Study of Condensed Matter by Neutron Scattering and Complementary Methods

Impulse and Resonant Response in Systems Showing Secular and Leaky Excitations

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A model consisting of a string embedded in an elastic medium and terminated by a harmonic oscillator has been studied in the frequency and time domains to elucidate the physical effects of supersonic and subsonic leaky waves as well as that of true surface waves. A supersonic leaky wave manifests itself by a resonant maximum of the local density of states within the band of bulk waves and by an anomalous dispersion of the real part of the frequency dependent response function. The time domain impulse response then contains mainly resonant contribution from the poles of the response function in analogy to ordinary resonances. True surface waves show generally analogous behaviour. Here, however, the phenomenon is governed by dissipation mechanisms different from the radiation into the bulk. An important difference is that the impulse response contains equilibrated contributions due to the poles and due to the stop frequency gap in the case of true surface waves. The main manifestation of a subsonic leaky wave, i.e. a surface resonance with the frequency situated in the stop gap, is a sharp peak of the real part of the frequency-dependent response function just at the bottom of the bulk waves band. This is in certain analogy with a large reactive power in electric circuits. A strong destructive interference of the resonant part of the impulse response with the part due to the gap makes the time domain response fast attenuated.

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1. Introduction

Defects of spatial periodicity in solids may give rise to localized excitations in addition to the Bloch waves propagating in the bulk of the material. No simple rules exist to predict if and in what number such excitations will really occur given the equations of motion for the bulk and those describing the boundary conditions imposed by the physical properties of the defect [1]. Generally, the excitations with frequencies outside the bands of bulk waves show infinite life times because no mechanism allows their energy to be radiated into the bulk. In contrast, the radiation by the bulk waves is usually responsible for an evanescent character of the defect excitations with frequencies situated within the bulk bands. In cases where the defect is a surface the infinitely lived excitations propagate along the surface and, therefore, are called true surface waves (TSW). In turn, the excitations of finite life times are termed pseudo surface waves (PSW), surfaces resonances (in some analogy with the resonances that are distinguished from particles in the high energy physics) or surface leaky waves to account for the "leakage" of energy into the bulk.

There are, however, examples known of the true surface waves within bulk bands. Such effects reflect a decoupling of the surface excitation from the bulk waves [2, 3]. Interestingly enough, it may also happen that the real part of the frequency of a surface leaky wave lie outside the bulk band, even though the imaginary part of the frequency, i.e. the inverse of the life time, may be different from zero. If excitations of this kind occur below the bands of the bulk acoustic waves they are called subsonic surfaces resonances or subsonic leaky waves. A number of examples of this type have been found on the inner surface of cylindrical cavity in an elastic medium [4]. It might seem that the limit of the imaginary part of frequency of a subsonic surface leaky wave tending to zero corresponds to a true surface wave. This is, however, not the case, because the amplitude of a leaky wave increases with the depth into the bulk up to the wave front, whereas the amplitude of a true surface wave decreases that is equivalent to it being a localized excitation.

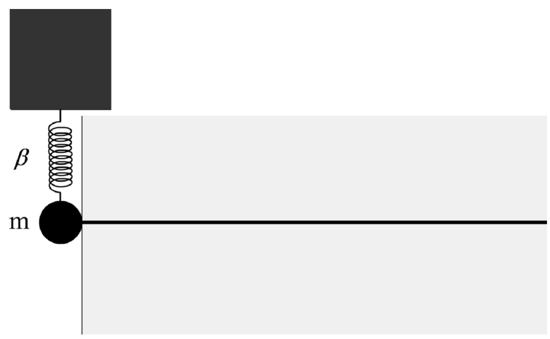

Fig. 1. Model: string of sound velocity c embedded in an elastic medium producing a stop frequency gap below frequency ω_0 terminated by a mass m in a harmonic potential with force constant β. In modeling of true surface waves additional damping of the end oscillator is considered.

