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New Concepts of the Study of an Infinite Beam on Elastic Foundation Based on the Distribution Theory

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The new approach to investigation the bending of the infinite length beam on the elastic foundation, applying the theory of distributions, is presented.

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1. Introduction

The purpose of this paper is to present a new approach to the study of the bending of a beam with the infinite length on the elastic foundation, based on an application of the distribution theory.

To justify our treatment we present briefly the traditional method [1, 2] of determining the maximal bending torque.

The starting point of the traditional approach is the differential equation

$$y''''(x) + 4\beta^4 y(x) = 0, \tag{1.1}$$

where $4\beta^4 = k/(EJ)$, EJ is the bending stiffness, the coefficient k characterizes the value of the unit force of the elastic supporting medium, i.e. $ky(x)$ describes the y coordinate of the elastic force acting on the beam at a point x, see Fig. 1. Equation (1.1) does not take into consideration the load acting on the beam.

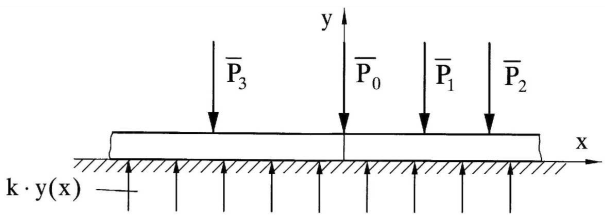

Fig. 1. The beam on the elastic supporting medium.

In the case of a single vertical concentrated force $\bar{P}$ with a magnitude P acting on a beam at the point $x = 0$ the general solution of (1.1) has the form

$$y(x) = \mathrm{e}^{\beta x}(A\sin(\beta x) + B\cos(\beta x))$$
$$+\mathrm{e}^{-\beta x}(C\sin(\beta x) + D\cos(\beta x)). \tag{1.2}$$

It is assumed that $A = B = 0$. Then to include the effect of an external force, the following conditions, permitting to determine constants C and D, are introduced:

$$y'(0) = 0, \quad EJy'''(0) = -P/2. \tag{1.3}$$

Conditions (1.3) together with conditions $A = B = 0$ imply that

$$C = D = \frac{P}{8EJ\beta^3}. \tag{1.4}$$

Hence the equation of the bending line of a beam under the action of a single external force assumes the form

$$y(x) = \frac{P}{8EJ\beta^3}\mathrm{e}^{-\beta x}(\sin(\beta x) + \cos(\beta x)), \tag{1.5}$$

while the bending torque and the transversal force are given respectively by the formulae

$$M(x) = EJy''(x) = \frac{P}{4\beta}\mathrm{e}^{-\beta x}(\cos(\beta x) - \sin(\beta x)), \tag{1.6}$$

$$Q(x) = EJy'''(x) = -\frac{P}{2}\mathrm{e}^{-\beta x}\cos(\beta x). \tag{1.7}$$

The imaginary bending line of a beam under the action of a single force is depicted in Fig. 2, while the graph of $y(x)$ is presented in Fig. 3.

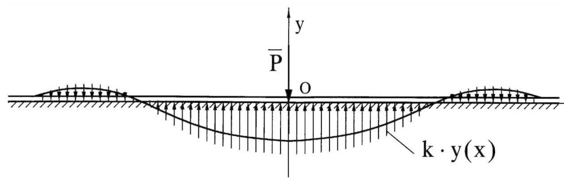

Fig. 2. The imaginary bending line of a beam under the influence of a single force.

The wavy line on Fig. 3 represents decreasing amplitudes. The length of the half-wave is $l_0 = \pi/\beta$ and it is seen from the Table 24 published in [1] that at the distance $2l_0$ from the origin, the maximal amplitude is approximately 0.2% of the magnitude of the displacement under the acting force, i.e. it practically equals zero.

In the case of a system of concentrated forces no formula of the bending line is available. The formulae (1.6) and (1.7) are successively applied to each force and the maximal torque is calculated. This is done by moving the origin to the point of attachment of the consecutive force and the maximal torque is searched among solutions obtained this way.

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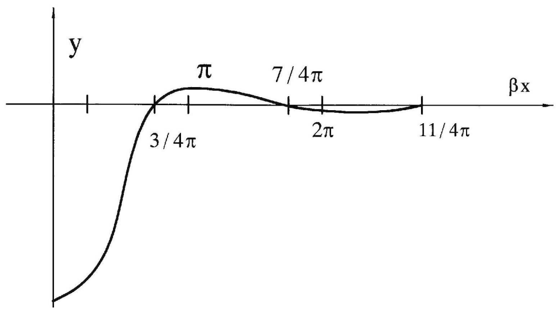

Fig. 3. The graph of $y(x)$, see Eq. (1.5).

