Green Function on a Quantum Disk for the Helmholtz Problem

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In this work, we present a new result which concerns the derivation of the Green function relative to the time-independent Schrödinger equation in two-dimensional space. The system considered in this work is a quantal particle that moves in an axi-symmetric potential. At first, we have assumed that the potential $V(r)$ to be equal to a constant V_0 inside a disk (radius a) and to be equal to zero outside the disk. We have used, to derive the Green function, the continuity of the solution and of its first derivative, at $r = a$ (at the edge). Secondly, we have assumed that the potential $V(r)$ is equal to zero inside the disk and is equal to V_0 outside the disk (the inverted potential). Here, also we have used the continuity of the solution and its derivative to obtain the associate Green function showing the discrete spectra of the Hamiltonian.

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1. Introduction

The method of Green's function is a very powerful tool in solving problems of mathematical physics. In the general case the Green function is a distribution that was introduced by Green in electromagnetism, and later used by Neuman in the theory of Newtonian potential and by Helmholtz in acoustics. Feynman also used this function in quantum field theory with a different name that is the "propagator". It has also been used in the numerical solution of boundary integral equations for potential flows in fluid mechanics, remote sensing of periodic surfaces, periodic gratings, and infinite arrays of resonators coupled to a waveguide, in many contexts of simulating systems of charged particles, in molecular dynamics, for the description of quasi-periodic arrays of point interactions in quantum mechanics, and in various *ab initio* first-principle multiple-scattering theories for the analysis of diffraction of classical and quantum waves.

There are usually several Green functions associated with the same equation. These different functions are distinguished from each other by the boundary conditions. Thus it is important, when computing the Green function of the linear differential equation to specify with the boundary conditions.

Before dealing with the description of our problem we include some works that are closely related to our problem, namely the Helmholtz equation on a disk. In Ref. [1] the author treats the problem of a thin circular Kirchhoff Poisson-plate. The plate edge is assumed to be elastically supported so that the boundary values are that the radial bending moment equals zero, whereas the strength is proportional to the function of the deflection on the boundary. The Green function is also studied by [2] in circular, annular and exterior circular domain. In Refs. [3, 4] the Green function was studied for the elliptic domain. The quantum problem relative to the scattering in two dimensions was also treated in [5].

In our work, we address the problem of Helmholtz on a disk but with new boundary conditions. These boundary conditions are useful in quantum mechanics to problems of diffusion and also for bound states. In quantum mechanics, if the potential is constant in the disk and is zero outside (or vice versa) the solution of the Helmholtz equation (the Schrödinger equation in our case) and the derivative of the solution are continuous on the boundary (the edge) of the disc. Specify one else in our problem: the Helmholtz equation takes two different forms depending on whether it is inside the disk or outside. This type of problem matches in quantum mechanics to the study of a particle is subject to a potential which is a positive constant to the interior of the disk and is zero on the outside.

In Sect. 2, we will expose the Schrödinger equation for the quantum particle moving in a potential defined as a positive constant inside the disk and zero outside the disk. It turns out that this equation is that of Helmholtz. We also define in this section the boundary conditions to which the solution of the problem must satisfy.

In Sect. 3, we calculate the Green functions associated with the problem. We have divided this section into two subsections 3.1 and 3.2. In Sect. 3.1, we considered that the quantum particle has an energy $E > V_0$ and then we calculated the Green function using the continuity of the solution and its first derivative on the frontier. In Sect. 3.2, we considered that the particle has energy $0 < E < V_0$ and calculated the Green function using the same boundary conditions as in Sect. 3.1.

In Sect. 4, we consider another problem on the disk that is to take the zero potential inside the disk and equal to $V_0 > 0$ outside the disk. Again, we calculated the Green function using the same boundary conditions for

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the two cases $E > V_0$ and $0 < E < V_0$. In this case, the spectra are given by a transcendental equation. If we make V_0 to infinity, we found the well-known result for an infinite well. The spectrum is calculated by the zeros of the Bessel function. Finally, we finish this work by a conclusion in Sect. 5.