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Now it is interesting to study the behavior of systems showing all the kinds of excitations under external perturbation both in the frequency and time domain. The model treated in this note is a semi-infinite string embedded in an elastic medium. It obeys, therefore, the Klein–Gordon equation of motion [5]. A point-like harmonic oscillator constitutes the termination of the string. The system is schematically shown in Fig. 1. The band of the bulk waves in the string is limited from below so that there is a stop gap at sufficiently low frequencies. The equations of motion are equivalent to those of a 2D membrane displacing out of plane and terminated by a straight edge of a non-zero linear density placed in an external harmonic potential. The width of the gap then is a function of the wave vector parallel to the edge [6]. Therefore, the solution of the present model gives at the same time the solutions of the dynamics of a more general two-dimensional system.

2. Response in frequency domain

The equation of motion of the string is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial y^2} - \omega_0^2 u, \quad y > 0, \tag{1}$$

where ω_0 is the lower limit of the bulk band and $c = (T/\rho)^{1/2}$ is the short-wavelength limit of the phase sound velocity in the string. The string is supposed to show a linear homogeneous mass density ρ and to be subjected to a tension T. ny solution of Eq. (1) can be composed of harmonic waves of the form

$$u(y,t) = u_0 \exp(-\mathrm{i}\omega t + \mathrm{i}ky), \tag{2}$$

where the wave vector is

$$k(\omega) = \pm \frac{1}{c}\sqrt{\omega^2 - \omega_0^2}, \quad \text{if} \quad \omega > \omega_0,$$

$$k(\omega) = \mathrm{i}\frac{1}{c}\sqrt{\omega_0^2 - \omega^2}, \quad \text{if} \quad \omega < \omega_0. \tag{3}$$

In the case of an imaginary wave vector only one sign should be considered for a real frequency ω to ensure a decrease in amplitude with the depth y into the bulk.

The boundary condition at $y = 0$ is the Newton second law for the oscillator attached to the string

$$\frac{\partial^2 u}{\partial t^2} = -\beta^2 u + \frac{T}{m}\frac{\partial u}{\partial y}\bigg|_{y=0} + f(t), \quad y = 0. \tag{4}$$

Here m is the mass and β is the eigenfrequency of the oscillator while $f(t)$ is an external perturbation counted in the units of force per mass.

The problem is particularly simple in the frequency domain, i.e. with the oscillating perturbation

$$f(t) = f_0(\omega)\exp(-\mathrm{i}\omega t). \tag{5}$$

After all the transient effects die out the motion of the end mass is also oscillatory

$$u(0,t) = u_0(\omega)\exp(-\mathrm{i}\omega t) \tag{6}$$

and is related with the applied perturbation by the Green function $G(\omega)$ also called response function

$$u_0(\omega) = G(\omega) f_0(\omega), \tag{7}$$

where

$$G(\omega) = \frac{1}{\beta^2 - \omega^2 - \mathrm{i}\gamma(\omega^2 - \omega_0^2)^{1/2}} \quad \text{if} \quad \omega > \omega_0,$$

$$G(\omega) = \frac{1}{\beta^2 - \omega^2 + \gamma(\omega_0^2 - \omega^2)^{1/2}} \quad \text{if} \quad \omega < \omega_0, \tag{8}$$

where $\gamma = T/mc$.

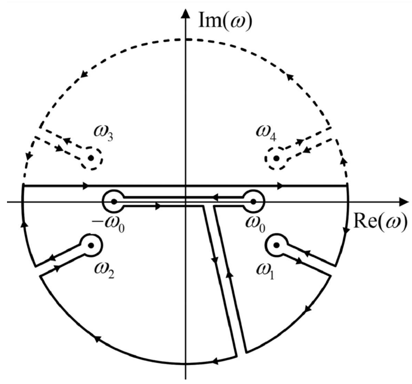

Fig. 2. Typical analytical structure of the Green function of Eq. (8) and paths of contour integration for $t < 0$ (dashed line), $t > 0$ (solid line).

When analytically continued on the complex plane ω the Green function shows four poles and a cut that can be put between $-\omega_0$ and ω_0 as shown in Fig. 2. The position of the poles depends on the parameters of the model. Of course the poles on the real axis correspond to secular excitations (true surface waves) and those with negative imaginary parts of frequency to finite-lived resonance excitations. A positive imaginary part of frequency is not physically acceptable because it would imply an infinite response in the long time limit. Given the complex frequency of a pole one obtains the corresponding wave vector of the excitation from the dispersion relation according to Eq. (3).

