Proceedings of the 41st Polish Seminar on Positron Annihilation, Lublin, September 9–13, 2013

Electron–Positron Annihilation in Ultra-Strong Magnetic Fields. Comparison of One- and Two-Photon Annihilation at Middly Relativistic Regime

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We consider the one- and two-photon e^+e^- annihilation processes in an ultra-strong magnetic field, at middly relativistic regime. Such conditions are reached in neutron stars and especially in magnetars, where electrons and positrons flow within the magnetar corona with average momenta of the order of $p \sim m_0 c$, where m_0 — electron mass, c — light speed. We pay special attention to the ratio of the total annihilation rates of both processes. The well-known result is that in the limit $p \to 0$ and for magnetic induction above the critical Schwinger value $B_0 = 4.41 \times 10^9$ T, one-photon annihilation dominates over the two-photon process. Results presented in this article verify the current knowledge about this ratio in magnetars; the calculations indicate that for particles moving with middly relativistic momenta both processes can be equally important.

DOI: 10.12693/APhysPolA.125.688
PACS: 11.10.Ef, 12.20.Ds, 95.60.–k, 97.60.Gb

1. Introduction

The process of e^+e^- pair annihilation in super-strong magnetic fields has been studied e.g. in [1–5]. One of the major results was that the annihilation into a single photon dominates over the two-photon process just for above $0.25 \times B_0$; moreover, the latter one significantly declines with increasing magnetic induction B. However, the considerations in the mentioned articles were focused on annihilation rate and total emission for particles at rest, $p \to 0$. A question arises, though: is it possible that in super-strong magnetic fields charged particles would move so slowly, by average? A very new result, reported in [6], provides the answer. The author studied electron–positron flows around magnetars, where the magnetic induction may even exceed 10^{12} T, and estimated dynamic distribution of electron and positron momenta. He demonstrated that the contribution of high speed particles (with the Lorentz factor $\gamma_L \gg 1$) is significant. Moreover, the maxima, especially in the outer magnetar corona, are located in the middly relativistic regime. Therefore we suspect that annihilation process as dependent on both the magnetic field B and momentum p should be taken into account. It was shown in paper [7] that the single-photon annihilation diminishes for relativistic momentum of the positron. In our recent study we show an opposite behaviour for the two-photon process [8]. Therefore, the major aim of this article is to reconsider the ratio of both processes, one- and two-photon annihilation in a super-strong magnetic field, but for any momentum of the annihilating particles.

2. Theory

By an ultra-strong magnetic field we mean the magnetic induction near (and above) the critical Schwinger value $B_0 = 4.41 \times 10^9$ T. As it was indicated in Introduction, our aim is to verify current knowledge about the ratio of one- and two-photon annihilation, in particular, to perform calculations in the center-of-mass frame for any momentum at middly relativistic regime.

The annihilation process of e^-e^+ pair into one or two photons is represented by reactions

$$e^-(p_1) + e^+(p_2) \to \gamma_1(k_1, \varepsilon_1), \quad (1)$$

$$e^-(p_1) + e^+(p_2) \to \gamma_1(k_1, \varepsilon_1) + \gamma_2(k_2, \varepsilon_2), \quad (2)$$

where p_1 and p_2 are 4-momenta of the electron and positron, respectively. The particles have spin projections onto z axis labeled s_1 and s_2. The photon in (1) carries polarization ε_1 and its 4-momentum is $k_1 = (\omega_1, k_1^x, k_1^y, k_1^z)$, whereas in (2) we have two photons with 4-momenta $k_1 = (\omega_1, k_1^x, k_1^y, k_1^z)$ and $k_2 = (\omega_2, k_2^x, k_2^y, k_2^z)$, which carry polarizations ε_1 and ε_2, respectively. We consider a frame with opposed electron and positron beams (with special consideration of the center-of-mass frame, CM). In order to simplify analytical calculations we assume that the directions of the magnetic field and the motion of the colliding particles are parallel, thus the 4-momenta of e^-, e^+ are: $p_1 = (E_p, 0, 0, p)$ and $p_2 = (E_q, 0, 0, -q)$ with energies E_p and E_q for the electron and positron, respectively. The scaterring amplitudes calculated with methods of quantum electrodynamics, see e.g. [9], are as follows: for one-photon annihilation (one Feynman diagram)