2. Two-dimensional quantum problem

Let us consider a quantum particle moving in an azimuthal symmetrical potential (independent of the azimutal angle θ) defined on a disk:

$$\phi(r,\theta) = \begin{cases} V_0 & \text{if} \quad 0 \le r \le a, \\ 0 & \text{if} \quad r > a. \end{cases} \tag{1}$$

The dynamics of this particle is governed by the time--independent Schrödinger equation

$$\hat{H}(r,\theta)\Psi(r,\theta) = E\Psi(r,\theta), \tag{2}$$

which is written in the natural polar coordinates (r,θ) and where $\hat{H}(r,\theta)$ is the Hamiltonian of the particle, with a mass M, moving in this potential. Equation (2) is merely an eigenvalue E and eigenfunctions equation $\Psi(r,\theta)$. The explicit form of the Hamiltonian of the system is

$$\hat{H} = -\frac{\hbar^2}{2M}\Delta_{r,\theta} + \phi(r,\theta), \tag{3}$$

where

$$\Delta_{r,\theta} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{4}$$

is the Laplacian in polar coordinates. Equation (2) writes as

$$\left(-\frac{\hbar^2}{2M}\Delta_{r,\theta} + \phi(r,\theta) - E\right)\Psi(r,\theta) = 0, \tag{5}$$

or, with respect of the definition of $\phi(r,\theta)$:

$$\begin{cases} \left(\frac{\hbar^2}{2M}\Delta_{r,\theta} + E\right)\Psi_{\text{out}}(r,\theta) = 0 & \text{if} \quad r > a, \\ \left(\frac{\hbar^2}{2M}\Delta_{r,\theta} - V_0 + E\right)\Psi_{\text{in}}(r,\theta) = 0 & \text{if} \quad 0 \le r \le a. \end{cases} \tag{6}$$

This system is subjected to the boundary conditions defined as $\Psi(r,\theta)$ and $\frac{\mathrm{d}}{\mathrm{d}r}\Psi(r,\theta)$ are to be continuous at $r = a$ for all values of the azimutal angle θ. The separation variables method leads to transform the last two equations as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\Psi_{\text{out}}\right) + \left(\frac{2M}{\hbar^2}Er - \frac{m^2}{r}\right)\Psi_{\text{out}}(r) = 0 \quad \text{if} \quad r > a, \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\Psi_{\text{in}}\right) + \left(\frac{2M}{\hbar^2}(E - V_0)r - \frac{m^2}{r}\right)\Psi_{\text{in}}(r) = 0 \quad \text{if} \quad 0 \le r \le a, \tag{8}$$

with the boundary conditions

$$\Psi_{\text{out}}(a) = \Psi_{\text{in}}(a). \tag{9}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}r}\Psi_{\text{out}}(r)\right|_{r=a} = \left.\frac{\mathrm{d}}{\mathrm{d}r}\Psi_{\text{in}}(r)\right|_{r=a} \tag{10}$$

and $m = \ldots -2, -1.0, +1, +2, \ldots$

The global Green function of the problem (6) augmented by the boundary conditions (9), (10) is given by

$$G(\boldsymbol{r},\boldsymbol{r}',E) = G(r,\theta,r',\theta',E) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} G(l:r,r',E)\exp(\mathrm{i}m(\theta-\theta')),$$

where $G(l:r,r',E) \equiv G(l:r,r')$ is the radial Green function that we shall calculate in the subsequent sections.

3. Construction of the Green function

3.1. The case $E > V_0$

3.1.1. The case $0 \le r \le r' \le a$ (inside the disk)

Following Eq. (8) the Green function is given by

$$G_{\text{in}}(l:r,r') \equiv G^{1,1}(l:r,r') = \begin{cases} A(r')J_l(\mu r), & 0 < r \le r', \\ B(r')\left[Y_l(\mu r) - \alpha(r')J_l(\mu r)\right], & r' \le r \le a, \end{cases} \tag{11}$$

where $\mu^2 = \frac{2M}{\hbar^2}(E - V_0)$. To compute the coefficients $A(r')$, $B(r')$ and $\alpha(r')$, we use the continuity of the Green function at $r = r'$:

$$G^{1,1}(l:r'_+,r') - G^{1,1}(l:r'_-,r') = 0 \iff B(r')Y_l(\mu r') - [A(r') + \alpha(r')B(r')]J_l(\mu r') = 0 \tag{12}$$

and use the discontinuity of the first derivative with respect to r at $r = r'$:

$$\frac{\mathrm{d}}{\mathrm{d}r}G'^{1,1}(l:r'_+,r') - \frac{\mathrm{d}}{\mathrm{d}r}G'^{1,1}(l:r'_-,r') = \frac{2}{\pi r'} \iff B(r')Y'_l(\mu r') - [A(r') + \alpha(r')B(r')]J'_l(\mu r') = \frac{2}{\pi\mu r'}. \tag{13}$$

