Gaussian Beam Diffraction in Inhomogeneous and Nonlinear Saturable Media

P. Berczynski

Institute of Physics, West Pomeranian University of Technology, al. Piastów 48, 70-310 Szczecin, Poland

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The method of complex geometrical optics is presented, which describes Gaussian beam diffraction and self-focusing in smoothly inhomogeneous and nonlinear saturable media of cylindrical symmetry. Complex geometrical optics reduces the problem of Gaussian beam diffraction and self-focusing in inhomogeneous and nonlinear media to the system of the first order ordinary differential equations for the complex curvature of the wave front and for Gaussian beam amplitude, which can be readily solved both analytically and numerically. As a result, complex geometrical optics radically simplifies the description of Gaussian beam diffraction and self-focusing effects as compared to the other methods of nonlinear optics such as: variational method approach, method of moments, and beam propagation method. The power of complex geometrical optics method is presented on the example of Gaussian beam width evolution in saturable fibre with either focusing and defocusing refractive profiles. Besides, the influence of initial curvature of the wave front on Gaussian beam evolution in nonlinear saturable medium is discussed in this paper.

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1. Introduction

Complex geometrical optics (CGO) has two equivalent forms: the ray-based form, which deals with complex rays [1–6], that is with trajectories in a complex space, and the eikonal-based form, which uses complex eikonal instead of complex rays [6, 7]. A surprising feature of CGO is its ability to describe Gaussian beam (GB) diffraction in both ray-based and eikonal-based approaches. Recently, CGO method has been applied to describe GB evolution in inhomogeneous media [8, 9] and nonlinear media of the Kerr type [10] including nonlinear fibres [11].

This paper describes the advantages of the eikonal-based form of CGO method for description of Gaussian beam diffraction and self-focusing in nonlinear saturable media with special attention to the influence of refractive profile and initial wave front curvature. First of all CGO is new method among commonly accepted approaches based on parabolic equation [12–15]. From practical point of view this paper models the light propagation in nonlinear saturable fibres and generalizes the results of previous papers [16–20], where authors considered mainly the light beam propagation in nonlinear media without contribution of the linear refraction and the effect of initial wave front curvature. Section 2 presents the basic equations of CGO method. Generalization of CGO for arbitrary nonlinear media beyond the Kerr type one is presented in Sect. 3. Section 4 outlines the ability of CGO method to describe GB propagation in nonlinear saturable fibre, where the influence of initial curvature of the wave front and fiber refractive profile on GB propagation is discussed. Finally, Sect. 5 formulates conditions for uniform waveguide with minimum radius taking into account either focusing and defocusing refraction.

2. Basic equations of CGO

2.1. Riccati equation for complex parameter B

For axially symmetric wave beam propagating along z direction in axially symmetric medium CGO method suggests solution of the form

$$\begin{aligned} u(\zeta, z) &= A \exp\left(\mathrm{i} k_0 \psi\right) \\ &= A(z) \exp\left(\mathrm{i} k_0 \left(\sqrt{\varepsilon_0} z + B(z)\zeta^2/2\right)\right), \end{aligned} \tag{1}$$

where $u = u(\zeta, z)$ is wave function of the beam, $A = A(z)$ is complex amplitude, $k_0 = 2\pi/\lambda_0$, where λ_0 is the wavelength of the beam in vacuum and ψ is complex-valued eikonal, which in accordance with (1) has the form

$$\psi = \sqrt{\varepsilon_0} z + B(z)\zeta^2/2, \tag{2}$$

where $\zeta = \sqrt{x^2 + y^2}$ is the distance from the fibre axis z. In above equation B is the complex curvature of the wave front [10] and ε_0 is permittivity of the medium measured along z axis. We assume that parameter ε_0 is constant along z axis. The real and imaginary parts of parameter $B = B_{\mathrm{R}} + \mathrm{i} B_{\mathrm{I}}$ determine the real curvature of the wave front κ and the beam width w (1/e point of the wave intensity) correspondingly

$$B_{\mathrm{R}} = \kappa, \quad B_{\mathrm{I}} = \frac{1}{k_0 w^2}. \tag{3}$$

The eikonal equation

$$(\nabla\psi)^2 = \varepsilon \tag{4}$$

in (ζ, z) coordinates takes the form

$$\left(\frac{\partial\psi}{\partial\zeta}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 = \varepsilon(z, \zeta). \tag{5}$$