3. Response in the time domain: impulse response

The Green function of Eqs. (8) is the Fourier transform of the response $u_{\mathrm{imp}}(t)$ of the system to a delta-like force $f(t) = \delta(t)$ applied to the mass m as given in Eq. (4). Thus, the kind of response, legitimately called impulse response [7], is given by the integral $u_{\mathrm{imp}}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}\omega$ which can be calculated by contour integration on the plane of complex ω. For $t < 0$ the contour should be closed on the upper half plane ω. The encircled residues then turn out to vanish, which reflects the causality of the problem: no response can precede the impulse. The integral obtained for $t > 0$ by closing the contour on the lower half plane consists of two contributions. The contribution originating from the poles has a simple analytical form of a sum of exponential functions $\mathrm{e}^{-\mathrm{i}\omega_n t}$ ($n = 1$, 2 in terms of Fig. 2), where the negative complex parts of the frequencies ω_n ensure a decrease in the response with time. The contribution originating from the "dog-bone" cannot be expressed analytically. In fact it is proportional to a convolution of a function $(t) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}\sqrt{\omega_0^2-\omega^2}\,\mathrm{e}^{-\mathrm{i}\,\omega t}\,\mathrm{d}\omega = \frac{\omega_0}{2t}J_1(\omega_0 t)$ with another exponentially decreasing function $B(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\,\omega t}}{\prod_{n=1}^{4}(\omega-\omega_n)}\,\mathrm{d}\omega$ which can be easily evaluated by the contour integral ($t \in -\infty \ldots \infty$ in the convolution). Therefore the resulting total impulse response is a decreasing function. In the limit $\omega_0 \to 0$ one deals with an ordinary damped oscillator. A non-zero stop band $\omega_0 > 0$ corresponds to a specific damping in which the higher frequencies are strongly damped in an analogy to a low pass filter. The model can also provide instance of true surface (defect) waves. A possibility for that is to put $\beta = 0$. With some additional damping of the end oscillator the poles on the lower half plane occur on the back Riemann surface leaf. Then the wave vector acquires a positive imaginary part which corresponds to an attenuated wave. The true surface wave is obtained in the limit of the additional damping tending to zero in which limit the poles on the back Riemann leaf approach the real axis. The spatially attenuated character of the waves is preserved in this limit.

4. Resonant response

In summary, one kind of resonances in the system under study corresponds to poles situated on the front leaf of the Riemann surface on the complex frequency plane. Such resonances describe leaky waves. The real part of the resonant frequency of such resonance above the lower limit of the bulk band $\mathrm{Re}\omega_1 > \omega_0$ (in terms of Fig. 2) mark supersonic leaky wave whereas $\mathrm{Re}\omega_1 < \omega_0$ is characteristic of subsonic leaky wave. The fact that the resonances are situated on the front Riemann leaf indicates a negative value of the imaginary part of the wave vector which amounts to an increase in amplitude with the depth into the bulk. The resonant frequencies situated on the back Riemann leaf in some proximity of the cut correspond to true surface waves, possibly damped by an additional viscous friction which is not explicitly included in the present model but can be added to give the resonance a finite width. The right hand side of the equation of motion (4) then contains an additional term $-\gamma_z\frac{\partial u}{\partial t}$. In the limit of this friction tending to zero, i.e. $\gamma_z \to 0$, the resonance frequencies lie exactly on the cut and the corresponding local density of states shows a delta-like peak. The resonances are qualitatively different. The negative imaginary part of the wave vector of the leaky waves, be they supersonic or subsonic, results in an increase in the amplitude with the depth. Therefore, the limit of vanishing imaginary part of frequency of such waves corresponds to temporarily undamped solutions that diverge with increasing depth. Such solutions are, therefore, unphysical in this limit. In contrast, the true surface waves, being the limit case of vanishing additional damping are evanescent and have clear physical meaning of localized excitations. Generally the most efficient way of exciting a resonance is to apply a signal which is identical/proportional with/to the impulse response. Since the applied frequency then is equal to the frequency of the corresponding pole of the Green function, the response-to-signal ratio is infinite. However, because the signal itself is a decreasing function the resulting response also tends to zero in the long time limit. Mathematically the resonant response is a convolution of the impulse response with itself. The response to a signal which is equal to the impulse response is called here resonant response.