Combining (12) and (13) we obtain

$$A(r') = \frac{B(r')\left[Y_l(\mu r') - \alpha(r')J_l(\mu r')\right]}{J_l(\mu r')} \tag{14}$$

and

$$B(r')Y'_l(\mu r') - \left[\frac{B(r')\left[Y_l(\mu r') - \alpha(r')J_l(\mu r')\right]}{J_l(\mu r')} + \alpha(r')B(r')\right]J'_l(\mu r') = \frac{2}{\pi\mu r'}. \tag{15}$$

Using the Bessel Wronskian

$$W(J_l(\mu r'), Y_l(\mu r')) = J_l(\mu r')Y'_l(\mu r') - Y_l(\mu r')J'_l(\mu r') = \frac{2}{\pi\mu r'}, \tag{16}$$

we get the coefficients

$$B(r') = J_l(\mu r') \tag{17}$$

and

$$A(r') = \left[Y_l(\mu r') - \alpha(r')J_l(\mu r')\right]. \tag{18}$$

Then, the Green function inside the disk is given by

$$G^{1,1}(l:r,r') = \begin{cases} \left[Y_l(\mu r') - \alpha(r')J_l(\mu r')\right]J_l(\mu r), & 0 < r \le r', \\ \left[Y_l(\mu r) - \alpha(r')J_l(\mu r)\right]J_l(\mu r'), & r' \le r \le a. \end{cases} \tag{19}$$

It rests to determine the coefficient $\alpha(r')$. To do this, we use the symmetry properties of $G(l:r,r')$:

$$G(l:r,r') = G(l:r',r), \tag{20}$$

$$[Y_l(\mu r') - \alpha(r')J_l(\mu r')]\, J_l(\mu r) = [Y_l(\mu r') - \alpha(r)J_l(\mu r')]\, J_l(\mu r). \tag{21}$$

By identifying in the last equation we find

$$\alpha(r') = \alpha(r) = \alpha. \tag{22}$$

Then the Green function inside the disk is given by

$$G^{1,1}(l:r,r') = \begin{cases} [Y_l(\mu r') - \alpha J_l(\mu r')]\, J_l(\mu r), & 0 < r \le r', \\ [Y_l(\mu r) - \alpha J_l(\mu r)]\, J_l(\mu r'), & r' \le r \le a. \end{cases} \tag{23}$$

We mention here that the coefficient α will be determined later when we explore the region outside of the disk.

3.1.2. The case $a \le r \le r' < \infty$ (outside the disk)

In the outside of the disk, the Green function can be written as

$$G^{2,2}(l:r,r') = \begin{cases} C(r')\,[Y_l(kr) - \beta(r')J_l(kr)], & a \le r \le r', \\ D(r')\, J_l(kr), & r' \le r < \infty, \end{cases} \tag{24}$$

where $k^2 = \frac{2M}{\hbar^2}E$. Using the continuity of the Green function at $r = r'$:

$$G^{2,2}(l:r'_+,r') - G^{2,2}(l:r'_-,r') = 0 \iff -C(r')\,Y_l(kr') + [D(r') + \beta(r')C(r')]\, J_l(kr') = 0 \tag{25}$$

and the discontinuity of the first derivative with respect to r at $r = r'$, we find

$$\frac{\mathrm{d}}{\mathrm{d}r}G'^{2,2}(l:r'_+,r') - \frac{\mathrm{d}}{\mathrm{d}r}G'^{2,2}(l:r'_-,r') = \frac{2}{\pi r'} \iff -C(r')\,Y'_l(kr') + [D(r') + \beta(r')C(r')]\, J'_l(kr') = \frac{2}{\pi k r'}. \tag{26}$$

Following (25) we check that

$$D(r') = \frac{C(r')\,[Y_l(kr') - \beta(r')J_l(kr')]}{J_l(kr')} \tag{27}$$

and using the Bessel Wronskian

$$W(J_l(kr'), Y_l(kr')) = J'_l(kr')\,Y_l(kr') - J_l(kr')\,Y'_l(kr') = \frac{-2}{\pi k r'}, \tag{28}$$

we find after substituting (27) and (28) in (26):

$$\frac{-2C(r')}{\pi k r'} = \frac{2J_l(kr')}{\pi k r'} \Rightarrow C(r') = -J_l(kr') \tag{29}$$

and after substituting (29) and (27) in (24) we find

$$G^{2,2}(l:r,r') = -\begin{cases} [Y_l(kr) - \beta(r')J_l(kr)]\, J_l(kr'), \\ \quad a \le r \le r', \\ [Y_l(kr') - \beta(r')J_l(kr')]\, J_l(kr), \\ \quad r' \le r < \infty. \end{cases} \tag{30}$$