In the following we will consider the beam of an infinite length on the elastic supporting medium under the action of n concentrated forces $\bar{P}_i$ with values $P_i < 0$ acting on the beam at points x_i. The bending of the beam is described by function $y(x)$, $x \in \mathbb{R}$ which satisfies the differential equation with the distributional right hand side

$$y''''(x) + 4\beta^4 y(x) = \sum_{i=1}^{n} F_i \delta(x - x_i), \tag{1.8}$$

where $F_i = P_i/(EJ)$ and $\delta(x - x_i)$ is a Dirac distribution concentrated at x_i, $(0 \le x_1 < x_2 < \cdots < x_n)$.

By the solution of Eq. (1.8) in the distributional sense we mean the function $y(x)$ of class C^4 in $\mathbb{R} \setminus \{x_1, \ldots, x_n\}$, such that $\int_{\mathbb{R}}(y''''(x) + 4\beta^4 y(x) - \sum_{i=1}^{n} F_i\delta(x - x_i))\varphi(x)dx = 0$ for any "test function" φ i.e. for any C^∞ function with a compact support.

It is known that for an arbitrary x_0 the distributional solution $y(x)$ is uniquely determined by initial values $y^{(j)}(x_0)$, $j = 0, 1, 2$, $(y^{(3}(x_0-) + y^{(3)}(x_0+)))/2$ [3].

In practical problems to make it possible to get the optimal choice of the solution method, three conceptions are proposed. In each of the first two conceptions, two different forms of particular integrals of (1.8) are considered. In the last one the elastic continuous supporting medium is replaced by the discrete one.

2. The first conception

First variant. In the first variant we consider the solution of (1.8) as a generalized function given by the formula [4, 5]

$$\begin{aligned} y(x) &= \mathrm{e}^{\beta x}(A\sin(\beta x) + B\cos(\beta x)) \\ &+ \mathrm{e}^{-\beta x}(C\sin(\beta x) + D\cos(\beta x)) \\ &- \frac{1}{4\beta^3}\sum_{i=1}^{n} F_i[\sinh\beta(x - x_i)\cos\beta(x - x_i) \\ &- \cosh\beta(x - x_i)\sin(\beta(x - x_i)]H(x - x_i), \end{aligned} \tag{2.1}$$

where

$$H(x - a) = \begin{cases} 1, & \text{for } x > a, \\ 0 & \text{for } x \le a. \end{cases}$$

Let $A = B = 0$ and assume that

$$y'(0) = 0, \quad y(\tilde{x}) = 0, \quad \tilde{x} > x_n, \tag{2.2}$$

where the choice of $\tilde{x}$ is given below.

The distributional derivatives [6] of (2.1) are given by

$$\begin{aligned} y'(x) &= \beta\,\mathrm{e}^{-\beta x}[-(C + D)\sin(\beta x) + (C - D)\cos(\beta x)] \\ &+ \frac{1}{2\beta^2}\sum_{i=1}^{n} F_i \sinh\beta(x - x_i)\sin\beta(x - x_i) \\ &\times H(x - x_i), \end{aligned} \tag{2.3}$$

$$\begin{aligned} y''(x) &= 2\beta^2\,\mathrm{e}^{-\beta x}[D\sin(\beta x) - C\cos(\beta x)] \\ &+ \frac{1}{2\beta}\sum_{i=1}^{n}[F_i \sinh\beta(x - x_i)\cos\beta(x - x_i) \\ &+ \cosh\beta(x - x_i)\sin\beta(x - x_i)]H(x - x_i), \end{aligned} \tag{2.4}$$

$$\begin{aligned} y'''(x) &= 2\beta^3\,\mathrm{e}^{-\beta x}[(C - D)\sin(\beta x) \\ &+ (C + D)\cos(\beta x)] + \sum_{i=1}^{n} F_i \cosh\beta(x - x_i) \\ &\times \cos(\beta(x - x_i)H(x - x_i). \end{aligned} \tag{2.5}$$

From Eq. (2.3) it follows that if $C = D$ then $y'(0) = 0$. By Eq. (2.1) and the second condition of Eq. (2.2),

$$\begin{aligned} C(\tilde{x}) &= \frac{\mathrm{e}^{\beta\tilde{x}}}{4\beta^3(\sin(\beta\tilde{x}) + \cos(\beta\tilde{x}))} \\ &\times \sum_{i=1}^{n} F_i[\sinh\beta(\tilde{x} - x_i)\cos\beta(\tilde{x} - x_i) \\ &- \cosh\beta(\tilde{x} - x_i)\sin\beta(\tilde{x} - x_i)], \end{aligned} \tag{2.6}$$

provided $A = B = 0$ and $C = D$.

