Infinite Body Centered Cubic Network of Identical Resistors

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We express the equivalent resistance between the origin $(0,0,0)$ and any other lattice site (n_1, n_2, n_3) in an infinite body centered cubic network consisting of identical resistors each of resistance R rationally in terms of known values b_0 and π. The equivalent resistance is then calculated. For large separations two asymptotic formulae for the resistance are presented and some numerical results with analysis are given.

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1. Introduction

The lattice Green function (LGF) is a basic physical term. Many quantities of interest in solid-state physics can be expressed in terms of it. For example, statistical model of ferromagnetism such as Ising model [1], Heisenberg model [2], spherical model [3], lattice dynamics [4], random walk theory [5, 6], and band structure [7, 8]. In Economou's book [9] one can find an excellent introduction to the LGF, where a review of the LGF of the so-called tight-binding Hamiltonian (TBH) used for describing the electronic band structures of crystal lattices is presented. The LGF defined in this paper is related to the GF of the TBH. Many efforts have been paid on studying the LGF of cubic lattices [10–25].

The LGF for the bcc lattice has been expressed as a sum of simple integrals of the complete elliptic integral of the first kind [10], Morita and Horiguci [11] presented formulae which are convenient for the evaluation of the LGF for the face centered cubic (fcc), bcc and rectangular lattices. These formulae involve the complete elliptic integral of the first kind with complex modulus. Morita [12] derived a recurrence relation, which gives the values of the LGF along the diagonal direction from a couple of the elliptic integrals of the first and second kind for the square lattice with discussions of how to apply the result to the bcc lattice. Finally, Glasser and Boersma [23] expressed the values of the LGF of the bcc lattice rationally. One can find more works in these works and references within therein.

The calculation of the equivalent resistance in infinite networks is a basic problem in the electric circuit theory. It is of extreme interest for physicists and electrical engineers.

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There are mainly three approaches to solve such a problem.

The first approach is a superposition of current distribution which has been used to calculate the effective resistance between adjacent sites on infinite networks [26–28].

The second one employs mapping between random walk and resistor-network problems as was carried out by Jeng [29]. In his method he calculated the effective resistance between any two sites in an infinite two-dimensional square lattice of unit resistors.

The third educational important method based on the LGF of the lattices has been used in calculating the equivalent resistance [30–38]. This method has been applied to both perfect and perturbed square, simple cubic (sc) networks and recently to the fcc network.

The present work is organized as follows: In Sect. 2, we briefly introduce the basic formulae of interest for the LGF of the bcc network. In Sect. 3, an application to the LGF of the bcc network is applied to express the equivalent resistance between the origin and the lattice site (n_1, n_2, n_3) in the infinite bcc network rationally in terms of some constants, and the asymptotic behavior for the resistance is also investigated as the separation between the two sites goes to infinity. Finally, we close this paper (Sect. 4) with a discussion of the results obtained.

2. Basic definitions and preliminaries

The LGF for the bcc lattice appears in many areas of physics (e.g., Ising model [1, 39, 40] Heisenberg model [2, 41, 42], and spherical models [43–45]) and it is defined as [13, 23]:

$$B(E; n_1, n_2, n_3) = \frac{1}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\frac{\cos(n_1 u)\cos(n_2 v)\cos(n_3 w)}{E-\cos u\cos v\cos w}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w, \quad (1)$$

where $E \geq 1$, n_1, n_2 and n_3 are either all even or all odd integers.

The LGF for the bcc lattice at the site $(0,0,0)$ which represents the origin of the lattice for $E = 1$ (i.e., $B(1;0,0,0) = b_0$) was of so interests in physics and it was carried out first by Van Peijpe [46] and later on by Watson [47] . They showed that

$$b_0 = B(1;0,0,0) = \frac{4}{\pi^2}\left[K\left(\frac{1}{\sqrt{2}}\right)\right]^2 = \frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^3} = 1.3932039297, \tag{2}$$

where K is the complete elliptic integral of the first kind, and Γ is the gamma function.

In a recent work the LGF for the infinite bcc lattice has been expressed rationally as [23]:

$$B(1;n_1,n_2,n_3) = \sigma_1 b_0 + \frac{\sigma_2}{\pi^2 b_0} + \sigma_3, \tag{3}$$

where, σ_1, σ_2 and σ_3 are rational numbers.

