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2D Cellular Automata with an Image Processing Application

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This paper investigates the theoretical aspects of two-dimensional linear cellular automata with image applications. We consider geometrical and visual aspects of patterns generated by cellular automata evolution. The present work focuses on the theory of two-dimensional linear cellular automata with respect to uniform periodic and adiabatic boundary cellular automata conditions. Multiple copies of any arbitrary image corresponding to cellular automata find so many applications in real life situation e.g. textile design, DNA genetics research, etc.

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1. Introduction

Cellular automata (CAs for brevity) introduced by Ulam and von Neumann [1] in the early 1950's, have been systematically studied by Hedlund from purely mathematical point of view. One-dimensional CA has been investigated to a large extent. However, little interest has been given to two-dimensional cellular automata (2DCA). Von Neumann [1] showed that a cellular automaton can be universal. Due to its complexity, von Neumann rules were never implemented on a computer. In the beginning of the eighties, Wolfram [2] has studied in much detail a family of simple one-dimensional (1D) CA rules and showed that even these simplest rules are capable of emulating complex behavior. Some basic and precise mathematical models using matrix algebra over the binary field which characterize the behavior of 2D nearest neighborhood linear CA with null and periodic boundary conditions have been seen in the literature [3, 4]. CA has received remarkable attention in the last few decades [4–6]. Due to its structure CA has given the opportunity to model and understand many behaviors in nature easier. Most of the work for CA is done for one--dimensional case. The set of papers [4, 7] deals with the behavior of the uniform 2D CA over binary fields.

In this paper, we study the theory of 2-dimensional uniform periodic and adiabatic boundary CA (2D PCA, ACA) of the all linear rules (e.g. von Neumann, Moore neighborhood and the others) and applications of image processing for self replicating patterns (see Figs. 1–8). We present some illustrative examples and figures to explain the method in detail. Using the rule matrices obtained in this work, the present paper contributes further to the algebraic structure of these CA and relates its applications studied by different authors previously (i.e. [2, 8]). The linear combination of the neighboring cells on which each cell value is dependent is called the rule number of the 2D CA over the field Z_2.

Regarding the neighborhood of the extreme cells, there exist four different approaches:

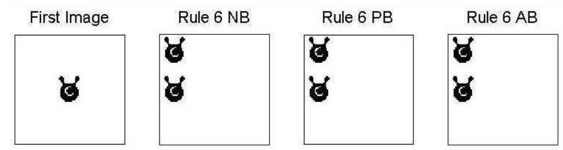

Fig. 1. An application of rule 8 with null (NB), periodic (PB) and adiabatic (AB) boundary, respectively, after 32 iterations of the first image.

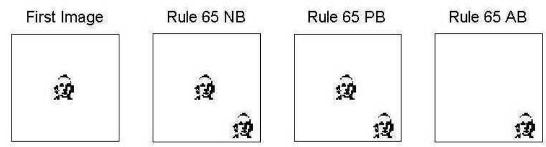

Fig. 2. An application of rule 65 after 32 iterations of the first image.

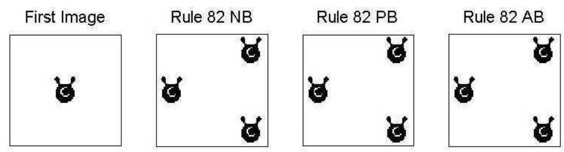

Fig. 3. An application of rule 82 after 32 iterations of the first image.

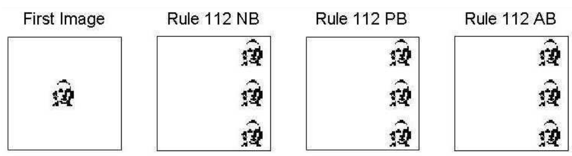

Fig. 4. Application of rule 112 after 32 iterations of the first image.

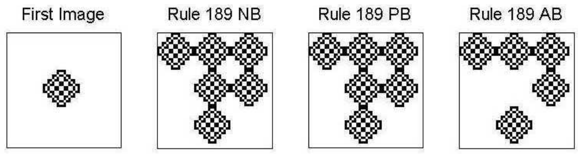

Fig. 5. An application of rule 189 after 16 iterations of the first image.