$$S_{\text{fi}}^{\text{I}} = \text{i}\,\text{e} \int \text{d}^4 x \bar{\psi}_{\text{f}}(x) \gamma_\mu A_1^\mu(x) \psi_{\text{i}}(x), \quad (3)$$

and for two-photon annihilation (which is a sum of two scattering amplitudes for two noncongruent Feynman diagrams)

$$S_{\text{fi}}^{\text{II}(a)} = \text{i}\,\text{e}^2 \int \text{d}^4 x\, \text{d}^4 x' \bar{\psi}^{\text{f}}(x') \gamma_\nu A_2^\nu(x') \times G_{\text{F}}(x' - x) \gamma_\mu A_1^\mu(x) \psi^{\text{i}}(x), \quad (4)$$

$$S_{\text{fi}}^{\text{II}(b)} = \text{i}\,\text{e}^2 \int \text{d}^4 x\, \text{d}^4 x' \bar{\psi}^{\text{f}}(x') \gamma_\nu A_1^\nu(x')$$

$$\times G_{\mathrm{F}}(x'-x)\gamma_\mu A_2^\mu(x)\psi^{\mathrm{i}}(x), \quad (5)$$

where $\psi^{\mathrm{i}}(x)$ and $\psi^{\mathrm{f}}(x')$ are the wave functions of the annihilating particles (in our case electron and positron, respectively). The quantities $A_j(x)$, where $j = 1,2$, are the photon wave functions defined as $A_j^\mu(x) = \sum_{r=1,2}(2\omega_j L^3)^{-1/2} \times \exp(\mathrm{i}k_j x)\varepsilon_{j,r}^\mu$, ω_j is the energy of the photon γ_j, k_j — its 4-momentum, and $\varepsilon_{j,r}$ — its polarization (the wave-function is then summed over two basis polarization vectors), while L — the length of the normalization box. Of course the full amplitude for the two-photon annihilation is a sum, $S_{\mathrm{fi}}^{\mathrm{II}} = S_{\mathrm{fi}}^{\mathrm{II}(a)} + S_{\mathrm{fi}}^{\mathrm{II}(b)}$. The amplitudes for both considered processes are calculated in the position representation; the momentum representation is unavailable due to capability of the magnetic field to bind the charged particles in oscillatory states. The fermion propagator in (4) and (5), in the case of external magnetic field has to be calculated from

$$G_{\mathrm{F}}(x'-x) = \left(\frac{L}{2\pi}\right)^2 \sum_{n=0}^{\infty}\sum_{s_1} \int \frac{\mathrm{d}A}{\lambda^2}\int \mathrm{d}P$$
$$\times \Big[-\mathrm{i}\,\Theta(t'-t)\,\mathrm{e}^{-\mathrm{i}E_n(t'-t)}u_n^{(s_1)}(x')\bar{u}_n^{(s_1)}(x)$$
$$+\mathrm{i}\,\Theta(t-t')\,\mathrm{e}^{\mathrm{i}E_n(t'-t)}v_n^{(s_1)}(x')\bar{v}_n^{(s_1)}(x)\Big], \quad (6)$$