3. Application: evaluation of the resistance $R(n_1,n_2,n_3)$ in an infinite bcc network

The aim of this section is to express the equivalent resistance between the origin $(0,0,0)$ and the lattice site (n_1,n_2,n_3) in the infinite bcc network which is consisting of identical resistors rationally in terms of b_0 and π.

First of all, it has been shown that for a 3D infinite network consisting of identical resistors each of resistance R, the equivalent resistance between the origin and any other lattice site is [30]:

$$R(\boldsymbol{r}) = 2[G(\boldsymbol{0}) - G(\boldsymbol{r})], \tag{4}$$

where $\boldsymbol{r}$ is the position vector of the lattices point, and for a d-dimensional lattice it has the following form:

$$\boldsymbol{r} = n_1\boldsymbol{a}_1 + n_2\boldsymbol{a}_2 + \ldots + n_d\boldsymbol{a}_d, \tag{5}$$

where n_1, $n_2, \ldots, n_d$ are integers, and $\boldsymbol{a}_1$, $\boldsymbol{a}_2, \ldots, \boldsymbol{a}_d$ are independent primitive translation vectors.

Also, the equivalent resistance between the origin and any other lattice site can be expressed in an integral form as [30]:

$$R(n_1,n_2,\ldots,n_d) = R\int_{-\pi}^{\pi}\frac{\mathrm{d}x_1}{2\pi}\cdots\int_{-\pi}^{\pi}\frac{\mathrm{d}x_d}{2\pi} \times \frac{1-\exp(\mathrm{i}n_1x_1 + \mathrm{i}n_2x_2 + \ldots + \mathrm{i}n_dx_d)}{\sum_{i=1}^{d}(1-\cos x_i)}. \tag{6}$$

On the other hand, the LGF for a 3D hypercube reads as [30]:

$$G(n_1,n_2,\ldots,n_d) = \int_{-\pi}^{\pi}\frac{\mathrm{d}x_1}{2\pi}\cdots\int_{-\pi}^{\pi}\frac{\mathrm{d}x_d}{2\pi} \times \frac{\exp(\mathrm{i}n_1x_1 + \mathrm{i}n_2x_2 + \ldots + \mathrm{i}n_dx_d)}{2\sum_{i=1}^{d}(1-\cos x_i)}. \tag{7}$$

For cubic lattices, $d = 3$. Then substituting $d = 3$ into Eqs. (6) and (7) and comparing them with Eq. (4) one gets

$$R(n_1,n_2,n_3) = R\left[b_0 - B(1;n_1,n_2,n_3)\right]. \tag{8}$$

Now making use of Eq. (3) and Eq. (8) one yields

$$\frac{R(n_1,n_2,n_3)}{R} = r_1 b_0 + \frac{r_2}{\pi^2 b_0} + r_3, \tag{9}$$

where $r_1 = 1 - \sigma_1$, $r_2 = -\sigma_2$ and $r_3 = -\sigma_3$ are rational numbers. These rational numbers, for the sites from $(0,0,0)$ to $(8,8,8)$, can be gathered from ([13], appendix A). In Table below we present these rational numbers.

Based on the recurrence formula presented in ([13], Eq. (5.8)), we have calculated additional rational values for the sites from $(9,1,1)$ to $(10,0,0)$ and arranged them in Table below.

Since the LGF is an even function (i.e., $B(1;n_1,n_2,n_3) = B(1;-n_1,-n_2,-n_3)$) and due to the fact that the infinite bcc network is pure and symmetric, then as a result $R(n_1,n_2,n_3) = R(-n_1,-n_2,-n_3)$.

Finally, it is interesting to study the asymptotic behavior of the equivalent resistance for large separation between the origin $(0,0,0)$ and any arbitrary lattice site (n_1,n_2,n_3).

The asymptotic form of $B(1;0,0,n_3)$, as $n_3 \to \infty$, is given by [13]:

$$B(1,0,0,2n_3) \to \frac{1}{\pi n_3}\left(1 - \frac{1}{8n_3^2} + \frac{1}{128n_3^4} - \frac{173}{1024n_3^6}\right). \tag{10}$$

While, for large value of $|n| = \sqrt{n_1^2+n_2^2+n_3^2}$ it has been shown that [13] $B(1;n_2,n_2,n_3)$ has the following asymptotic formula:

$$B(1;n_1,n_2,n_3) \approx \frac{2}{\pi}|n|^{-1}\left[1 - \frac{9}{8}|n|^{-2} + \frac{5}{8}|n|^{-6}\left(n_1^4+n_2^4+n_3^4\right) + \frac{15}{4}|n|^{-6}\left(n_1^2n_2^2+n_2^2n_3^2+n_3^2n_1^2\right)\right]. \tag{11}$$