$\tilde{x}$ is chosen to satisfy $\sin(\beta\tilde{x}) + \cos(\beta\tilde{x}) \neq 0$, which implies the inequality

$$\tilde{x} \neq \frac{\pi(k - 1/4)}{\beta}, \quad k = 1, 2, \ldots. \tag{2.7}$$

Taking into consideration the graph of Eq. (1.5) (see Fig. 3) and condition $\tilde{x} > x_n$ we assume that

$$\beta x_n + \pi/4 \le \beta\tilde{x} < \pi(k - 1/4), \quad k = 1, 2, \ldots. \tag{2.8}$$

Number $k = k_0$ should be chosen as small as possible to satisfy: $x_n + \pi/(4\beta) \le \tilde{x} < \pi(k_0 - 1/4)/\beta$.

From conditions $A = B = 0$, $C = D$ it follows that the equation of line of bending of the beam has the form

$$\begin{aligned} y(x) &= C(\tilde{x})\,\mathrm{e}^{-\beta x}(\sin(\beta x) + \cos(\beta x)) \\ &- \frac{1}{4\beta^3}\sum_{i=2}^{n} F_i[\sinh\beta(x - x_i)\cos\beta(x - x_i) \\ &- \cosh\beta(x - x_i)\sin(\beta(x - x_i)]H(x - x_i), \end{aligned} \tag{2.9}$$

where $C(\tilde{x})$ is defined by Eq. (2.6). The summation in Eq. (2.9) starts from 2, since $\bar{P}_1$ is included in the condition $y'(0) = 0$.

Remark 1. *For any $\tilde{x}$ satisfying $\tilde{x} = \pi(2s + 1/4)/\beta$ $(s = 0, 1, \cdots)$ from Eq. (2.6) it follows that*

$$C(\tilde{x}) = \frac{P_1}{8\beta^3 EJ} \tag{2.10}$$

For $n = 1$, $x_1 = 0$ from Eq. (2.6) we get the known formula (see Eq. (1.5))

$$y(x) = \frac{P_1}{8EJ\beta^3}\,\mathrm{e}^{-\beta x}(\sin(\beta x) + \cos(\beta x)). \tag{2.11}$$

Usually the expressions for the bending torque $M(x) = EJy''(x)$ and the transversal force $Q(x)$ are also required.

Recalling that $\sin\beta z - \cos\beta z = \sqrt{2}\sin(\beta z - \pi/4)$, $\sin\beta z + \cos\beta z = \sqrt{2}\cos(\beta z - \pi/4)$, they are given by the formulae

$$M(x) = \frac{1}{2\beta}\Big(\frac{e^{\beta(\tilde{x}-x)}\sin(\beta x - \pi/4)}{\cos(\beta\tilde{x} - \pi/4)} \times \sum_{i=1}^{n} P_i[\sinh\beta(\tilde{x}-x_i)\cos\beta(\tilde{x}-x_i) - \cosh\beta(\tilde{x}-x_i)\sin\beta(\tilde{x}-x_i)] + \sum_{i=2}^{n} P_i[\sinh\beta(x-x_i)\cos\beta(x-x_i) + \cosh\beta(x-x_i)\sin\beta(x-x_i)]H(x-x_i)\Big), \tag{2.12}$$

and

$$Q(x_i) = T(x_i^+) - T(x_i^-), \tag{2.13}$$

where $T(x_i^{\pm})$ denotes the right(left)-hand limit of T at x_i, $Q(x) = T(x)$ for $x \neq x_i$, and

$$T(x) = \frac{e^{\beta(\tilde{x}-x)}\cos\beta x}{\cos(\beta\tilde{x} - \pi/4)}\sum_{i=1}^{n} P_i[\sinh\beta(\tilde{x}-x_i) \times \cos\beta(\tilde{x}-x_i) - \cosh\beta(\tilde{x}-x_i)\sin\beta(\tilde{x}-x_i)] + \sum_{i=2}^{n} P_i[\cosh\beta(x-x_i) \times \cos\beta(x-x_i)H(x-x_i). \tag{2.14}$$

Second variant. In this variant, assuming $A = B = 0$, we will consider the solution of Eq. (1.8) written in the form (equivalent to Eq. (2.9))

$$y(x) = e^{-\beta x}(C\sin(\beta x) + D\cos(\beta x)) + \frac{1}{4\beta^3}\sum_{i=1}^{n} F_i e^{-\beta(x-x_i)}[\cos\beta(x-x_i) + \sin(\beta(x-x_i)]H(x-x_i). \tag{2.15}$$