5. Examples and discussion

Figure 3 shows the impulse response (Fig. 3a) and the resonant response (Fig. 3b) for a typical supersonic leaky wave. The parameters of the system are $\mathrm{Re}(\omega_1) = 1.32958$, $\mathrm{Im}(\omega_1) = -0.0709063$, $\omega_0 = 1$, $\beta = 1.11803$, $\gamma = 0.15$, $m = 0.7$. Figure 4 represents the response of the same system in the frequency domain. In particular, the red (continuous) line gives the local density of states (LDOS) which is proportional to the power transmitted to the system by the oscillating applied perturbation. In terms of the electric circuits theory LDOS corresponds to the true power or dispersive power [7]. A resonant maximum visible in the LDOS is centred around the resonant frequency and its width is determined by the effective damping constant γ. The real part of the Green function, here represented by a green (dashed) line, is a reactive power, indicating the exchange of power between the system and the stimulus within each period of oscillations without, however, any net work transmitted to the system. The ratio of the response amplitude to the amplitude of the stimulus is the hypotenuse of the power triangle and is represented here by a blue (dotted) line. s one can see, the behaviour of all the quantities resembles that of an ordinary resonance: the maximum of LDOS coincides with zero of the reactive power. The response in the time domain (Fig. 3b) shows a maximum characteristic for the damped resonance response. The response-to-signal ratio in the maximum depends on the value of the effective damping constant γ. Interesting is that the contribution from the dog-bone is hardly present in this case, the majority of the response being provided by the residue part.

The case of a true surface wave artificially damped by an additional viscous friction $\gamma_z = 0.05$ is illustrated in Figs. 5 and 6. The parameters of the model are $\mathrm{Re}(\omega_1) = 0.919464$, $\mathrm{Im}(\omega_1) = -0.0146668$, $\omega_0 = 1$, $\beta = 0$, $\gamma = 1$, $m = 0.5$. The strong peak in LDOS in Fig. 6 corresponds to zero of the reactive power as it should be in any damped resonance. The time-domain impulse (Fig. 5a) and resonant response (Fig. 5b) show interesting constructive interference between the contributions from the residues and from the dog-bone. This interference is at the origin of the maximum of the resonant response being about 15 times stronger than the amplitude of stimulus. This is just the most common case whereof surface wave, e.g. the Rayleigh wave, whose

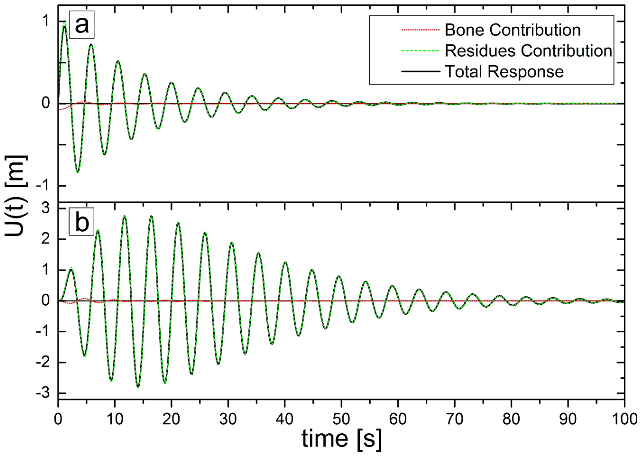

Fig. 3. Impulse response (a) and resonant response (b) in case of supersonic leaky wave ($\mathrm{Re}(\omega_1) = 1.32958$, $\mathrm{Im}(\omega_1) = -0.0709063$, $\omega_0 = 1$, $\beta = 1.11803$, $\gamma = 0.15$, $m = 0.7$).