In accordance with paraxial approximation radius ζ should be small enough. Therefore parameter $\varepsilon(z, \zeta)$ in Eq. (5) can be expanded in the Taylor series in ζ in the vicinity of symmetry axis z, obtaining this way the expression

$$\varepsilon(z,\zeta)=\varepsilon(\zeta=0)+\left(\frac{\partial\varepsilon}{\partial\zeta}\Big|_{\zeta=0}\right)\zeta+\left(\frac{\partial^2\varepsilon}{\partial\zeta^2}\Big|_{\zeta=0}\right)\frac{\zeta^2}{2}. \tag{6}$$

Substituting (2) and (6) into eikonal Eq. (5) and comparing coefficients of ζ^0, ζ and ζ^2 we obtain relations

$$\varepsilon(\zeta=0)=\varepsilon_0,\quad \frac{\partial\varepsilon}{\partial\zeta}\Big|_{\zeta=0}=0 \tag{7}$$

and the Riccati equation for complex curvature B:

$$\sqrt{\varepsilon_0}\frac{\mathrm{d}B}{\mathrm{d}z}+B^2=\gamma. \tag{8}$$

Parameter γ for axially symmetric medium equals

$$\gamma=\frac{1}{2}\frac{\partial^2\varepsilon}{\partial\zeta^2}\Big|_{\zeta=0}. \tag{9}$$

Substituting (3) into Eq. (1), we obtain the Gaussian beam of the form

$$u(\zeta,z)=A(z)\exp\left(-\frac{\zeta^2}{2w^2}\right)\times\exp\left(\mathrm{i}k_0\left(\sqrt{\varepsilon_0}z+\kappa\frac{\zeta^2}{2}\right)\right). \tag{10}$$

Solution (10) reflects the general feature of CGO, which in fact deals with the Gaussian beams.

2.2. The equation for GB complex amplitude

In the framework of paraxial approximation the amplitude $A=A(z)$ is complex-valued within CGO method and satisfies the transport equation

$$\mathrm{div}(A^2\mathrm{grad}\,\psi)=0, \tag{11}$$

which for axially symmetric beam in (ζ, z) coordinates takes the following form:

$$\frac{\mathrm{d}A^2}{\mathrm{d}z}\frac{\partial\psi}{\partial z}+\left[\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial\psi}{\partial\zeta}\right)+\frac{\partial^2\psi}{\partial z^2}\right]A^2=0. \tag{12}$$

In accordance with Eq. (2), assuming that ζ is small parameter and assuming that first derivative $\frac{\mathrm{d}B}{\mathrm{d}z}$ is limited we obtain that $\frac{1}{2}\frac{\mathrm{d}B}{\mathrm{d}z}\zeta^2\ll\sqrt{\varepsilon_0}$. Based on above assumption we obtain that

$$\frac{\partial\psi}{\partial z}\cong\sqrt{\varepsilon_0},\quad \frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial\psi}{\partial\zeta}\right)=2B\ \text{ and }\ \frac{\partial^2\psi}{\partial z^2}\cong 0. \tag{13}$$

As a result Eq. (12) reduces to the ordinary differential equation in the form

$$\sqrt{\varepsilon_0}\frac{\mathrm{d}A}{\mathrm{d}z}+BA=0. \tag{14}$$

The above equation for GB complex amplitude (14), as well as the Riccati equation for complex curvature B (8) are the basic CGO equations. CGO reduces the problem of GB diffraction to the domain of ordinary differential equations. Having calculated the complex parameter B from Riccati Eq. (8), one can readily determine complex amplitude A by integration of Eq. (14). As a result the complex amplitude of cylindrically symmetric GB takes the form

$$A(z')=A_0\exp\left(-\int_0^{z'}B(z')\,\mathrm{d}z'\right), \tag{15}$$

where $A_0=A(0)$ is an initial amplitude and $z'=z/\sqrt{\varepsilon_0}$.

