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# Characteristics of Complexity in Selected Economic Models in the Light of Nonlinear Dynamics

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The catastrophe theory and deterministic chaos constitute the basic elements of economic complexity. Elementary catastrophes were the first remarkable form of nonlinear, topological complexity that were thoroughly studied in economics. Another type of catastrophe is the complexity catastrophe, namely an increase in the complexity of a system beyond a certain threshold which marks the beginning of a decrease in a system's adaptive capacity. As far as the ability to survive is concerned, complex adaptive systems should function within the range of optimal complexity which is neither too low or too high. Deterministic chaos and other types of complexity follow from the catastrophe theory. In general, chaos is seemingly random behavior of a deterministic system which stems from its high sensitivity to the initial condition. The theory of nonlinear dynamical systems, which unites various manifestations of complexity into one integrated system, runs contrary to the assumption that markets and economies spontaneously strive for a state of equilibrium. The opposite applies: their complexity seems to grow due to the influence of classical economic laws.

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## 1. Introduction

The definition of complexity proposed by Day is popularly accepted in economic theory [1]. According to Day, an economic system is dynamically complex if its deterministic endogenous processes lead to aperiodic behavior or structural changes. Other types of behavior, such as stationary states, cyclic movements and balanced growth, are classified as simple dynamic phenomena. The studied objects may be described by nonlinear difference or differential equations with an option of incorporating stochastic elements. This definition of complexity is gradually becoming the norm in economics research focusing on nonlinear dynamics [2, 3]. This definition is relatively broad and open, and it may be used in analyses of different forms of complexity [4, 5].

## 2. Cusp catastrophe and chaotic hysteresis in economic transformation model

One of the greatest challenges faced by the global economy is the transformation of centrally planned economies into market economies. The reforms initiated in 1989 are still under way, but we can attempt to formulate a theory of system transformation already today. Our main assumption should be rooted in the existence of common and shared market principles regardless of the degree of variation encountered across different economies. Such an approach is required to develop a general theory of system transformation. The first step in the process involves the construction of a chaotic hysteresis model [6, 7]. At the same time, we have an application of two basic methods of the theory of nonlinear dynamical systems: elementary catastrophes and deterministic chaos. The starting point is a socialist economy. According to

the Marxist convention, economy was divided into two sectors: the consumer goods sector and the capital goods sector. The notions of technological gap and cusp catastrophe were used to describe social and economic crises. The attractor in the form of chaotic hysteresis that appears in a reformed economy results from the activity of a two-stage nonlinear accelerator. A nonlinear version of the multiplier-accelerator model formulated by Puu has been used [8, 9].

The economic system is described by the following equations:

$$I_t = I_t^C + I_t^K, \quad (1)$$

$$I_t^C = (1 - v)I_{t-1}, \quad (2)$$

$$I_t^K = u(I_{t-1} - I_{t-2}) - u(I_{t-1} - I_{t-2})^3, \quad (3)$$

where  $I_t$  — total investment in time  $t$ ,  $I_t^C$  — investment in the consumer goods sector in time  $t$ ,  $I_t^K$  — investment in the capital goods sector in time  $t$ ,  $u$  — accelerator coefficient in the capital goods sector,  $v$  — accelerator coefficient in the consumer goods sector.

When a new variable is introduced to represent the increment in total investment,

$$Z_t = I_t - I_{t-1}, \quad (4)$$

the model (1)–(3) is reduced to a two-dimensional map with the following form:

$$I_t = I_{t-1} + Z_t, \quad (5)$$

$$Z_t = u(Z_{t-1} - Z_{t-1}^3) - vI_{t-1}. \quad (6)$$

These formulae cannot be solved analytically, but they can be subjected to numerical explorations.

The next element of the theory is the technological gap ( $G$ ) which stems from a higher rate of the capital-intensive nature of production in socialism in comparison with a capitalist economy. Paradoxically, this phe-

nomenon reflects postulates the production stability and full employment which were to make socialism a more bearable system than capitalism with its chronic unemployment and crises. The technological gap can be given by

$$G(T) = \int_{t=0}^T \Phi [D(t)] dt = \frac{Y_m(t)}{K_m(t)} - \frac{Y_s(t)}{K_s(t)}, \quad (7)$$

on the assumption that

$$\frac{\partial \Phi [D(t)]}{\partial D(t)} > 0, \quad (8)$$

where  $D$  — percent of output controlled by the central planner,  $Y$  — output,  $K$  — capital stock, and subscripts  $m$  and  $s$  indicate market capitalism and command socialism, respectively.