As we must have the symmetry property

$$G(l:r,r') = G(l:r',r), \tag{31}$$

we deduce

$$\beta(r') = \beta(r) \Rightarrow \beta \tag{32}$$

and then

$$G^{2,2}(l:r,r') = -\begin{cases} [Y_l(kr) - \beta J_l(kr)]\, J_l(kr'), \\ \quad a \le r \le r', \\ [Y_l(kr') - \beta J_l(kr')]\, J_l(kr), \\ \quad r' \le r < \infty. \end{cases} \tag{33}$$

where β is a constant that we start to compute in the following section.

3.1.3. The coefficients α and β

To find the coefficients α and β we use the continuity of the Green function and the continuity of its derivative at $r = a$:

$$G^{1,1}(l:r,a) = G^{2,2}(l:r,a) \Leftrightarrow [Y_l(\mu a) - \alpha J_l(\mu a)] \times J_l(\mu a) = -[Y_l(ka) - \beta J_l(ka)]\, J_l(ka), \tag{34}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r}G'^{1,1}(l:r,a)\rfloor_{r=a} = \frac{\mathrm{d}}{\mathrm{d}r}G^{2,2}(l:r,a)\rfloor_{r=a} \tag{35}$$

or

$$\mu\,[Y'_l(\mu a) - \alpha J'_l(\mu a)]\, J_l(\mu a) = -k\,[Y'_l(ka) - \beta J'_l(ka)]\, J_l(ka). \tag{36}$$

After simplifications we get the coefficient α:

$$\alpha = \Big\{ -2J_l(ka) + \pi a J_l(\mu a)\big[kY_l(\mu a)J'_l(ka) - \mu J_l(ka)Y'_l(\mu a)\big]\Big\} \Big/ \Big\{\pi a J_l(\mu a)\big[kJ_l(\mu a)J'_l(ka) - \mu J_l(ka)J'_l(\mu a)\big]\Big\} = \frac{-2J_l(ka) + \pi a J_l(\mu a)V(k,\mu)}{\pi a J_l(\mu a)U(k,\mu)} \tag{37}$$

such as

$$V(k,\mu) = kY_l(\mu a)J'_l(ka) - \mu J_l(ka)Y'_l(\mu a),$$
$$U(k,\mu) = kJ_l(\mu a)J'_l(ka) - \mu J_l(ka)J'_l(\mu a).$$

In the same way, we find

$$\beta = \big\{ -2J_l(\mu a) + \pi a J_l(ka)\big[\mu Y_l(ka)J'_l(\mu a) - kJ_l(\mu a)Y'_l(ka)\big]\big\} \Big/ \big\{\pi a J_l(ka)\big[\mu J_l(ka)J'_l(\mu a) - kJ_l(\mu a)J'_l(ka)\big]\big\} = \frac{2J_l(\mu a) + \pi a J_l(ka)\mathsf{F}(k,\mu)}{\pi a J_l(ka)U(k,\mu)} \tag{38}$$

where $\mathsf{F}(k,\mu) = \mu Y_l(ka)J'_l(\mu a) - kJ_l(\mu a)Y'_l(ka)$. Finally, the Green function inside the disk is given by

$$G^{1,1}(l:r,r') = \begin{cases} Y_l(\mu r')\,J_l(\mu r) - \left[\frac{-2J_l(ka)+\pi a J_l(\mu a)V(k,\mu)}{\pi a J_l(\mu a)U(k,\mu)} J_l(\mu r')\right] J_l(\mu r), \\ \quad 0 < r \le r', \\ Y_l(\mu r)J_l(\mu r') - \left[\frac{-2J_l(ka)+\pi a J_l(\mu a)V(k,\mu)}{\pi a J_l(\mu a)U(k,\mu)} J_l(\mu r)\right] J_l(\mu r'), \\ \quad r' \le r \le a, \end{cases} \tag{39}$$

and outside the disk

$$G^{2,2}(L:r,r') = -\begin{cases} Y_l(kr)J_l(kr') \\ \quad -\left[\frac{2J_l(\mu a)+\pi a J_l(ka)\mathsf{F}(k,\mu)}{\pi a J_l(ka)U(k,\mu)}J_l(kr)\right]J_l(kr'), \\ \quad a \le r \le r', \\ Y_l(kr')J_l(kr) \\ \quad -\left[\frac{2J_l(\mu a)+\pi a J_l(ka)\mathsf{F}(k,\mu)}{\pi a J_l(ka)U(k,\mu)}J_l(kr')\right]J_l(kr), \\ \quad r' \le r < \infty. \end{cases} \tag{40}$$

