.9) we know the balance number is $N = 1$ so if we substitute this number into (3.1), our solution can be expressed as follows:

$$u(\xi) = a_0 + a_1\left(\frac{1}{G'}\right). \tag{4.2.21}$$

If we substitute (4.1.19) into ODE (4.2.8) using (3.2) the left hand side of ODE becomes a polynomial in $(1/G')$. Setting the each coefficient of equation to zero yields a system of algebraic equations which can be solved by Maple to find the following results:

$$a_0 = \lambda + \sqrt{-2\alpha + \lambda^2}, \quad a_1 = 6\mu,$$
$$c = \sqrt{-2\alpha + \lambda^2}, \quad \beta = 0. \tag{4.2.22}$$

By substituting (4.2.22) into (4.2.21) using (4.1.21) we obtain

$$u(\xi) = \lambda + \sqrt{-2\alpha + \lambda^2} + \frac{2\mu\lambda}{-\mu + c_1\lambda[\cosh(\xi\lambda) - \sinh(\xi\lambda)]}, \tag{4.2.23}$$

where $\xi = x - \sqrt{-2\alpha + \lambda^2}t$.

5. Conclusion

In this work, we have used the two-variable $(G'/G, 1/G)$-expansion method and $(1/G')$-expansion method to derive new exact solutions of the Boussinesq equation and the system of variant Boussinesq equations. We show that the solutions we found in this article are different from the solutions presented by other authors in recent papers. We foresee that our results can be found potentially useful for applications in mathematical physics and engineering. All solutions in this paper have been found by aid of Maple packet program. Thus, we conclude that the proposed method can be extended to solve the nonlinear problems which arise in the theory of solitons and other areas.

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