see [3]. E_n is the energy of the Landau level $n = 0,1,2\ldots$, electron and positron bispinors ($u_n^{(s_1)}$ and $v_n^{(s_1)}$, respectively) are the solutions of Dirac's equation in the magnetic field: $-\gamma^\mu(\mathrm{i}\partial_\mu \pm \mathrm{e}A_\mu - m_0)\psi^\pm(x) = 0$, $\psi^\pm(x)$ are full electron/positron wave functions (in this equation +e is for a positron, −e for an electron), A_μ is the 4-potential, here chosen as $A = (0,0,Bx,0)$. The exact form of the solutions can be found e.g. in [7]. Here we only stress that the components of the bispinors contain oscillator wave functions for n-th Landau level, for which the energy equals $E_n = \sqrt{m_0^2 + p^2 + 2n/\lambda^2}$ (we have defined $\lambda \equiv (m_0\sqrt{b})^{-1}$ and $b \equiv B/B_0$). After some calculus one finds that squared scattering amplitude for one-photon annihilation reads

$$|S_{\mathrm{fi}}^{\mathrm{I}}|^2 = \frac{T}{L^5}\frac{4\pi^4\alpha}{E_pE_q\omega_1}\,\mathrm{e}^{-\frac{(a_- - a_+)^2}{2\lambda^2}}\,\mathrm{e}^{-\frac{1}{2}\lambda^2(k_1^x)^2}$$
$$\times \sin^2\theta_1(E_p+m_0)(E_q+m_0)$$
$$\times\left(1-\frac{p}{E_p+m_0}\frac{q}{E_q+m_0}\right)^2\delta\left(\omega_1 - E_p - E_q\right)$$
$$\times\,\delta\left(k_1^y - \frac{a_- - a_+}{\lambda^2}\right)\delta\left(k_1^z - p + q\right), \quad (7)$$

where the quantities $a_\pm$ are the distances of the orbit centers of the electron (a_-) or positron (a_+) from z axis, and we used the definition of the fine structure constant α in the natural system of units: $e^2 = 4\pi\alpha$; T and L are the length of the normalization box for the electron/positron waves along the time and space coordinate, respectively. The amplitude for the two-photon annihilation, in turn, has a form of

$$|S_{\mathrm{fi}}^{\mathrm{II}}|^2 = \frac{e^4\pi^3}{E_pE_q\omega_1\omega_2}\frac{T}{8L^8}\delta\left(E_p+E_q-\omega_1-\omega_2\right)$$
$$\times\,\delta\left(k_1^z+k_2^z-p+q\right)\delta\left(k_1^y+k_2^y+\frac{a_- - a_+}{\lambda^2}\right)$$
$$\times\exp\left(\frac{-\lambda^2}{2}\left((k_1^x)^2+(k_1^y)^2+(k_2^x)^2+(k_2^y)^2\right)\right)$$
$$\times\sum_{r_1r_2}\Big|\,\mathrm{e}^{\mathrm{i}\frac{\lambda^2}{2}(k_1^xk_2^y-k_1^yk_2^x)}\mathcal{N}^{(1)}$$
$$+\,\mathrm{e}^{-\mathrm{i}\frac{\lambda^2}{2}(k_1^xk_2^y-k_1^yk_2^x)}\mathcal{N}^{(2)}\Big|^2, \quad (8)$$

where the quantities $\mathcal{N}^{(1,2)}$ can be found for example in [3] and [7]. It is worth stressing here that $\mathcal{N}^{(1,2)}$ depend on photon polarizations r_1 and r_2. In general case, both quantities can be calculated only numerically, however in some special cases, e.g. in CM frame, one is able to find its analytic form.

When we already have scattering amplitudes for one- and two-photon annihilation, we can now calculate the cross-sections and total rates for these processes. We focus here on the total cross-sections, whose definition can be found e.g. in [3], and in our case reads

$$\sigma^{\mathrm{I}} = \frac{2\pi^2\alpha\lambda^2}{(p+q)(E_p+E_q)}$$
$$\times\frac{\left[(E_p+m_0)(E_p+E_q)-p(p+q)\right]^2}{(E_p+m_0)(E_p+E_q)-p^2}$$
$$\times\,\mathrm{e}^{-\frac{\lambda^2}{2}\left((E_p+E_q)^2-(p-q)^2\right)}, \quad (9)$$