The distributional derivatives of $y(x)$ are

$$y'(x) = \beta e^{-\beta x}[-(C+D)\sin(\beta x) + (C-D)\cos(\beta x)] - \frac{1}{2\beta^2}\sum_{i=1}^{n} F_i e^{-\beta(x-x_i)}\sin\beta(x-x_i)H(x-x_i) + \frac{1}{4\beta^3}\sum_{i=1}^{n} F_i\delta(x-x_i), \tag{2.16}$$

$$y''(x) = 2\beta^2 e^{-\beta x}(D\sin(\beta x) - C\cos(\beta x)) - \frac{1}{2\beta}\sum_{i=1}^{n} F_i e^{-\beta(x-x_i)}[-\sin\beta(x-x_i) + \cos(\beta(x-x_i)]H(x-x_i) + \frac{1}{4\beta^3}\sum_{i=1}^{n} F_i\delta'(x-x_i), \tag{2.17}$$

$$y'''(x) = 2\beta^3 e^{-\beta x}((C-D)\sin(\beta x) + (C+D)\cos(\beta x)) + \sum_{i=1}^{n} F_i e^{-\beta(x-x_i)}\cos\beta(x-x_i)H(x-x_i) + \sum_{i=1}^{n} F_i[-\frac{1}{2\beta}\delta(x-x_i) + \frac{1}{4\beta^3}\delta''(x-x_i)], \tag{2.18}$$

$$y''''(x) = -4\beta^4 e^{-\beta x}(C\sin(\beta x) + D\cos(\beta x)) - \beta\sum_{i=1}^{n} F_i e^{-\beta(x-x_i)}[\cos\beta(x-x_i) + \sin\beta(x-x_i)]H(x-x_i) + \sum_{i=1}^{n} F_i[\delta(x-x_i) - \frac{1}{2\beta}\delta'(x-x_i) + \frac{1}{4\beta^3}\delta'''(x-x_i)]. \tag{2.19}$$

By Eq. (2.15), if $C = D$, then $y'(0) = 0$.

We have $\lim_{x\to 0^-}\lim_{n\to\infty}\delta_n(x) - \lim_{x\to 0^+}\lim_{n\to\infty}\delta_n(x) = 0$. The similar observation holds true for $\delta'(x)$ and $\delta''(x)$.

Here $\{\delta_n(x)\}$ denotes a δ-sequence of C^∞ functions with compact supports contained in $(-\varepsilon_n, \varepsilon_n)$, $\lim_{n\to\infty}\varepsilon_n = 0$, satisfying $\delta_n(-x) = \delta_n(x)$, $\int_{-\varepsilon_n}^{\varepsilon_n}\delta_n(x)dx = 1$.

Remark 2. *If Eq. (2.15) and Eq. (2.19) are inserted into Eq. (1.8), then on the left-hand side of the resulting equation sums with terms $\delta'(x-x_i)$ and $\delta'''(x-x_i)$ will remain. Since the solution is understood in the distributional sense, the above terms can be neglected. This follows from the observation that* $\int_{x_i-\varepsilon_n}^{x_i+\varepsilon_n}\delta_n'(x-x_i)dx = 0$, $\int_{x_i-\varepsilon_n}^{x_i+\varepsilon_n}\delta_n'''(x-x_i)dx = 0$.

Proceeding as in the first variant and assuming $C = D$, from the second condition Eqs. (2.2) and (2.15) we obtain

$$C(\tilde{x}) = -\frac{e^{\beta\tilde{x}}}{4\beta^3(\sin(\beta\tilde{x}) + \cos(\beta\tilde{x}))}\sum_{i=1}^{n} F_i e^{-\beta(x-x_i)} \times[\sin\beta(\tilde{x}-x_i) - \cos\beta(\tilde{x}-x_i)] \tag{2.20}$$

and the formula for $y(x)$:

$$y(x) = C(\tilde{x})e^{-\beta x}(\sin(\beta x) - \cos(\beta x)) + \frac{1}{4\beta^3}\sum_{i=2}^{n} F_i e^{-\beta(x-x_i)}[\sin\beta(x-x_i) + \cos\beta(x-x_i)]H(x-x_i), \tag{2.21}$$

where $C(\tilde{x})$ is defined by Eq. (2.20).

Remark 3. *As in the first variant, we obtain the formulae, equivalent to Eqs. (2.12) and (2.14):*

$$M(x) = -\frac{1}{2\beta}\Big(e^{-\beta(\tilde{x}-x)}\frac{\sin\beta(x-\pi/4)}{\cos\beta(\tilde{x}-\pi/4)} \times\sum_{i=1}^{n} P_i e^{-\beta(\tilde{x}-x_i)}[\sin\beta(\tilde{x}-x_i) + \cos\beta(\tilde{x}-x_i)] + \sum_{i=2}^{n} P_i e^{-\beta(x-x_i)}[\cos\beta(x-x_i) - \sin\beta(x-x_i)]H(x-x_i)\Big), \tag{2.22}$$

$$T(x) = -\frac{\sqrt{2}}{2}e^{\beta(\tilde{x}-x)}\frac{\sin(\beta x - \pi/4)}{\cos(\beta\tilde{x} - \pi/4)} \times\sum_{i=1}^{n} P_i e^{-\beta(\tilde{x}-x_i)}[\sin\beta(\tilde{x}-x_i) + \cos\beta(\tilde{x}-x_i)] + \sum_{i=2}^{n} P_i e^{-\beta(x-x_i)}\cos\beta(x-x_i)H(x-x_i). \tag{2.23}$$