Inserting Eq. (10) and Eq. (11) into Eq. (8), one gets the following two equations:

$$\frac{R(0,0,2n_3)}{R} \to b_0 - \frac{1}{\pi n_3}\left(1 - \frac{1}{8n_3^2} + \frac{1}{128n_3^4} - \frac{173}{1024n_3^6}\right), \tag{12}$$

$$\frac{R(n_1,n_2,n_3)}{R} \to b_0 - \frac{2}{\pi}|n|^{-1}\left[1 - \frac{9}{8}|n|^{-2} + \frac{5}{8}|n|^{-6} \times \left(n_1^4+n_2^4+n_3^4\right) - \frac{15}{4}|n|^{-6}(n_1^2n_2^2+n_2^2n_3^2+n_1^2n_3^2)\right]. \tag{13}$$

The last asymptotic formula agrees with Eq. (12) for $n_1 = 0$, $n_2 = 0$ and for $n_3 = 2n_3$. In addition, the above two asymptotic formulae can be used to check the results obtained in Table below. For example

$$\frac{R(8,0,0)}{R} \approx 1.31425, \quad \frac{R(8,8,6)}{R} \approx 1.34413, \quad \frac{R(8,8,8)}{R} \approx 1.34778, \quad \frac{R(9,9,9)}{R} \approx 1.35273, \quad \frac{R(10,0,0)}{R} \approx 1.32986. \tag{14}$$

From the above two asymptotic formulae, one can see that as $n_3 \to \infty$, or as $|n| \to \infty$ then the resistance goes to a finite value (i.e., goes to b_0).

TABLE

Values for selected rational numbers r_1, r_2, r_3 and $R(n_1, n_2, n_3)$ for sites $(0,0,0)$ to $(10,0,0)$.

(n_1, n_2, n_3)	r_1	r_2	r_3	$R(n_1, n_2, n_3)/R$
000	0	0	0	0
111	0	0	1	1.0000
002	1	−4	0	1.1023
022	0	16	0	1.16360
222	−3	−36	8	1.20228
113	2	−8	−1	1.20461
133	−4	80	1	1.24521
333	−18	−504	63	1.26877
004	8/9	0	0	1.23840
024	25/9	−36	0	1.25190
224	104/9	16	−16	1.26285
044	−112/9	256	0	1.28003
244	−407/9	444	32	1.28626
444	−360/9	−5376	448	1.30059
115	−2/9	8	1	1.27220
135	120/9	−224	−1	1.28558
335	810/9	920	−191	1.29564
155	−652/9	1392	1	1.30374
355	−4266/9	1192	575	1.30990
555	7650/9	−48840	2369	1.31934
006	1	−36/25	0	1.28848
026	−16/9	1296/25	0	1.29327
226	−155/9	444/25	24	1.29753
046	409/9	−21316/25	0	1.30487
246	1112/9	−42224/25	−48	1.30795
446	5481/9	379804/25	−1952	1.31569
066	−288	138384/25	0	1.31802
266	−5147/9	249596/25	72	1.31998
466	−165600/36	−563056/25	8048	1.32510
666	169317/9	−9083844/25	216	1.33175
117	20/9	−272/25	−1	1.30476
137	−808/36	10856/25	1	1.31055
337	−7616/36	−29888/25	383	1.31541
157	9480/36	−125320/25	−1	1.31962
357	12620/9	−275440/25	−1151	1.32317
557	20584/9	4757600/25	−17025	1.32903
177	−57824/36	769376/25	1	1.32912
377	−207128/36	1964312/25	2303	1.33153
577	−1396840/36	−2848376/5	95489	1.33566
777	9331056/36	−42996912/25	−236033	1.34058
008	1664/1764	0	0	1.31422
028	8164/1764	−1764/25	0	1.31641
228	48256/1764	−1648/25	−32	1.31845
048	−183136/1764	50176/25	0	1.32213
448	−3604288/1764	−737024/25	4992	1.32815
068	2029540/1764	−550564/25	0	1.32949
268	3543104/1764	−928496/25	−96	1.33070
468	28134948/1764	−664004/25	−20288	1.33398
668	−38421504/1764	49955984/25	−114976	1.33849
088	−12686080/1764	3444736/25	0	1.33687
288	−19609372/1764	5280412/25	128	1.33772
488	−115817696/1764	13932544/25	50944	1.34006
688	−436242204/1764	−216804644/25	975232	1.34340
888	5048578944/1764	165654528/25	−4469248	1.34721
119	−148/441	272/25	1	1.32369
139	17230/441	−17912/25	−1	1.32666
339	19368/49	30816/25	−639	1.3293
159	−306598/441	66616/5	1	1.33168
359	−1425500/441	177776/5	1919	1.33381
559	−6683968/441	−475264	55681	1.33751
179	41833/63	−3179392/25	−1	1.33754
379	1200158/63	−7803464/25	−3839	1.33914
579	10571410/63	4255912/5	−295681	1.34199
779	−678572	435693424/25	−322047	1.34551
199	−17834440/441	19368352/25	1	1.3433
399	−4533062/49	42106968/25	6399	1.34446
599	−335991178/441	10935752/5	902401	1.34657
799	−2907724/7	−2645823088/25	8275455	1.34938
999	1270018116/49	7998622128/25	−59378175	1.35244
0010	1	−196/225	0	1.32985