2.3. The equation for GB width evolution

Riccati Eq. (8) is equivalent to the set of two equations for the real and imaginary parts of the complex curvature B:

$$\begin{cases}\sqrt{\varepsilon_0}\frac{\mathrm{d}B_\mathrm{R}}{\mathrm{d}z}+B_\mathrm{R}^2-B_\mathrm{I}^2=\gamma,\\ \sqrt{\varepsilon_0}\frac{\mathrm{d}B_\mathrm{I}}{\mathrm{d}z}+2B_\mathrm{R}B_\mathrm{I}=0.\end{cases} \tag{16}$$

Substituting (3) into (16), we obtain the expression

$$\sqrt{\varepsilon_0}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{w^2}\right)=-\frac{2\kappa}{w^2}, \tag{17}$$

which leads to the known relation between the beam width w and the wave front curvature κ in the form [12]:

$$\kappa=\sqrt{\varepsilon_0}\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}z}. \tag{18}$$

Substituting now relation (18) into the first equation of the system (16), we obtain the ordinary differential equation of the second order for GB width evolution

$$\varepsilon_0\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-\gamma w=\frac{1}{k_0^2w^3}. \tag{19}$$

3. CGO of inhomogeneous and arbitrary nonlinear media

In this section, the CGO method is applied for beam propagation in inhomogeneous and cylindrically symmetric nonlinear medium with permittivity profile of the form

$$\varepsilon=\varepsilon_\mathrm{LIN}+\varepsilon_\mathrm{NL}g_\mathrm{I}(I), \tag{20}$$

where ε_NL is nonlinear coefficient. Introducing the parameter of characteristic inhomogeneity scale L we can present linear part of the medium permittivity as equal to

$$\varepsilon_\mathrm{LIN}=\varepsilon_0\pm\frac{\zeta^2}{L^2}. \tag{21}$$

In Eq. (20) g_I is an arbitrary function of the beam intensity $I=\frac{c}{4\pi}uu^*$ and ζ is a distance from the fibre axis. The beam intensity for wave function in Eq. (1) taking into account also Eq. (3) takes the form

$$I=\frac{c}{4\pi}|A(z)|^2\exp\left(-\frac{\zeta^2}{2w^2}\right). \tag{22}$$

One can notice that

$$\varepsilon(\zeta=0)=\varepsilon_\mathrm{LIN}(\zeta=0)+\varepsilon_\mathrm{NL}g_\mathrm{I}[I]_{\zeta=0}=\varepsilon_0+\varepsilon_\mathrm{NL}g_\mathrm{I}\left[\frac{c}{4\pi}|A(z)|^2\right]. \tag{23}$$

In accordance with CGO method boundary applicability [9, 10] the following condition must be satisfied $\varepsilon_\mathrm{NL}g_\mathrm{I}\left[\frac{c}{4\pi}|A(z)|^2\right]\ll\varepsilon_0$ and resulting in first condition in Eq. (7) is satisfied. One can notice that first derivative

$$\frac{\partial\varepsilon}{\partial\zeta}=\frac{\partial\varepsilon_\mathrm{LIN}}{\partial\zeta}-\varepsilon_\mathrm{NL}\frac{\mathrm{d}g_\mathrm{I}}{\mathrm{d}I}\frac{c}{4\pi w^2}|A(z)|^2\exp\left(-\frac{\zeta^2}{2w^2}\right)\zeta \tag{24}$$

is equal to zero when $\zeta=0$. Thus, the second condition in Eq. (7) is also satisfied. The characteristic inhomogeneity scale of the fibre mentioned above is related with fibre core radius r_c by the relation $L=r_\mathrm{c}/\delta$, where δ is the difference of the constant refractive indexes between core and cladding and ε_0 is permittivity along symmetry axis. In Eq. (21) positive sign in the expression corresponds to linear defocusing refraction whereas negative sign describes focusing inhomogeneity of the fibre. For the permittivity in Eq. (20) and Eq. (21) the Riccati Eq. (8) can be presented as

$$\frac{\mathrm{d}B}{\mathrm{d}z'} + B^2 = \gamma_{\mathrm{LIN}} + \gamma_{\mathrm{NL}}, \tag{25}$$

and equation for GB width evolution takes the following form:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z'^2} - (\gamma_{\mathrm{LIN}} + \gamma_{\mathrm{NL}})\, w = \frac{1}{k_0^2 w^3}, \tag{26}$$