3. The second conception

The approach presented above is now applied to solution $y(x)$ Eq. (2.1) of equation Eq. (1.8) assuming that $A=-C$, $B=D$ and under conditions

$$y'(\tilde{x})=0, \quad y(\tilde{x})=0, \quad x_1<x_2<\dots<x_n,$$
$$0<x_n<\tilde{x}. \tag{3.1}$$

First variant. In this case solution of Eq. (1.8) is considered in the form

$$y(x)=2(A\sinh(\beta x)\sin(\beta x)+B\cosh(\beta x)\cos(\beta x))$$
$$-\frac{1}{4\beta^3}\sum_{i=1}^{n}F_i[\sinh\beta(x-x_i)\cos\beta(x-x_i)$$
$$-\cosh\beta(x-x_i)\sin(\beta(x-x_i)]H(x-x_i)$$
$$=2(A\sinh(\beta x)\sin(\beta x)$$
$$+B\cosh(\beta x)\cos(\beta x)-g_0(x). \tag{3.2}$$

Derivatives of Eq. (3.2) are:

$$y'(x)=2\beta((A-B)\cosh(\beta x)\sin(\beta x)$$
$$+(A+B)\sinh(\beta x)\cos(\beta x))$$
$$+\frac{1}{2\beta^2}\sum_{i=1}^{n}F_i\sinh\beta(x-x_i)\sin\beta(x-x_i)H(x-x_i)$$
$$=2\beta((A-B)\cosh(\beta x)\sin(\beta x)$$
$$+(A+B)\sinh(\beta x)\cos(\beta x))-g_1(x), \tag{3.3}$$

$$y''(x)=-4\beta^2(-B\sinh(\beta x)\sin(\beta x)+A\cosh(\beta x)$$
$$\times\cos(\beta x))+\frac{1}{2\beta}\sum_{i=1}^{n}F_i[\cosh\beta(x-x_i)\sin\beta(x-x_i)$$
$$+\sinh\beta(x-x_i)\cos(\beta(x-x_i)]H(x-x_i)=$$
$$2\beta^2(-B\sinh(\beta x)\sin(\beta x)$$
$$+A\cosh(\beta x)\cos(\beta x))-g_2(x), \tag{3.4}$$

$$y'''(x)=-4\beta^3(-(A+B)\cosh(\beta x)\sin(\beta x)$$
$$+(A-B)\sinh(\beta x)\cos(\beta x))+\sum_{i=1}^{n}F_i\cosh\beta(x-x_i)$$
$$\times\cos\beta(x-x_i)H(x-x_i). \tag{3.5}$$

From Eq. (3.1), one can determine values of A and B. To simplify computations we select

$$\tilde{x}=(2\pi s+\pi/4)/\beta, \tag{3.6}$$

hence $\sin\beta\tilde{x}=\cos\beta\tilde{x}=1/\sqrt{2}$. The value of s is chosen that $\tilde{x}>x_n$.

Substituting $x=\tilde{x}$ in Eqs. (3,2) and (3.3) and taking into consideration Eq. (3.1), one gets the system

$$\sqrt{2}\begin{pmatrix}\sinh\beta\tilde{x} & \cosh\beta\tilde{x}\\ \beta(\cosh\beta\tilde{x}+\sinh\beta\tilde{x}) & \beta(\sinh\beta\tilde{x}-\cosh\beta\tilde{x})\end{pmatrix}$$
$$\times\begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}g_0(\tilde{x})\\g_1(\tilde{x})\end{pmatrix} \tag{3.7}$$

Since

$$\det\begin{pmatrix}\sinh\beta\tilde{x} & \cosh\beta\tilde{x}\\ \beta(\cosh\beta\tilde{x}+\sinh\beta\tilde{x}) & \beta(\sinh\beta\tilde{x}-\cosh\beta\tilde{x})\end{pmatrix}=$$
$$-\beta(1+2\cosh\beta\tilde{x}\sinh\beta\tilde{x})<0$$

for all $\tilde{x}>x_n$, the system Eq. (3.7) has a solution for any $\tilde{x}>x_n$.