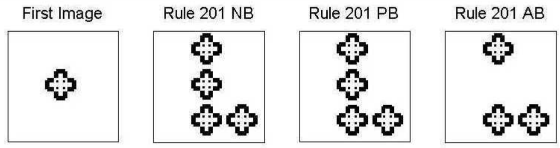

Fig. 6. Application of rule 201 after 16 iterations of the first image.

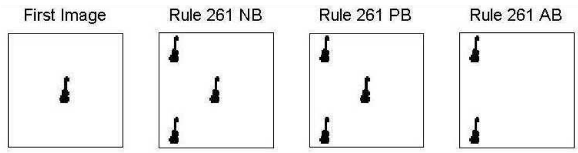

Fig. 7. Application of rule 261 after 32 iterations of the first image.

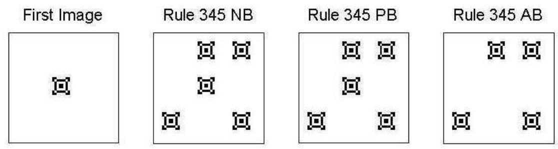

Fig. 8. Application of rule 345 after 16 iterations of the first image.

- A null boundary (NB) CA is the one whose extreme cells are connected to 0-state.
- A periodic boundary (PB) CA is the one whose extreme cells are adjacent to each other.
- An adiabatic boundary (AB) CA is duplicating the value of the cell in an extra virtual neighbor.

2. Rule matrices with primary rules

The auxiliary matrices T_1 and T_2 are as follows:

$$T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Lemma 1 [3]. *The representation of the next state of all primary rules (*$1, 2, 4, 8, 16, 32, 64, 128,$ *and* 256*) under the null boundary condition can be given by using the auxiliary matrices* T_1 *and* T_2 *defined above in the following way:*

Rule $1N$: $[X_{t+1}] = [X_t]$,

Rule $2N$: $[X_{t+1}] = [X_t][T_2]$,

Rule $4N$: $[X_{t+1}] = [T_1][X_t][T_2]$,

Rule $8N$: $[X_{t+1}] = [T_1][X_t]$,

Rule $16N$: $[X_{t+1}] = [T_1][X_t][T_1]$,

Rule $32N$: $[X_{t+1}] = [X_t][T_1]$,

Rule $64N$: $[X_{t+1}] = [T_2][X_t][T_1]$,

Rule $128N$: $[X_{t+1}] = [T_2][X_t]$,

Rule $256N$: $[X_{t+1}] = [T_2][X_t][T_2]$.

2.1. Rule matrices under periodic boundary

The matrices T_{1p} and T_{2p} are as follows:

$$T_{1p} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_{2p} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Theorem 2. *(Periodic case) The matrix for any periodic boundary CA rule (PB) can be represented as*

$(T_{PB})_{mn\times mn} =$

$$\begin{pmatrix} A_p & B_p & O & O & \dots & \dots & O & D_p \\ C_p & A_p & B_p & O & \dots & \dots & O & O \\ O & C_p & A_p & B_p & O & \dots & O & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & \dots & O & C_p & A_p & B_p & O \\ O & O & \dots & \dots & O & C_p & A_p & B_p \\ E_p & O & \dots & \dots & O & O & C_p & A_p \end{pmatrix},$$

where A_p, B_p, C_p, D_p, E_p are one of the following matrices of the order of $n\times n$: 0, $\boldsymbol{I}$, T_{1p}, T_{2p}, $\boldsymbol{I}+T_{1p}$, $\boldsymbol{I}+T_{2p}$, $T_{1p}+T_{2p}$ and $\boldsymbol{I}+T_{1p}+T_{2p}$.