$$\sigma^{\mathrm{II}} = \int \mathrm{d}\omega_1\,\mathrm{d}\Omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\Omega_2\frac{\alpha^2}{32\pi}\frac{\omega_1\omega_2}{pE_q+qE_p}$$
$$\times\,\delta\left(E_p+E_q-\omega_1-\omega_2\right)\delta\left(k_1^z+k_2^z-p+q\right)$$
$$\times\,\mathrm{e}^{\frac{-\lambda^2}{2}\left[(k_1^x)^2+(k_2^x)^2+(k_1^y)^2+(k_2^y)^2\right]}$$
$$\times\sum_{r_1r_2}\Big|\,\mathrm{e}^{\mathrm{i}\frac{\lambda^2}{2}(k_1^xk_2^y-k_1^yk_2^x)}\mathcal{N}^{(1)}$$
$$+\,\mathrm{e}^{-\mathrm{i}\frac{\lambda^2}{2}(k_1^xk_2^y-k_1^yk_2^x)}\mathcal{N}^{(2)}\Big|^2. \quad (10)$$

The total rates for both annihilation processes are defined as $R_{\mathrm{I}} = n^-n^+\int\mathrm{d}\omega_1\,\mathrm{d}\Omega_1\int\mathrm{d}p\mathrm{d}q\eta^-(p)\eta^+(q)\,\mathrm{d}\sigma^{\mathrm{I}}|p/E_p + q/E_q|$ and $R_{\mathrm{II}} = n^-n^+\int\mathrm{d}\omega_1\,\mathrm{d}\Omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\Omega_2\int\mathrm{d}p\mathrm{d}q\,\eta^-(p)\eta^+(q)\,\mathrm{d}\sigma^{\mathrm{II}}|p/E_p+q/E_q|$, respectively, where $\mathrm{d}\sigma^{\mathrm{I,II}}$ is the differential cross-section for one- or two-photon annihilation and $|p/E_p+q/E_q|$ is the relative relativistic velocity, n^- — density of the electron plasma, n^+ — density of the positron plasma, $\eta^-(p)$ and $\eta^+(q)$ — momentum distributions of electrons and positrons, respectively. In our calculations we assumed constant densities $n^\pm$ and Gaussian momentum distributions $\eta^\pm(P) = (\Delta P\sqrt{\pi})^{-1}\exp(-(P-P_0)/\Delta P^2)$, where $P = p$ for η^- and $P = q$ for η^+. Further calculations have to be performed numerically.

3. Results and discussion

Quantitatively we describe annihilation processes by providing the cross-section, spectral functions, total emission and total annihilation rates $R_{\rm I}$ and $R_{\rm II}$ (on which we focus in this manuscript).

Figure 1 shows $R_{\rm I}$ for one-photon and $R_{\rm II}$ for two--photon annihilation calculated in the center-of-mass frame, when electrons and positrons have Gaussian momentum distributions with the width $\Delta p = 1$ MeV and the center $p_0 = 0$. The rates are normalized to R_0 — a value of total annihilation rate for particles moving in no external field, i.e. in the free case. R_0 is calculated in the same manner as in the definition of $R_{\rm II}$, however by inserting Dirac's cross-section $\sigma_{\rm D}$ [9].

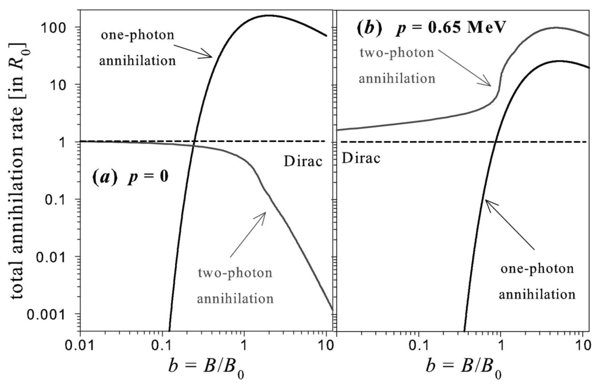

Fig. 1. Total annihilation rates for one- and two--photon annihilation ($R_{\rm I}$ and $R_{\rm II}$, respectively) in a super-strong magnetic field, normalized to the free case R_0 calculated from Dirac's cross-section; (a) in the limit of $p = 0$, (b) for particles moving with relativistic momentum $p = 0.65$ MeV.