Remark 4. *Considering, instead of Eq. (3.1), conditions*

$$y(\tilde{x})=0, \quad y''(\tilde{x})=0, \tag{3.8}$$

we obtain equation for constants A and B:

$$\sqrt{2}\begin{pmatrix}\sinh\beta\tilde{x} & \cosh\beta\tilde{x}\\ \beta^2\cosh\beta\tilde{x} & -\beta^2\sinh\beta\tilde{x}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=$$
$$\begin{pmatrix}g_0(\tilde{x})\\g_2(\tilde{x}),\end{pmatrix} \tag{3.9}$$

which is uniquely solvable, since its determinant $\Delta=\beta^2(\sinh^2\beta\tilde{x}-\cosh^2\beta\tilde{x})=-(1/2)\beta^2\cosh\beta\tilde{x}<0$ for all $\tilde{x}$.

Second variant. We proceed with the general solution of the form different from Eq. (3.2).

We will apply the following form of the general solution

$$y(x)=2(A\sinh\beta x\sin\beta x+B\cosh\beta x\cos\beta x)$$
$$+\frac{1}{4\beta^3}\sum_{i=1}^{n}F_i\,\mathrm{e}^{-\beta(x-x_i)}[\sin\beta(x-x_i)+\cos\beta(x-x_i)]$$
$$\times H(x-x_i)=2(A\sinh\beta x\sin\beta x$$
$$+B\cosh\beta x\cos\beta x)+k_0(x). \tag{3.10}$$

Consequently, derivatives of $y(x)$ assume the form

$$y'(x)=2\beta((A-B)\cosh\beta x\sin\beta x+(A+B)\sinh\beta x$$
$$\times\cos\beta x)-\frac{1}{2\beta^2}\sum_{i=1}^{n}F_i\,\mathrm{e}^{-\beta(x-x_i)}\sin\beta(x-x_i)$$
$$\times H(x-x_i)+\frac{1}{4\beta^3}\sum_{i=1}^{n}F_i\delta(x-x_i)$$
$$=2\beta((A-B)\cosh\beta x\sin\beta x$$
$$+(A+B)\sinh\beta x\cos\beta x+k_1(x), \tag{3.11}$$

$$y''(x)=2\beta^2(-B\sinh\beta x\sin\beta x+A\cosh\beta x\cos\beta x)$$
$$+\frac{1}{2\beta}\sum_{i=1}^{n}F_i\,\mathrm{e}^{-\beta(x-x_i)}[\sin\beta(x-x_i)$$
$$-\cos\beta(x-x_i)]H(x-x_i)+\frac{1}{4\beta^3}\sum_{i=1}^{n}F_i\delta'(x-x_i)$$
$$=2\beta^2(-B\sinh\beta x\sin\beta x$$
$$+A\cosh\beta x\cos\beta x)+k_2(x), \tag{3.12}$$

$$y'''(x)=2\beta^3[-(A+B)\cosh\beta x\sin\beta x+(A-B)$$
$$\times\sinh\beta x\cos\beta x]+\sum_{i=1}^{n}F_i\,\mathrm{e}^{-\beta(x-x_i)}\cos\beta(x-x_i)$$

$$\times H(x-x_i) - \frac{1}{2\beta}\sum_{i=1}^{n} F_i\delta(x-x_i)$$
$$+\frac{1}{4\beta^3}\sum_{i=1}^{n} F_i\delta''(x-x_i), \qquad (3.13)$$
$$y''''(x) = -4\beta^4(A\sinh\beta x\sin\beta x + B\cosh\beta x$$
$$\times\cos\beta x) - \sum_{i=1}^{n} \mathrm{e}^{\beta(x-x_i)}F_i[\sin\beta(x-x_i)$$
$$+\cos\beta(x-x_i)]H(x-x_i) + \sum_{i=1}^{n} F_i\delta(x-x_i) - \frac{1}{2\beta}$$
$$\times\sum_{i=1}^{n} F_i\delta'(x-x_i) + \frac{1}{4\beta^3}\sum_{i=1}^{n} F_i\delta'''(x-x_i). \qquad (3.14)$$

Assume Eq. (3.6) is satisfied. Then proceeding as in the first version, we obtain a system of linear equations for A and B corresponding to conditions Eq. (3.1)

$$\sqrt{2}\begin{pmatrix}\sinh\beta\tilde{x} & \cosh\beta\tilde{x}\\ \beta^2(\cosh\beta\tilde{x}+\sinh\beta\tilde{x}) & \beta^2(\sinh\beta\tilde{x}-\cosh\beta\tilde{x})\end{pmatrix}$$
$$\times\begin{pmatrix}A\\B\end{pmatrix} = \begin{pmatrix}k_0(\tilde{x})\\k_1(\tilde{x})\end{pmatrix} \qquad (3.15)$$

and a system corresponding to Eq. (3.8)

$$\sqrt{2}\begin{pmatrix}\sinh\beta\tilde{x} & \cosh\beta\tilde{x}\\ \beta^2\cosh\beta\tilde{x} & -\beta^2\sinh\beta\tilde{x}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix} =$$
$$\times\begin{pmatrix}k_0(\tilde{x})\\k_2(\tilde{x})\end{pmatrix}. \qquad (3.16)$$

4. The third conception

The continuous elastic support of the beam is replaced by the suitable chosen discrete support. Discrete models of such beams of both finite and infinite length are easier to handle mathematically.