Lemma 3. *The next state of all primary rules $(1, 2, 4, 8, 16, 32, 64, 128, 256)$ of 2D periodic cellular automaton over Z_2 can be represented as follows:*

Rule $1P$: $[X_{t+1}] = [X_t]$,

Rule $2P$: $[X_{t+1}] = [X_t][T_{2p}]$,

Rule $4P$: $[X_{t+1}] = [T_{1p}][X_t][T_{2p}]$,

Rule $8P$: $[X_{t+1}] = [T_{1p}][X_t]$,

Rule $16P$: $[X_{t+1}] = [T_{1p}][X_t][T_{1p}]$,

Rule $32P$: $[X_{t+1}] = [X_t][T_{1p}]$,

Rule $64P$: $[X_{t+1}] = [T_{2p}][X_t][T_{1p}]$,

Rule $128P$: $[X_{t+1}] = [T_{2p}][X_t]$,

Rule $256P$: $[X_{t+1}] = [T_{2p}][X_t][T_{2p}]$,

2.2. *Rule matrices under adiabatic boundary*

The auxiliary matrices T_{1a} and T_{2a} are defined as follows:

$$T_{1a} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{2a} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence we get the following general rule matrix result for the adiabatic case as a theorem.

Theorem 4. *(Adiabatic case) The rule matrix for any adiabatic boundary CA rule (AB) can be represented as*

$(T_{AB})_{mn\times mn} =$

$$\begin{pmatrix} A_a & B_a & O & O & \dots & \dots & O & O \\ C_a & A_a & B_a & O & \dots & \dots & O & O \\ O & C_a & A_a & B_a & O & \dots & O & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & \dots & O & C_a & A_a & B_a & O \\ O & O & \dots & \dots & O & C_a & A_a & B_a \\ O & O & \dots & \dots & O & O & C_a & A_a \end{pmatrix},$$

where A_a, B_a, C_a are one of the following matrices of the order of $n\times n$: 0, $\boldsymbol{I}$ T_{1a}, T_{2a} $\boldsymbol{I}+T_{1a}$, $\boldsymbol{I}+T_{2a}$, $T_{1a}+T_{2a}$ and $\boldsymbol{I}+T_{1a}+T_{2a}$.

Lemma 5. *The next state of all primary rules $(1, 2, 4, 8, 16, 32, 64, 128, 256)$ of 2D adiabatic cellular automaton with Z_2 can be represented as follows:*

Rule $1AB$: $[X_{t+1}] = [X_t]$,

Rule $2AB$: $[X_{t+1}] = [X_t][T_{1a}]^t$,

Rule $4AB$: $[X_{t+1}] = [T_{1a}][X_t][T_{1a}]^t$,

Rule $8AB$: $[X_{t+1}] = [T_{1a}][X_t]$,

Rule $16AB$: $[X_{t+1}] = [T_{1a}][X_t][T_{2a}]^t$,

Rule $32AB$: $[X_{t+1}] = [X_t][T_{2a}]^t$,

Rule $64AB$: $[X_{t+1}] = [T_{2a}][X_t][T_{2a}]^t$,

Rule $128AB$: $[X_{t+1}] = [T_{2a}][X_t]$,

Rule $256AB$: $[X_{t+1}] = [T_{2a}][X_t][T_{1a}]^t$.

3. Application of image processing

Self replicating pattern generation is one of the most interesting topic and research area in nonlinear science. A *motif* is considered as a basic sub-pattern. Pattern generation is the process of transforming copies of the motif about the array (1D), plane (2D) or space (3D) in order to create the whole repeating pattern with no overlaps and blank [1, 2]. These patterns have some mathematical properties which make generating algorithm possible. A cellular automaton is a good candidate algorithmic approach used for pattern generation.

Creating algorithmic approach for generating self replicating patterns of digital images (motif as in first image) is important and sometimes difficult task. Meanwhile many researchers face with many challenges in building and developing tiling algorithms such as providing simple and applicable algorithm to describe high complex patterns model. Growth from simple motif in 2D CAs can produce self replicating patterns with complicated boundaries (null, periodic, adiabatic and reflexive), characterized by a variety of growth dimensions. The approach given here leads to an accurate algorithm for generating different patterns.