4. Results and discussion

In this work we have expressed the equivalent resistance between the origin $(0,0,0)$ and any arbitrary lattice site (n_1, n_2, n_3) in an infinite bcc network consisting of identical resistors each of resistance R rationally in terms of the two known values b_0 and π. The rational number r_1, r_2, and r_3 presented in Eq. (14) were calculated using some recurrence formulae. In Figs. 1 and 2 the equivalent resistance is plotted against the lattice site.

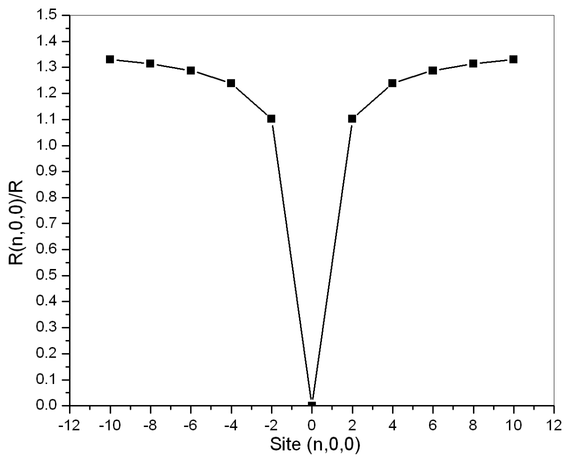

Fig. 1. Resistance between the origin $(0,0,0)$ and the site $(n,0,0)$ along [100] direction for bcc network.

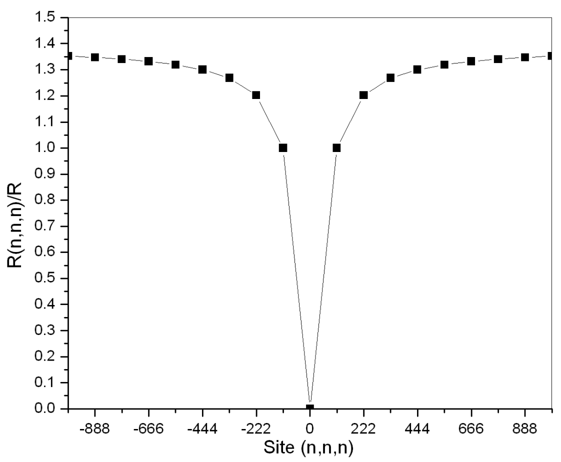

Fig. 2. Resistance between the origin $(0,0,0)$ and the site (n,n,n) along [111] direction for bcc network.

Figure 1 shows the resistance in an infinite bcc lattice against the site (n_1, n_2, n_3) along the [100] direction. From this figure it is clear that the resistance is symmetric.

Figure 2 shows the resistance in an infinite bcc lattice against the site (n_1, n_2, n_3) along the [111] direction. From this figure it is clear that the resistance is symmetric.

The above figures indicate that as the separation between the origin and the lattice site (n_1, n_2, n_3) increases, the equivalent resistance approaches a finite value (i.e., $b_0 = 1.3932039297$) as explained above.

It is worth mentioning that for other cubic networks (i.e., sc and face centered cubic, fcc) the resistance approaches a finite value for large separation between the two sites [30, 33, 37], whereas for the infinite square lattice it goes to infinity [30, 34].

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