Here the model with the discrete support is considered. The corresponding equation has the form [2, 7]

$$y''''(x) = \frac{1}{EJ}(\varphi_0(x) + \varphi_1(x)), \qquad (4.1)$$

(see Fig. 4), where $\varphi_0(x)$ is a distribution of a system of discrete elastic supports and $\varphi_1(x)$ represents the external loads acting on a beam:

$$\varphi_0(x) = \sum_{i=-n}^{n} S_i\delta(x-x_i),$$
$$\varphi_1(x) = \sum_{i=1}^{m} P_i\delta(x-z_i). \qquad (4.2)$$

P_i is the y-coordinate of the external force $\bar{P}_i$, $P_i < 0$, $S_i = ky(x_i)$, $x_i = hi$. We assume that the number of elastic supports is odd, otherwise an extra support at the point x_n is added.

Points z_i of action of $\bar{P}_i$ upon a beam are situated between the extreme points of action of the support, i.e. $x_{-n} \le z_i \le x_{n-1}$, $i = 1,\dots,m$ if a number of elastic supports is even, or $x_{-n} \le z_i \le x_n$, $i = 1,\dots,m$ if a number of x_i is odd.

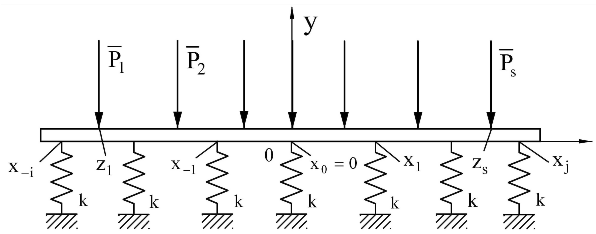

Fig. 4. The beam on the discrete support.

By Eqs. (4.1) and (4.2), the differential equation of the beam has the form

$$y''''(x) = \frac{1}{EJ}[k\sum_{i=-n}^{n} y(x_i)\delta(x-x_i)$$
$$+\sum_{i=1}^{m} P_i\delta(x-z_i)]. \qquad (4.3)$$

We assume that for a sufficiently large n

$$y(x_{-n}) = y'(x_{-n}) = 0, \quad y(x_n) = y'(x_n) = 0. \qquad (4.4)$$

Note that $z_i \in (x_{-n}, x_n)$ for $i = 1,\dots,m$. From Eq. (4.3) we get

$$y'''(x) = \frac{1}{EJ}(k\sum_{i=-n}^{n} y(x_i)H(x-x_i)$$
$$+\sum_{i=1}^{m} P_iH(x-z_i) + C_1), \qquad (4.5)$$
$$y''(x) = \frac{1}{EJ}(k\sum_{i=-n}^{n} y(x_i)(x-x_i)H(x-x_i)$$
$$+\sum_{i=1}^{m} P_i(x-z_i)H(x-z_i) + C_1x + C_2), \qquad (4.6)$$
$$y'(x) = \frac{1}{EJ}(k\sum_{i=-n}^{n} y(x_i)(1/2)(x-x_i)^2H(x-x_i)$$
$$+\sum_{i=1}^{m} P_i(1/2)(x-z_i)^2H(x-z_i)$$
$$+C_1(x^2/2) + C_2x + C_3), \qquad (4.7)$$
$$y(x) = \frac{1}{EJ}(k\sum_{i=-n}^{n} y(x_i)(1/6)(x-x_i)^3H(x-x_i)$$
$$+\sum_{i=1}^{m} P_i(1/6)(x-z_i)^3H(x-z_i)$$
$$+C_1(x^3/6) + C_2(x^2/2) + C_3x + C_4). \qquad (4.8)$$

Let $p(x)$ and $q(x)$ denote the right hand sides of Eqs. (4.7) and (4.8) respectively. Then, by Eq. (4.4), the values of $y(x_i)$ and integration constants C_i are determined by equations

$$p(x_{-n}) = 0, \quad q(x_{-n}) = 0, \quad p(x_n) = 0,$$
$$q(x_n) = 0 \qquad (4.9)$$

and conditions

$$y(x_i) - q(x_i) = 0, \quad \text{for } i = -n+1,\dots,n-1. \qquad (4.10)$$

Set $a = 6EJ/k$, $R_i = C_i/k$, $F_i = P_i/k$, $y_i = y(x_i)$, $I(s) = \{r : x_s - \eta_r > 0\}$. Denote by w the polynomial (with unknown coefficients R_i): $w(x) = R_1x^3 + 3R_2x^2 + 6R_3x + 6R_4$.

