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On the Flexural Vibration of Pre-Stressed Nanobeams Based on a Nonlocal Theory

O. Rahmani

Mechanical Engineering Department, University of Zanjan, Zanjan, Iran

In this study, a model based on a nonlocal theory is proposed to study the effect of external loads on the free vibration of nanobeams. The governing equations for the initially pre-stressed beam are obtained by using the Hamilton principle. Analytical solution of vibration is presented using the Eringen nonlocal theory to bring out the effect of the nonlocal behavior on natural frequencies. The performance of the present model is compared with that of others by the presentation of comparative results. It is suggested that the present model can be used as a benchmark in future studies.

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1. Introduction

In the recent years, beams have found important applications in nanoscale measurements e.g., in atomic force microscopes (AFMs). Missing of a material length scale parameter, the conventional beam theories cannot explain the size effect monitored in bending tests at nanoscales. Therefore, these classical beam models need to be extended to include material nanostructural features. Nonlocal theories contain supplementary material constants and are able to efficiently describe a variety of size effects. The vibration characteristics of nanobeams can be employed for nano electromechanical systems (NEMS) applications. Subsequent to the Eringen studies [1, 2] on the nonlocal elasticity in order to capture the small scale effect, several researchers have used the nonlocal elasticity concept for the vibration analyses of nanostructures [3–5]. According to the lack of study on the pre loads effect on the vibration of nanostructures with allowance for small scale effect, this paper presents a formulation of nonlocal Euler beam theory for the flexural vibration of simply supported nanobeams based on Eringen's nonlocal theory.

2. Pre-stressed Euler beam: equation of motion

Although different shape functions are applicable, only the one which converts the present theory to the corresponding Euler–Bernoulli beam theory is employed here. According to the Euler beam theory, the strain–displacement relation is given by [6]:

$$\varepsilon_{xx} = u_{,x} - z w_{,xx} = \varepsilon^0_{xx} - z\,\Psi^0, \tag{1}$$

where ε^0 and $\Psi^0 \varepsilon^0_{xx}$ are the extensional strain and the curvature of mid-plane, respectively.

The derivation of the governing equations and the boundary conditions is based on Hamilton's principle of the minimization of the Lagrangian L of the deformed system

$$\delta \int_{t_1}^{t_2} (\pi - T - W)\,\mathrm{d}t = 0, \tag{2}$$

where π is the strain energy of the beam, T is the kinetic energy of the system, W is the work done by external loads and t is the time coordinate between the times t_1 and t_2. Applying Hamilton's principle leads to the equation of motion of pre-stressed beam [3]:

$$N_{,x} + F = I_0 u_{,tt}, \tag{3}$$

$$M_{,xx} - (N_\mathrm{a} w_{,x})_{,x} + p = I_0 w_{,tt} - I_2 w_{,xxtt}, \tag{4}$$

where I_0 and I_2 are mass inertias of the beam, N and M are the stress and moment resultants, $F(x,t)$ and $p(x,t)$ are the axial and transverse distributed forces, and N_a is the in-plane external force.

3. Non-local Euler beam theory

Unlike the constitutive equation in classical elasticity, Eringen's nonlocal elasticity theory [1] states that the stress at a point x in a body depends not only on the strain at point x but also on those at all other points of the body. Thus, the nonlocal stress tensor σ at point x is expressed [3]:

$$\sigma = \int_V K(|\boldsymbol{x}' - \boldsymbol{x}|, \tau)\boldsymbol{S}(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}', \tag{5}$$

$$\boldsymbol{S}(\boldsymbol{x}) = C(\boldsymbol{x}) : \varepsilon(\boldsymbol{x}), \tag{6}$$