3.1.4. The case $0 < r' \le a \le r < \infty$ (r' inside and r outside the disk)

In this case the Green function writes as

$$G^{2,1}(l:r,r') = [Y_l(kr) - \lambda J_l(kr)]\, J_l(\mu r') \tag{41}$$

where λ is a constant to be determined using the continuity of the Green function at $r = a$:

$$G^{2,1}(l:r,a)\big|_{r=a} = G^{2,2}(l:r,a)\big|_{r=a}. \tag{42}$$

Then

$$[Y_l(ka) - \lambda J_l(ka)]\, J_l(\mu a) = -\left[Y_l(ka) - \frac{2J_l(\mu a) + \pi a J_l(ka)\mathsf{F}(k,\mu)}{\pi a J_l(ka)U(k,\mu)}J_l(ka)\right] \times J_l(ka) \tag{43}$$

or

$$\lambda = \frac{Y_l(ka)}{J_l(ka)} + \frac{4}{\pi a\,[\mu J_l(ka)J_l'(\mu a) - kJ_l(\mu a)J_l'(ka)]}, \tag{44}$$

$$= \frac{Y_l(ka)}{J_l(ka)} + \frac{4}{\pi a U(k,\mu)}. \tag{45}$$

Then we obtain the Green function (mixed):

$$G^{2,1}(l:r,r') = \left[Y_l(kr) - \left(\frac{Y_l(ka)}{J_l(ka)} + \frac{4}{\pi a U(k,\mu)}\right) \times J_l(kr)\right] J_l(\mu r'). \tag{46}$$

3.1.5. The case $0 < r \le a \le r' < \infty$ (r inside and r' outside the disk)

In this case the Green function writes as

$$G^{1,2}(l:r,r') = J_l(\mu r)\,[Y_l(kr') - \eta J_l(kr')], \tag{47}$$

where η is a constant to be determined using the continuity of the Green function at $r = a$:

$$G^{1,2}(l:r,a)\big|_{r=a} = G^{1,1}(l:r,a)\big|_{r=a}, \tag{48}$$

$$J_l(\mu a)\,[Y_l(ka) - \eta J_l(ka)] \tag{49}$$

$$= \left[Y_l(\mu a) - \frac{-2J_l(ka) + \pi a J_l(\mu a)V(k,\mu)}{\pi a J_l(\mu a)U(k,\mu)}\right] J_l(\mu a).$$

Then

$$\eta = \frac{Y_l(ka)}{J_l(ka)} - \frac{4}{\pi a\,[kJ_l(\mu a)J_l'(ka) - \mu J_l(ka)J_l'(\mu a)]}$$

$$= \frac{Y_l(ka)}{J_l(ka)} + \frac{4}{\pi a U(k,\mu)}. \tag{50}$$

Then the mixed Green functions become

$$G^{1,2}(L:r,r') = \left[Y_l(kr') - \left(\frac{Y_l(ka)}{J_l(ka)} + \frac{4}{\pi a U(k,\mu)}\right) \times J_l(kr')\right] J_l(\mu r). \tag{51}$$

3.2. The case $0 < E < V_0$

In this case μ becomes purely imaginary number $\mu' = \mathrm{i}\mu$:

$$\mu' = \mathrm{i}\sqrt{2(E - V_0)} = \mathrm{i}\mu. \tag{52}$$

Then the Bessel functions transform as

$$J_l(\mu' r) \to I_l(\mu r), \tag{53}$$

$$Y_l(\mu' r) \to K_l(\mu r). \tag{54}$$

Let us summarize all results in different regions now. In the region defined inside the disk:

3.2.1. $0 \le r \le r' \le a$ (inside the disk)

$$G^{1,1}(l:r,r') = \begin{cases} K_l(\mu r')\,I_l(\mu r) - \frac{-2J_l(ka)+\pi a I_l(\mu a)\Omega(k,\mu)}{\pi a I_l(\mu a)\Phi(k,\mu)} \\ \quad \times I_l(\mu r')\,I_l(\mu r), \quad 0 < r \le r', \\ K_l(\mu r)I_l(\mu r') - \frac{-2J_l(ka)+\pi a I_l(\mu a)\Omega(k,\mu)}{\pi a I_l(\mu a)\Phi(k,\mu)} \\ \quad \times I_l(\mu r)I_l(\mu r'), \quad r' \le r \le a, \end{cases} \tag{55}$$

where

$$\Omega(k,\mu) = kK_l(\mu a)J_l'(ka) - \mathrm{i}\mu J_l(ka)K_l'(\mu a),$$
$$\Phi(k,\mu) = kI_l(\mu a)J_l'(ka) - \mathrm{i}\mu J_l(ka)I_l'(\mu a).$$

3.2.2. $0 < r' \le a \le r < \infty$ (r' inside and r outside the disk)

$$G^{2,1}(l:r,r') = \left[Y_l(kr) - \left(\frac{Y_l(ka)}{J_l(ka)} - \frac{4}{\pi a \Phi(k,\mu)}\right) \times J_l(kr)\right] I_l(\mu r'). \tag{56}$$

3.2.3. $0 < r \le a \le r' < \infty$ (r' outside and r inside the disk)

$$G^{1,2}(l:r,r') = \left[Y_l(kr') - \left(\frac{Y_l(ka)}{J_l(ka)} - \frac{4}{\pi a \Phi(k,\mu)}\right) \times J_l(kr')\right] I_l(\mu r). \tag{57}$$

3.2.4. $a \le r \le r' < \infty$ (outside the disk)

$$G^{2,2}(l:r,r') = \begin{cases} Y_l(kr)J_l(kr') - \frac{2I_l(\mu a)+\pi a J_l(ka)\Psi(k,\mu)}{\pi a J_l(ka)\Phi(k,\mu)} \\ \quad \times J_l(kr)J_l(kr'), \quad a \le r \le r', \\ Y_l(kr')J_l(kr) - \frac{2I_l(\mu a)+\pi a J_l(ka)\Psi(k,\mu)}{\pi a J_l(ka)\Phi(k,\mu)} \\ \quad \times J_l(kr')J_l(kr), \quad r' \le r < \infty, \end{cases} \tag{58}$$

where

$$\Psi(k,\mu) = kI_l(\mu a)Y_l'(ka) - \mathrm{i}\mu Y_l(ka)I_l'(\mu a).$$

4. Two-dimensional problem for the inverted potential

Consider here the quantum particle moving in an azimuthal symmetrical potential (independent of the azimutal angle θ) defined now as follows:

$$\phi(r,\theta) = \begin{cases} 0 & \text{if} \quad 0 \le r \le a, \\ V_0 & \text{if} \quad r > a. \end{cases} \tag{59}$$

To compute the Green function for this problem, it suffices to reconsider the solutions obtained in the first section and inter-change in them the constants $\mu \leftrightarrow k$. In this inversion E becomes less than V_0 ($E < V_0$) and μ becomes equal to $\sqrt{2m(V_0 - E)}/\hbar$. For example in the case $0 \le r \le r' \le a$ where the potential $\phi(r,\theta)$ is zero and equal to V_0 outside the disk, the Green function becomes

$$G^{1,1}(l : r, r') = \begin{cases} Y_l(kr') J_l(kr) - \left[\frac{-2J_l(ka)+\pi a J_l(\mu a) V(k,\mu)}{\pi a J_l(\mu a) U(k,\mu)} J_l(kr')\right] J_l(kr), \\ \quad 0 < r \le r', \\ Y_l(kr) J_l(kr') - \left[\frac{-2J_l(ka)+\pi a J_l(\mu a) V(k,\mu)}{\pi a J_l(\mu a) U(k,\mu)} J_l(kr)\right] J_l(kr'), \\ \quad r' \le r \le a, \end{cases}$$

and outside the disk ($r \ge r' \ge a$):

$$G^{2,2}(l : r, r') = -\begin{cases} Y_l(\mu r) J_l(\mu r') - \left[\frac{2J_l(\mu a)+\pi a J_l(ka) \mathsf{F}(k,\mu)}{\pi a J_l(ka) U(k,\mu)} J_l(\mu r)\right] J_l(\mu r'), \\ \quad a \le r \le r', \\ Y_l(\mu r') J_l(\mu r) - \left[\frac{2J_l(\mu a)+\pi a J_l(ka) \mathsf{F}(k,\mu)}{\pi a J_l(ka) U(k,\mu)} J_l(\mu r')\right] J_l(\mu r), \\ \quad r' \le r < \infty. \end{cases} \tag{60}$$