Recalling that $x_s - x_i = (s-i)h$, the above formulae lead to the system of $2n+3$ linear equations with unknown y_i, R_j.

The first and the last two equations correspond to Eq. (4.9), the remaining ones represent Eq. (4.10).

$$w(-nh) = 0, \tag{4.11}$$

$$(1/3)w'(nh) = (1/3)\frac{\mathrm{d}}{\mathrm{d}z}w(z)|_{z=nh} = 0, \tag{4.12}$$

$$-ay(x_s) + \sum_{i=-n}^{s-1} y(x_s)(s-i)^3h^3 + \sum_{r\in I(s)} F_r(x_r - z_s) + w(sh) = 0, \tag{4.13}$$

$$\sum_{i=-n}^{n-1} y(x_i)(n-i)^3h^3 + \sum_{r\in I(n)} F_r(x_n - z_r) + w(nh) = 0, \tag{4.14}$$

$$\sum_{i=-n}^{n-1} y(x_i)(n-i)^2h^2 + \sum_{r\in I(n)} F_r(x_n - z_r) + (1/3)w'(nh) = 0, \tag{4.15}$$

which can be written in vector form

$$h^3\mathcal{A}(n)y + W = P, \tag{4.16}$$

where

$$\mathcal{A}(n) = \begin{pmatrix}
1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -a/h^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1^3 & -a/h^3 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2^3 & 1^3 & \dots & -a/h^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 3^3 & 2^3 & \dots & 1^3 & -a/h^3 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & (2n-1)^3 & (2n-2)^3 & \dots & 2^3 & 1^3 & -a/h^3 & 0 & 0 \\
0 & (2n)^3 & (2n-1)^3 & (2n-2)^3 & \dots & 3^3 & 2^3 & 1^3 & -a/h^3 & 0 \\
0 & (2n)^2 & (2n-1)^2 & (2n-2)^2 & \dots & 3^2 & 2^2 & 1^2 & 0 & -a/(3h^2)
\end{pmatrix},$$

$$y = \begin{pmatrix} 0 \\ 0 \\ y_{-n+1} \\ . \\ y_{n-1} \\ 0 \\ 0 \end{pmatrix}, \quad W = \begin{pmatrix} (1/3)w'(-nh) \\ w(-nh) \\ w((-(n-1)h) \\ . \\ w((n-1)h) \\ w(nh) \\ (1/3)w'(nh) \end{pmatrix},$$

$$P = \begin{pmatrix} 0 \\ 0 \\ P_{-n+1} \\ . \\ P_{n-1} \\ 0 \\ 0 \end{pmatrix},$$

where

$$P_s = \sum_{r\in I(s)} F_r(x_r - z_s), s = 0, \pm 1, \dots, \pm(n-1).$$

The particular form Eq. (4.16) of the system is used to reduce the size of $\mathcal{A}(n)$.

In the $(2n+3)\times(2n+3)$ matrix $\mathcal{A}(n)$ marked columns correspond to y'_{-n}, y_{-n}, y_{-n+1}, y_{-n+2}, y_{-n+3}, y_{-n+4}, y_{n-1}, y_n, y'_n.

Remark 5. *Number n in $\mathcal{A}(n)$ should be chosen as small as possible. Its value has to satisfy Eq. (4.10) and the inequality*

$$x_n > \max\{|z_1|, |z_m|\} + \frac{9\pi}{4\beta}.$$

The origin of coordinates should be placed at the centre of the interval $[z_{\min}, z_{\max}]$ containing points z_i, $i = 1, \dots, m$.

5. Conclusions

The paper presents three conceptions of the description of the bending line of the infinitely long elastic beam. The analytic formulae of the bending line given here, contrary to approaches known in the literature, take into consideration not one but the whole system of external forces acting on the beam.

The description of concentrated forces applying the Dirac measures makes it possible to obtain the desired equation of the bending line in the case of several external forces acting on a beam.

We use the term "conception" instead of "method" because the resulting formulae cannot be compared with the ones existing in the literature, where only the influence of a single external force on the bending is considered.

It is possible that some conceptions presented in the paper can be considered as new methods provided they will be verified by numerous examples. The verification can be done by sketching the bending line (which is not an easy task) and then making its analysis.

The distribution theory is used in other scientific works on the modeling of continuous and discrete systems [8–10]. The problem of such verification will be the subject of another work.

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