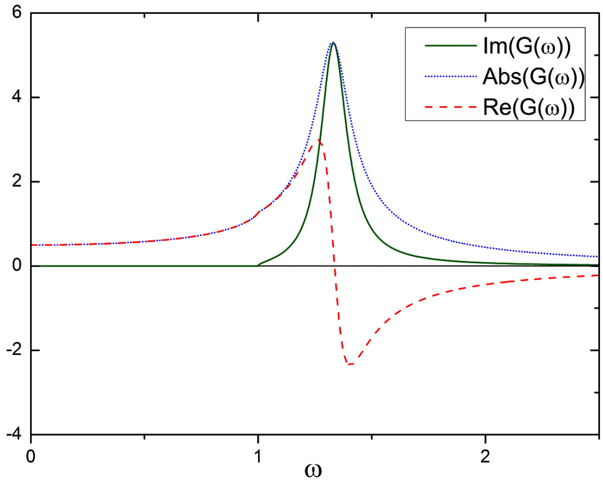

Fig. 4. Frequency response in case of supersonic leaky wave ($\mathrm{Re}(\omega_1) = 1.32958$, $\mathrm{Im}(\omega_1) = -0.0709063$, $\omega_0 = 1$, $\beta = 1.11803$, $\gamma = 0.15$, $m = 0.7$).

propagation is only affected by losses not related with the radiation of the power into the bulk. In both above cases the maximum of amplitude coincides with the maximum of LDOS.

The most interesting case is a subsonic leaky wave as shown in Figs. 7 and 8. Here the parameters of the system are $\mathrm{Re}(\omega_1) = 0.979507$, $\mathrm{Im}(\omega_1) = -0.0339038$, $\omega_0 = 1$, $\beta = 1.04881$, $\gamma = 0.5$, $m = 1.02$. The LDOS has no longer a maximum within the bulk band, then shows a strong edge singularity at the lower limit of the band as it is seen in Fig. 8. However, the reactive power does not show a zero there. Instead, a sharp maximum of the reactive power occurs at the edge of the bulk band.

The impulse response and, consequently, the resonant response shows a kind of destructive interference between the contributions from the residues and from the dog-bone. s a result the response is weakened, but an enve-

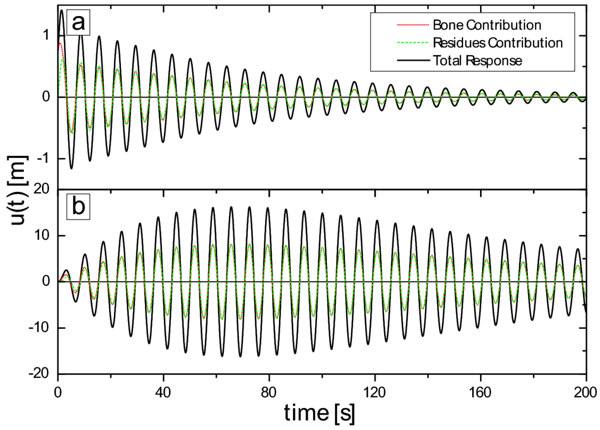

Fig. 5. Impulse response (a) and resonant response (b) in case of true surface wave artificially damped ($\mathrm{Re}(\omega_1) = 0.919464$, $\mathrm{Im}(\omega_1) = -0.0146668$, $\omega_0 = 1$, $\beta = 0$, $\gamma = 1$, $m = 0.5$, additional damping constant $\gamma_z = 0.05$).

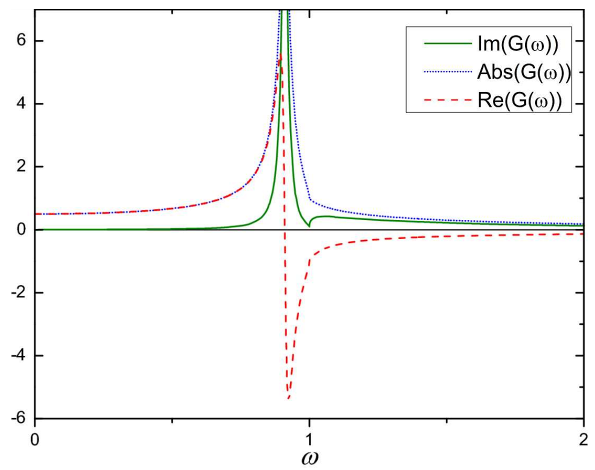

Fig. 6. Frequency response in case of true surface wave artificially damped ($\mathrm{Re}(\omega_1) = 0.919464$, $\mathrm{Im}(\omega_1) = -0.0146668$, $\omega_0 = 1$, $\beta = 0$, $\gamma = 1$, $m = 0.5$, additional damping constant $\gamma_z = 0.05$).