where

$$\gamma_{\mathrm{LIN}} = \frac{1}{2}\frac{\partial^2 \varepsilon_{\mathrm{LIN}}}{\partial \zeta^2}|_{\zeta=0} = \pm\frac{1}{L^2} \tag{27}$$

and

$$\gamma_{\mathrm{NL}} = \frac{\varepsilon_{\mathrm{NL}}}{2}\left[\frac{\mathrm{d}^2 g_{\mathrm{I}}}{\mathrm{d}I^2}\left(\frac{\partial I}{\partial \zeta}\right)^2 + \frac{\mathrm{d}g_{\mathrm{I}}}{\mathrm{d}I}\frac{\partial^2 I}{\partial \zeta^2}\right]\Bigg|_{\zeta=0}. \tag{28}$$

4. Solutions for GB propagation in saturable nonlinear fibres

Let us consider now axially symmetric medium with the permittivity

$$\varepsilon = \varepsilon_0 \pm \frac{\zeta^2}{L^2} + \frac{\varepsilon_{\mathrm{NL}} I}{1 + \varepsilon_{\mathrm{NL}} I/\varepsilon_{\mathrm{s}}}, \tag{29}$$

where ε_{s} denotes saturating permittivity. The permittivity profile in Eq. (29) models nonlinear optical fibres [20], which for low intensities $I \to 0$ has the Kerr type profile

$$\varepsilon = \varepsilon_0 \pm \frac{\zeta^2}{L^2} + \varepsilon_{\mathrm{NL}} I \tag{30}$$

and which saturates for $I \to \infty$, resulting in

$$\varepsilon = \varepsilon_0 \pm \frac{\zeta^2}{L^2} + \varepsilon_{\mathrm{s}}. \tag{31}$$

For permittivity in Eq. (29) Riccati equation and equation for GB width evolution take the form

$$\frac{\mathrm{d}B}{\mathrm{d}z'} + B^2 = \pm\frac{1}{L^2} - \frac{\varepsilon_{\mathrm{NL}}\left|A_0\right|^2 w_0^2}{\left(w^2 + \varepsilon_{\mathrm{NL}}\left|A_0\right|^2 w_0^2/\varepsilon_{\mathrm{s}}\right)^2}, \tag{32}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z'^2} + \frac{\varepsilon_{\mathrm{NL}}\left|A_0\right|^2 w_0^2 w}{\left(w^2 + \varepsilon_{\mathrm{NL}}\left|A_0\right|^2 w_0^2/\varepsilon_{\mathrm{s}}\right)^2} \pm \frac{w}{L^2} = \frac{1}{k_0^2 w^3}. \tag{33}$$

Introducing next dimensionless width of GB $f = w/w_0$, Eq. (33) can be presented in the form

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z'^2} + \frac{\varepsilon_{\mathrm{NL}}\left|A_0\right|^2 f}{w_0^2\,(f^2+p)^2} \pm \frac{f}{L^2} = \frac{1}{L_{\mathrm{D}}^2 f^3}, \tag{34}$$

where $L_{\mathrm{D}} = k_0 w_0^2$ is diffraction distance and $p = \varepsilon_{\mathrm{NL}}|A_0|^2/\varepsilon_{\mathrm{s}}$ is nonlinear parameter, which is a measure of the value of nonlinear part of fibre permittivity relative to saturating one. The first integral of Eq. (34) takes the form

$$\frac{1}{2}\left(\frac{\mathrm{d}f}{\mathrm{d}z'}\right)^2 - \frac{\varepsilon_{\mathrm{NL}}\left|A_0\right|^2}{2w_0^2\,(f^2+p)} \pm \frac{f^2}{2L^2} + \frac{1}{2L_{\mathrm{D}}^2 f^2} = C. \tag{35}$$

In accordance with Eq. (18) the value $\mathrm{d}f/\mathrm{d}z'$ at $z' = 0$ presents the squared initial wave front curvature

$$\left(\frac{\mathrm{d}f}{\mathrm{d}z'}\right)^2\Big|_{z'=0} = \frac{1}{w^2(0)}\left(\frac{\mathrm{d}w}{\mathrm{d}z'}\right)^2\Big|_{z'=0} = \kappa_0^2. \tag{36}$$