Figure 1a fully corresponds to Fig. 11 in Ref. [3] and has been obtained in the limiting case of $p = 0$. It confirms the dominating role of the one-photon annihilation when the magnetic induction $b > 0.25$, where $b = B/B_0$. Moreover, the two-photon annihilation rate in the presence of the magnetic field is always lesser than the free--case rate R_0 and strongly diminishes with increasing induction.

On the other hand, Fig. 1b presents opposite behavior. If the particle momentum reaches a middly relativistic value, here $p = 0.65$ MeV $\Leftrightarrow E_p \approx 1.6 m_0$, the two--photon annihilation becomes equally important as the one-photon process, and even $R_{\rm II} > R_{\rm I}$. This is an important result if we ask a question about the annihilation process in the magnetar corona, where not only the magnetic induction can exceed the critical value, but also the momentum distribution of electrons and positrons are shifted toward middly relativistic values [8]. However another question arises: how does the ratio of the annihilation rates behave as a function of the momentum, and how the momentum distributions of the particles in plasma affect the properties of the ratio?

The answer is included in Fig. 2. We have calculated the ratio of the total annihilation rates of the both processes $R_{\rm II}/R_{\rm I}$ as dependent on the collision energy $E_p = \sqrt{m_0^2 + p^2}$ and for distributions characteristics $p_0 = q_0 = 0$ (filled circles)or $p_0 = q_0 = 1$ MeV (hollow circles). The width $\Delta p = \Delta q = 1$ MeV on all plots. It is clearly visible that even for the magnetic induction $b \geq 1$, the two-photon annihilation may exceed or, at least, be comparable, to the one-photon process, providing that the collision energy E_p is middly relativistic and the momentum distribution is also shifted toward middly relativistic values (here about $2 \times m_0$). As we can see by comparing (a) and (b) in Fig. 2, the stronger increase in the ratio is observed for $b = 1$ and $p_0 = q_0 = 1$ MeV; for example, at $E_p = 2 \times m_0$ on (a) $R_{\rm II} \simeq 100 R_{\rm I}$, but for $b = 10$ and when $p_0 = q_0 = 0$ in $\eta^{\pm}$, the ratio is growing very slowly with increasing collision energy, see (b), e.g. for $E_p = 2.5 m_0 R_{\rm II} = 1.2 R_{\rm I}$. This outcome supplements the well known results reported in Ref. [3] where at $b \geq 1$ the one-photon annihilation is the dominating process, see Fig. 11 there.

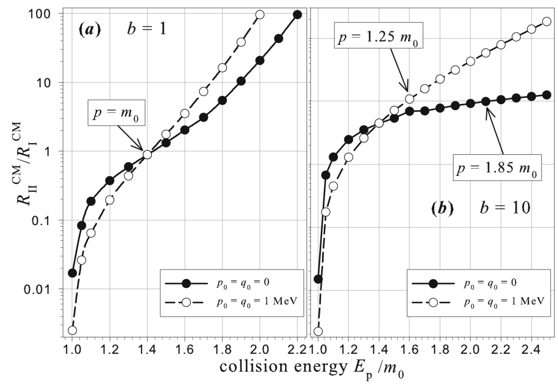

Fig. 2. Ratio of the total annihilation rates $R_{\rm II}$ to $R_{\rm I}$ in the case of $b = 1$ (a) and $b = 10$ (b).

4. Conclusions

The rates of one- and two-photon e^+e^- pair annihilation are equally significant when the magnetic induction exceeds the critical Schwinger value and the energy of the colliding particles is middly relativistic. It looks that in magnetar corona (where such conditions are reached) both processes take place with about the same probability and the hypothesis of the dominating role of one--photon annihilation should be revisited.

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