lope with a maximum is still clearly visible. The response also shows a waving of its amplitude whose effect is attributed to beats between the frequency of the resonance and that of the lower edge of the bulk band. In this example the life time of the subsonic leaky wave is long in comparison with the previous one. This manifests itself in a rather high amplitude of the response to a sinusoidal perturbation of frequency $\omega = \mathrm{Re}(\omega_1)$. The majority of the response is, however, in-phase with the perturbation, that corresponds to the vanishing of the LDOS for $\mathrm{Re}(\omega_1) < \omega_0$. Physically it means that although the impulse response is not very strong, the amplitude excited by the resonance frequency can be very high for the harmonic excitation. This may have significance in practical applications of such resonant systems.

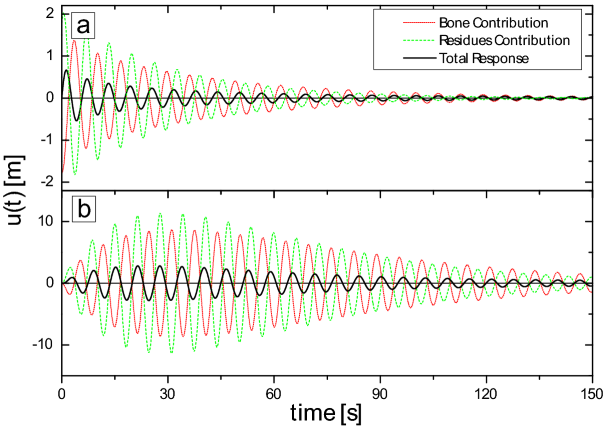

Fig. 7. Impulse response (a) and resonant response (b) in case of subsonic leaky wave ($\mathrm{Re}(\omega_1) = 0.979507$, $\mathrm{Im}(\omega_1) = -0.0339038$, $\omega_0 = 1$, $\beta = 1.04881$, $\gamma = 0.5$, $m = 1.02$).

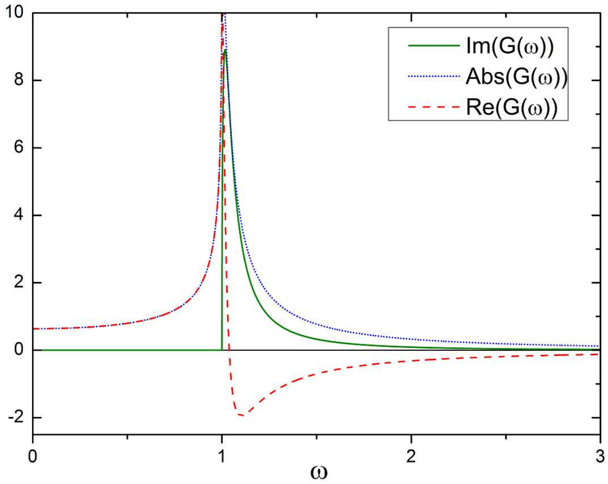

Fig. 8. Frequency response in case of subsonic leaky wave ($\mathrm{Re}(\omega_1) = 0.979507$, $\mathrm{Im}(\omega_1) = -0.0339038$, $\omega_0 = 1$, $\beta = 1.04881$, $\gamma = 0.5$, $m = 1.02$).

6. Conclusions

There exists a qualitative difference between supersonic leaky waves and true waves, on the one hand, and subsonic leaky waves, on the other hand. Whereas there is a resonant maximum in the local density of states accompanied by a zero of the real part of the frequency dependent response function in the former case, the LDOS is significantly shifted towards the lower border of the bulk band in the latter case but no maximum is present within the bulk band. The most pronounced effect of a subsonic leaky wave is a sharp maximum of the real part of the response function at this lower border of the bulk band. The system then is analogous to electric systems with high reactive power. A consequence of it is that the vibrational frequency at the defect can be rather high. A destructive interference of the resonant and "dog-bone" contributions of the impulse response in the case of the subsonic leaky wave makes the time domain response unexpectedly weak.

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