Thus Eq. (35) takes the following form:

$$f^2\left(\frac{\mathrm{d}f}{\mathrm{d}z'}\right)^2 - \frac{\varepsilon_{\mathrm{NL}}\left|A_0\right|^2 f^2}{w_0^2\,(f^2+p)} \pm \frac{f^4}{L^2} + \frac{1}{L_{\mathrm{D}}^2} = \left(\kappa_0^2 - \frac{\varepsilon_{\mathrm{NL}}\left|A_0\right|^2}{w_0^2(1+p)} \pm \frac{1}{L^2} + \frac{1}{L_{\mathrm{D}}^2}\right) f^2. \tag{37}$$

Taking advantage of differential relation $[(f^2)']^2 = 4f^2 f'^2$ to above equation, where $F = f^2$, one obtains

$$\frac{1}{4}\left(\frac{\mathrm{d}F}{\mathrm{d}z'}\right)^2 - \frac{\varepsilon_{\mathrm{NL}}\left|A_0\right|^2 F}{w_0^2(F+p)} \pm \frac{F^2}{L^2} + \frac{1}{L_{\mathrm{D}}^2} = \left(\kappa_0^2 - \frac{\varepsilon_{\mathrm{NL}}\left|A_0\right|^2}{w_0^2(1+p)} \pm \frac{1}{L^2} + \frac{1}{L_{\mathrm{D}}^2}\right) F \tag{38}$$

and differentiating once Eq. (38) we obtain mathematically simple equation which is useful especially for numerical simulations and has the form

$$\frac{\mathrm{d}^2 F}{\mathrm{d}z'^2} - \frac{2\varepsilon_{\mathrm{NL}}\left|A_0\right|^2 p}{w_0^2(F+p)^2} \pm \frac{4F}{L^2} = 2\left(\kappa_0^2 - \frac{\varepsilon_{\mathrm{NL}}\left|A_0\right|^2}{w_0^2(1+p)} \pm \frac{1}{L^2} + \frac{1}{L_{\mathrm{D}}^2}\right). \tag{39}$$

When $\varepsilon_{\mathrm{NL}}|A_0|^2 \ll \varepsilon_{\mathrm{s}}$, then p is small parameter ($p \to 0$) and as a result the above Eq. (39) reduces to the form

$$\frac{\mathrm{d}^2 F}{\mathrm{d}z'^2} \pm \frac{4F}{L^2} = 2\left(\kappa_0^2 - \frac{\varepsilon_{\mathrm{NL}}\left|A_0\right|^2}{w_0^2} \pm \frac{1}{L^2} + \frac{1}{L_{\mathrm{D}}^2}\right). \tag{40}$$

Noticing next that parameter $w_0^2/\varepsilon_{\mathrm{NL}}|A_0|^2$ denotes nonlinear length L_{NL}, above Eq. (40) has the following analytical solution for the case of nonlinear Kerr type fibre with linear defocusing refractive profile described by the factors $-4F/L^2$ and $-1/L^2$ in Eq. (40)

$$f^2 = \cosh^2\left(\frac{z'}{L}\right) + \left(\kappa_0^2 L^2 - \frac{L^2}{L_{\mathrm{NL}}^2} + \frac{L^2}{L_{\mathrm{D}}^2}\right)\sinh^2\left(\frac{z'}{L}\right) + \kappa_0 L \sinh\left(\frac{2z'}{L}\right). \tag{41}$$

For the nonlinear self-focusing fibre with focusing linear refractive profile (the positive factors $4F/L^2$ and $1/L^2$ in Eq. (40)) the solution has the form

$$f^2 = \cos^2\left(\frac{z'}{L}\right) + \left(\kappa_0^2 L^2 - \frac{L^2}{L_{\mathrm{NL}}^2} + \frac{L^2}{L_{\mathrm{D}}^2}\right)\sin^2\left(\frac{z'}{L}\right) + \kappa_0 L \sin\left(\frac{2z'}{L}\right). \tag{42}$$

Numerical analysis of Eq. (39), beyond the case $p \to 0$ shows that for focusing permittivity profile, assuming that $\kappa_0 = 0$ we obtain three types of solutions presented in Fig. 1: oscillatory self-focusing (trace 1), stationary solution (trace 2) and oscillatory diffracting (trace 3).