The spectra are given by the poles of $G^{1,1}(l : r, r')$. In the case where r is inside and r' is outside the disk

$$G^{2,1}(l : r, r') = \left[Y_l(\mu r) - \left(\frac{Y_l(\mu a)}{J_l(\mu a)} - \frac{4}{\pi a U(k,\mu)}\right) \times J_l(\mu r)\right] J_l(kr'). \tag{61}$$

In the case where r' is inside and r is outside the disk

$$G^{1,2}(l : r, r') = \left[Y_l(\mu r') - \left(\frac{Y_l(\mu a)}{J_l(\mu a)} - \frac{4}{\pi a U(k,\mu)}\right) \times J_l(\mu r')\right] J_l(kr), \tag{62}$$

we put

$$\begin{aligned} \alpha_1 &= \Big\{2J_l(\mu a) + \pi a J_l(ka)\big[kJ_l(\mu a)Y_l'(ka) - \mu Y_l(ka) J_l'(\mu a)\big]\Big\} \Big/ \Big\{\pi a J_l(ka)\big[kJ_l(\mu a)J_l'(ka) - \mu J_l(ka) J_l'(\mu a)\big]\Big\} \\ &= \left\{\frac{2J_l(\mu a)}{\pi a J_l(ka) \mu J_{l-1}(\mu a)} + k\frac{J_l(\mu a)}{\mu J_{l-1}(\mu a)}\left[Y_{l-1}(ka) - \frac{l}{ka} Y_l(ka)\right] - Y_l(ka)\left(1 - \frac{l}{a}\frac{J_l(\mu a)}{\mu J_{l-1}(\mu a)}\right)\right\} \\ &\quad \Big/ \left\{k\frac{J_l(\mu a)}{\mu J_{l-1}(\mu a)}\left[J_{l-1}(ka) - \frac{l}{ka} J_l(ka)\right] - J_l(ka)\left(1 - \frac{l}{a}\frac{J_l(\mu a)}{\mu J_{l-1}(\mu a)}\right)\right\}. \end{aligned} \tag{63}$$

If we make the limit μ going to + infinity that is to say V_0 goes to + infinity (infinity well), we obtain

$$J_l(\mu a)/[\mu J_{l-1}(\mu a)] \to 0, \tag{64}$$

then

$$\alpha_1 = Y_l(ka)/J_l(ka), \tag{65}$$

and the Green function, for the infinite two-dimensional well, becomes

$$G^{1,1}(l : r, r') = \begin{cases} \left[Y_l(kr') - \frac{Y_l(ka)}{J_l(ka)} J_l(kr')\right] J_l(kr), \\ \quad 0 < r \le r', \\ \left[Y_l(kr) - \frac{Y_l(ka)}{J_l(ka)} J_l(kr)\right] J_l(kr'), \\ \quad r' \le r \le a, \end{cases} \tag{66}$$

$$G^{2,2}(l : r, r') = 0, \quad r, r' > a, \tag{67}$$

$$G^{2,1}(l : r, r') = 0, \quad 0 < r' < a < r, \tag{68}$$

$$G^{1,2}(l : r, r') = 0, \quad 0 < r < a < r'. \tag{69}$$

This result is a well known quantum problem of a quantum particle moving in the infinite two-dimensional well presenting an azimuthal symmetry [2]. The spectra are then given by the root of $J_l(ka) = 0$.

5. Conclusion

In this work, we present a new result which concerns the derivation of the Green function relative to the time--independent Schrödinger equation in two-dimensional space. The system considered in this work is a quantal particle that moves in an axi-symmetric potential. At first, we have assumed that the potential $V(r)$ is equal to a constant V_0 inside a disk (radius a) and is equal to zero outside the disk. We have used to derive the Green function, the continuity of the solution and of its first derivative at $r = a$. Secondly, we have assumed that the potential $V(r)$ is equal to zero inside the disk and is equal to V_0 outside the disk. Here, also we have used the continuity of the solution and its derivative to obtain the associate Green function.

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