where $S(x)$ is the classical, macroscopic stress tensor at point x and the kernel function $K(|x' - x|, \tau)$ shows the nonlocal modulus, being the distance (in Euclidean norm) and τ is a material constant that relates to external and internal characteristic lengths. $C(x)$ is the linear elasticity tensor. The constitutive Eqs. (5) and (6) together represent the nonlocal constitutive behavior of a size dependent structure. $\tau = e_0 a/l$ is given as a small scale factor where e_0 is a constant to adjust the model to match the reliable results by experiments or other models, l is the external characteristic length, and a is internal characteristic length. The integral constitutive relation in (5) leads to difficulty in elasticity problems solving. On the other hand, it is possible [1, 2] to define the integral constitutive relations in a counterpart differential form as [3]:

$$(1 - \tau^2 l^2 \nabla^2)\sigma = S. \tag{7}$$

The nonlocal constitutive relation in Eq. (7), with Eqs. (5) and (6), gives Hooke's law for a uni-axial stress state for nanobeam

$$\sigma_{xx} - \mu\sigma_{xx,xx} = E\varepsilon_{xx}, \tag{8}$$

where $\mu = e_0^2 a^2$ and E is Young's modulus. Integrating Eq. (8) over the beam's cross-section area, we obtain the axial force–strain relation as

$$N - \mu N_{,xx} = EA\varepsilon_{xx}^0, \tag{9}$$

Also, multiplying Eq. (8) by z, integrating over the cross-section area, we obtain the moment curvature relation as

$$N - \mu N_{,xx} = EA\varepsilon_{xx}^0, \tag{10}$$

$$M - \mu M_{,xx} = EI\kappa^0. \tag{11}$$

Substituting for the second derivative of M from Eq. (4) into Eq. (11), and substituting the result into Eq. (4), gives

$$(-EIw_{,xx})_{,xx} + \mu[(N_{\mathrm{a}}w_{,x}) - p + I_0 - I_2 w_{,xxtt}]_{,xx} + p - (N_{\mathrm{a}}w_{,x})_{,x} = I_0 w_{,tt} - I_2 w_{,xxtt}. \tag{12}$$

4. Analytical solution

For the simply supported boundary condition, one can consider the following mode shapes function:

$$w(x,t) = X(x)(A\cos\omega t + B\sin\omega t). \tag{13}$$

The natural frequencies of the pre-stressed nanobeam obtain as following:

$$\omega_n = (n\pi/l)[(n\pi/l)^2(EI - \mu\hat{N}) - \hat{N}]^{1/2}(M_n\lambda_n)^{-1/2}, \tag{14}$$

where

$$M_n = I_0 + I_2(n\pi/l)^2, \tag{15}$$

$$\omega_n = 1 + \mu(n\pi/l)^2. \tag{16}$$

5. Numerical results

In this section, numerical results are given for analytical solutions given in previous section. Due to nondimensionalization, only the following material and geometrical properties are required in computations: $E = 30\times10^6$, $L/h = 20$ and $\rho = 1$. Table shows that the nondimensional natural frequency of nanobeam decreases with increase of nonlocal parameter. The present results are in good agreement with the earlier study. It can be seen that the fundamental frequency decreases by increasing the axial compressive loading. Moreover, the natural frequency is equal to zero when the axial force is equal to buckling load [3].

TABLE

Effect of the pre-load on the nondimensional natural frequency $\Omega = \omega_n L^2\ (\rho A/EI)^{1/2}$ of simply supported nanobeams.

μ	Ref. [3]	present paper for N/N_{cr}				
	$N=0$	0	0.25	0.5	0.95	1
3	8.6693	8.6604	6.2137	5.0735	1.6044	0
2	9.0195	9.0102	7.2299	5.9032	1.8667	0
1	9.4159	9.4062	8.1460	6.6512	2.1033	0
0	9.8696	9.8595	8.5386	6.9717	2.2046	0

6. Discussion

A generalized nonlocal Euler beam theory is used to study free vibration of simply supported nanobeams. Nonlocal constitutive equations of Eringen are used in the formulations. Effect of nonlocality and axial pre load are investigated in detail for each considered problem. It was shown that increase of the axial compressive loading leads to decrease of the fundamental frequency of nanobeams. Present formulation can be extended to other beam theories, as well as other boundary conditions.

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