In Fig. 2 it is shown that due to the presence of linear defocusing refraction the new type of solution appears

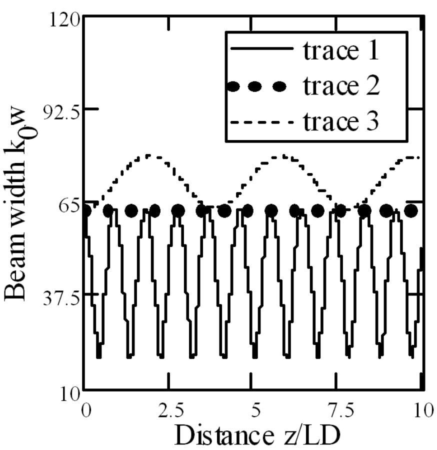

Fig. 1. Numerical solutions of Eq. (39) for the focusing refraction and parameters $w_0 = 10\lambda$, $\varepsilon_{\rm NL}|A_0|^2 = 0.0001$, $p = 10^{-4}$ and trace 1: $L = 0.3L_{\rm D}$, trace 2: $L = 1.285L_{\rm D}$, trace 3: $L = 2L_{\rm D}$.

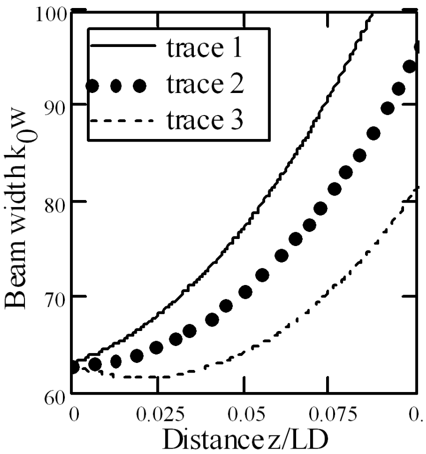

Fig. 2. Numerical solutions of Eq. (39) for the defocusing refraction and parameters $w_0 = 10\lambda$, $\varepsilon_{\rm NL}|A_0|^2 = 0.001$, $p = 10^{-3}$ and $L = 0.1L_{\rm D}$ and trace 1: $\kappa_0 = 2/L_{\rm D}$, trace 2: $\kappa_0 = 0$, trace 3: $\kappa_0 = -2/L_{\rm D}$.

where GB width increases monotonously and diffraction widening process is enhanced by linear defocusing refractive profile.

In Fig. 3 we notice that increase of total beam power makes that self-focusing effect becomes strong enough to overcome diffraction widening and linear defocusing processes. As a result GB width oscillates though initially dominates diffraction widening and defocusing refraction effect.

In Fig. 2 and Fig. 3 the influence of initial wave front curvature on GB width evolution is depicted. One can notice that when the initial wave front curvature is negative $\kappa_0 < 0$, the GB width first decreases approaching to minimum value and for the positive value of this parameter the GB width at once increases.

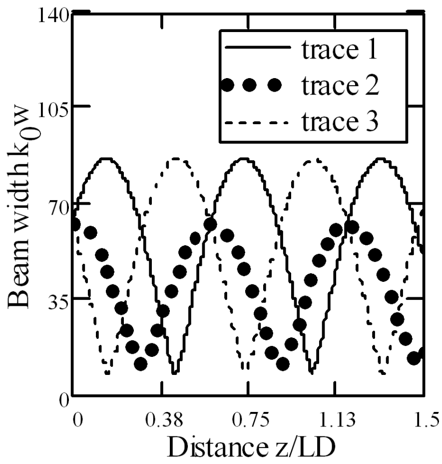

Fig. 3. Numerical solutions of Eq. (39) for the defocusing refraction and parameters $w_0 = 10\lambda$, $\varepsilon_{\rm NL}|A_0|^2 = 0.01$, $p = 10^{-2}$ and $L = 0.3L_{\rm D}$ and trace 1: $\kappa_0 = 5/L_{\rm D}$, trace 2: $\kappa_0 = 0$, trace 3: $\kappa_0 = -5/L_{\rm D}$.