In this paper we use the CAs with all the nearest neighborhoods to generate self replicate patterns of digital images. For applying 2D null, periodic and adiabatic CA linear rules in image processing, we take a binary matrix of size (100×100) due to computational limitations. We map each element of the matrix to a unique pixel on the screen (writing new MATLAB codes) and we color a pixel white for 0, black for 1 for the matrix elements. Then we take another image (as a motif) whose size is less than (30×30) for which patterns are to be generated and put it in the center of the binary matrix. This is the way how the image is drawn within an area of (100×100) pixels. It is observed from the figures that the self replicating patterns can be generated only when number of repetition is 2^n where $n = 4$. A neighborhood function that specifies which of the adjacent cells affects its state also determines how many copies will be obtained from the self-replicating process. In the two dimensional and eight neighborhoods case, this should be at most eight copies of the original image itself. This situation brings also some limitation over the matrix size of the images to be replicated. The matrix size of the original images should lower 30 percent of the display matrix in all directions. If the first image exceeds 30 percentage of the length of row or column of the display matrix self replication pattern when the iteration number t reaches to 16 does not occur. Also behaviors for different boundaries produce different shapes when $t = 16$. Hence we have a classification device and tables up to self replicating pattern number and for the case seed image less than 30 percentage of the display matrix (see Figs. 1–8), these will be presented in the next studies.

4. Conclusion

In this paper we discuss the theory 2-dimensional, uniform periodic and adiabatic boundary CAs of linear rules and applications of image processing. It is seen that CAs theory can be applied successfully in self replicating patterns of image processing. The some characterization and applications on a 2D finite CA by using matrix algebra built on Z_3 are planned to next studies. However after making use of the matrix representation of 2D CA, it will be provided an algorithm to obtain the number of Garden of Eden configurations for the 2D CA defined by some rules. Some other interesting results and further connections on this direction wait to be explored in 2D CA's, see Refs. [9–16].

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References

[1] J. von Neumann, in: *The Theory of Self-Reproducing Automata*, Ed. A.W. Burks, Univ. of Illinois Press, Urbana 1966.

[2] S. Wolfram, *Rev. Mod. Phys.* **55**, 601 (1983).

[3] P.P. Choudhury, B.K. Nayak, S. Sahoo, S.P. Rath, *Theory and Applications of Two-Dimensional, Null-Boundary, Nine-Neighborhood, Cellular Automata Linear Rules*, Tech. Report No. ASD/2005/4, 13 May 2005, arXiv/0804.2346.

[4] K. Dihidar, P.P. Choudhury, *Inf. Sci.* **165**, 91 (2004).

[5] H. Akın, *Appl. Math. Comput.* **170**, 339 (2005).

[6] H. Akın, I. Siap, *Inf. Proc. Lett.* **103**, 24 (2007).

[7] P. Chattopadhyay, P.P. Choudhury, K. Dihidar, *Comput. Math. Appl.* **38**, 207 (1999).

[8] P.P. Choudhury, S. Sahoo, S.S. Hassan, S. Basu, D. Ghosh, D. Kar, Ab. Ghosh, Av. Ghosh, A.K. Ghosh, *Int. J. Comp. Cogn.* **8**, 50 (2010).

[9] H. Akın, I. Siap, S. Uguz, *AIP Conf. Proc.* **1309**, 16 (2010).

[10] H.H. Chou, J.A. Reggia, *Physica D* **110**, 252 (1997).

[11] J. Gravner, G. Gliner, M. Pelfrey, *Physica D* **240**, 1460 (2011).

[12] I. Siap, H. Akın, S. Uguz, *Comput. Math. Appl.* **62**, 4161 (2011).

[13] I. Siap, H. Akın, F. Sah, *J. Frank. Inst.* **348**, 1258 (2011).

[14] I. Siap, H. Akın, F. Sah, *Inf. Sci.* **180**, 3562 (2010).

[15] S. Uguz, H. Akın, I. Siap, *Int. J. Bifur. Chaos* **23**, 1350101 (2013).

[16] S. Uguz, H. Akın, I. Siap, *Int. J. Bifur. Chaos* **24**, (2014), to be published.