5. Conditions for uniform waveguide in the presence of linear refraction

Demanding for the beam to propagate as a stationary mode when $F = f^2 = 1$ and $\kappa_0 = 0$, Eq. (39) takes the form

$$-\frac{2\varepsilon_{\rm NL}\left|A_0\right|^2 p}{w_0^2(1+p)^2} \pm \frac{4}{L^2} = 2\left(\frac{1}{L_{\rm D}^2} - \frac{\varepsilon_{\rm NL}\left|A_0\right|^2}{w_0^2(1+p)} \pm \frac{1}{L^2}\right). \tag{43}$$

After simple modifications we obtain the expression

$$W^2 = \frac{1}{\frac{p}{(1+p)^2} \pm \frac{w_0^2}{\varepsilon_{\rm s}L^2}}, \tag{44}$$

where $W^2 = \varepsilon_{\rm s}4\pi^2 w_0^2/\lambda^2$ denotes dimensionless radius of the waveguide. The parameter W depends on nonlinear parameter p which is shown for different values of fibre inhomogeneity scale L and different initial beam widths w_0 in Fig. 4.

In Fig. 5 it is shown how dimensionless radius of the waveguide depends on fibre inhomogeneity scale L for different values of nonlinear parameter p in the case of linear focusing refraction and in Fig. 6 in the presence of linear defocusing refraction. One can notice that radius of the waveguide is minimum $W_{\rm min}$ when the parameter p is equal to unity $p = 1$. It takes place when $\varepsilon_{\rm NL}|A_0|^2 = \varepsilon_{\rm s}$. As a result $W_{\rm min}$ takes the form

$$W_{\rm min} = 2\bigg/\sqrt{1 \pm \frac{4w_0^2}{\varepsilon_{\rm s}L^2}}. \tag{45}$$

One can notice that when the contribution of linear refraction is insignificant, then the minimum radius of the waveguide approaches to constant value $W_{\rm min}(L \to \infty) = 2$. In the case of focusing refraction (positive sign in Eq. (44)) the minimum radius of the waveguide becomes smaller comparing to the case $W_{\rm min}(L \to \infty) = 2$. In the case of defocusing linear process (negative sign in

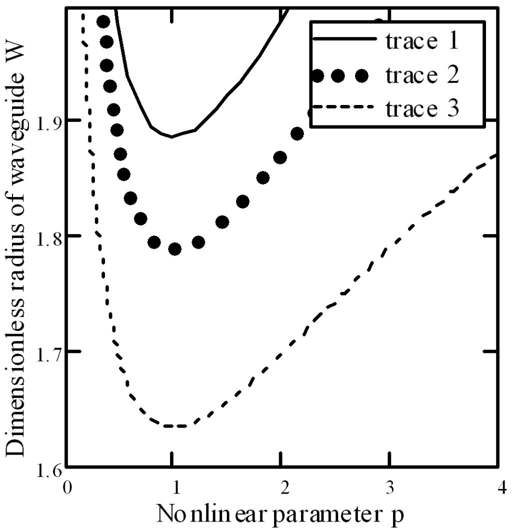

Fig. 4. Dependence of the dimensionless radius of the waveguide W on nonlinear parameter p in the case of linear focusing refraction for parameters: $w_0^2/\varepsilon_s L^2 = 1/32$ (trace 1), $w_0^2/\varepsilon_s L^2 = 1/16$ (trace 2), $w_0^2/\varepsilon_s L^2 = 1/8$ (trace 3).

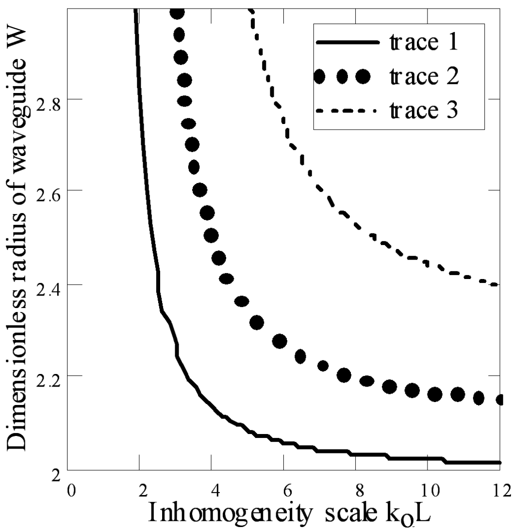

Fig. 6. Dependence of the dimensionless radius of the waveguide W on fibre inhomogeneity scale L in the case of defocusing refraction for the parameters $k_0^2 w_0^2/\varepsilon_s = 1/2$ and $p = 1$ (trace 1), $k_0^2 w_0^2/\varepsilon_s = 1$ and $p = 2$ (trace 2), $k_0^2 w_0^2/\varepsilon_s = 2$ and $p = 3$ (trace 3).

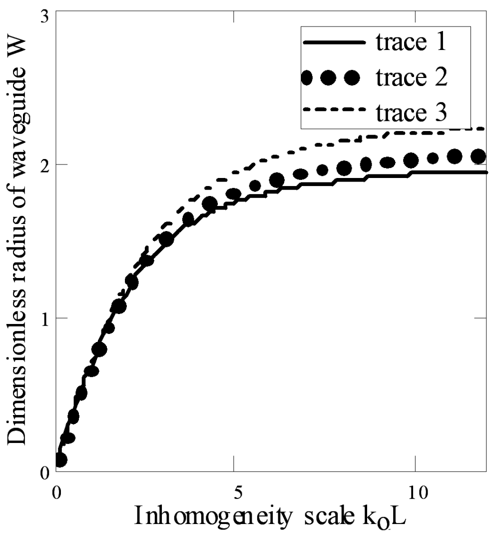

Fig. 5. Dependence of the dimensionless radius of the waveguide W on fibre inhomogeneity scale L in the case of linear focusing refraction for the parameters $k_0^2 w_0^2/\varepsilon_s = 2$ and $p = 1$ (trace 1), $k_0^2 w_0^2/\varepsilon_s = 2$ and $p = 2$ (trace 2), $k_0^2 w_0^2/\varepsilon_s = 2$ and $p = 3$ (trace 3).

Eq. (44)) this radius W becomes greater than $W_{\min}(L \to \infty) = 2$.

In Fig. 5 one can notice that for the focusing refraction the dimensionless radius of the waveguide W increases for small values of inhomogeneity scale and next approaches to constant value. In Fig. 6 one can distinguish that for the defocusing refraction the dimensionless radius of the waveguide W initially decreases and becomes constant, when the parameter L becomes greater. Let us calculate also the power of the beam in uniform waveguide. One can notice that standard form of total beam power is equal to [10]:

$$P = \frac{1}{8} c \sqrt{\varepsilon_0} w_0^2 \left|A_0\right|^2. \tag{46}$$

Taking into account the definitions of nonlinear parameter $p = \varepsilon_{\mathrm{NL}} |A_0|^2/\varepsilon_s$ and dimensionless radius of the waveguide $W^2 = \varepsilon_s 4\pi^2 w_0^2/\lambda^2$, we can calculate the power of the beam in uniform waveguide P_{UW}, which is equal to

$$P_{\mathrm{UW}} = \frac{c\sqrt{\varepsilon_0}\lambda^2 W^2 p}{32\pi^2 \varepsilon_{\mathrm{NL}}}. \tag{47}$$

6. Conclusion

The paper applies the method of CGO to the analysis of the GB evolution in smoothly inhomogeneous and nonlinear saturable media of cylindrical symmetry. The CGO method reduces diffraction and self-focusing problems for the Gaussian beam to a solution of ordinary differential equations, describing behaviour of the amplitude, the beam width, and the curvature of the wave front. CGO method readily provides a solution for inhomogeneous nonlinear saturable fibre in a simpler way than the standard methods of nonlinear optics such as: the variation method approach, method of moments and beam propagation method. Following analogously like in papers [16–20] we model the light propagation in nonlinear fibres by Gaussian beam, which is self-sustained solution within CGO method. Besides simplicity and affectivity CGO method supplies a number of new results.

Firstly, it is shown in the paper how focusing/defocusing refraction influences on GB width evolution in nonlinear saturable fibre.

Secondly, it is presented that the CGO method effectively describes joint influence of the refractive index profile and the initial curvature of the wave front.

It is also shown how linear refraction influences on the conditions of formation of the stationary mode and conditions for minimum radius of the waveguide. This way CGO method demonstrates high ability in further applications of nonlinear graded-index optics in both experimental and theoretical problems.

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