In the following step, we introduce the cusp catastrophe whose equilibrium surface meets condition

$$M_3 = \left[ (c_1, c_2, x) : \frac{df}{dx} = 0, \frac{d^2f}{dx^2} = x^3 + c_1x + c_2 \right]. \quad (9)$$

The potential function has the following form:

$$f : \mathbf{R}^2 \times \mathbf{R}^1 \rightarrow \mathbf{R}. \quad (10)$$

The function (10) has a simple multinomial representation:

$$f(c_1, c_2, x) = \frac{1}{4}x^4 + \frac{1}{2}c_1x^2 + c_2x, \quad (11)$$

where  $x$  represents the state variable, and  $c_1, c_2$  are the control parameters [10].

In the investigated theory, the state variable is the probability that market reforms will be introduced  $x = P(m)$ , the bifurcation parameter is the size of the technological gap  $c_1 = G$ , whereas the asymmetric parameter is the rate of investment growth  $c_2 = Z/I$ . The reforms involve a reduction of accelerator value in consumption, which implies a higher volume of investments in the consumer goods sector.

The exploration of the system in (5), (6) was performed on the assumption that the constant value of the accelerator in the capital goods sector is  $u = 2$ , whereas the value of parameter  $v$  was gradually decreased. For  $0.01 \leq v \leq 0.1395$  in the phase space of the system, there is an investment cycle in the form of a chaotic attractor. The attractor is transformed when the value of the accelerator in the consumer goods sector is lowered, and it ultimately takes on the form of chaotic hysteresis at  $v = 0.00005$ . The above points to chaos inside the cycle. During the transformation, the system's complexity is reduced, as illustrated by a decrease in the capacity dimension, information dimension and correlation dimension of subsequent chaotic attractors [11–13]. Chaos is gradually eliminated from the system. This is accompanied by larger-amplitude and longer-period oscillations in investments. The discussed process poses a difficult challenge for economic policy-makers. A reduction of complexity increases instability, which is manifested by large-amplitude and long-period oscillations, whereas decreased instability leads to greater complexity. Those correlations represent a trade-off effect between complex-

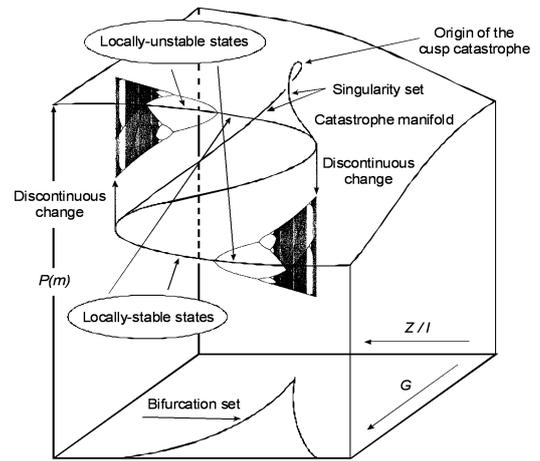


Fig. 1. A graphic interpretation of the chaotic hysteresis model combines the fundamentals of the catastrophe theory and the deterministic chaos theory.

ity and instability. It implies that the definitions of the two terms do not have to overlap in economic theory.

The graphic interpretation of a system transformation model in Fig. 1 combines the key modeled components: an attractor in the form of chaotic hysteresis, cusp catastrophe and technological gap. The equilibrium surface of a catastrophe describes various scenarios of economic crises and the corresponding reforms that were implemented as a remedy. For  $G = 0$  we have an exemplary market economy. The presence of the technological gap, which occurs after passage through the origin of a catastrophe, dissects the equilibrium surface into two layers — the upper layer and the bottom layer. Those layers point to the presence of discontinuous changes in the probability of transformation whenever the rate of investment growth reaches the necessary value. Sudden jumps take place when the asymmetric factor crosses the bifurcation set of a catastrophe in parameter space which is given by the following formula:

$$B_3 = [(c_1, c_2) : 4c_1^3 + 27c_2^2 = 0]. \quad (12)$$

The bifurcation set is a projection of the singularity set

$$S_3 = \left[ (c_1, c_2, x) : \frac{d^2f}{dx^2} = 0, \frac{d^2f}{dx^2} = 3x^2 + c_1 = 0 \right] \quad (13)$$

onto the parameter space.

A centrally planned economy can initially achieve a certain level of stability at the lower layer of the catastrophe manifold. The size of the technological gap acts as a stimulus for change. A large technological gap cannot be tolerated by central planners because it poses a considerable threat for the system. The reduction of the technology gap generally requires the implementation of new techniques and, consequently, higher investments. A jump to the upper layer of the catastrophe manifold can lead to a short period of economic stability. Socialist examples have demonstrated that this situation can easily lead to overinvestment, which would imply a return to

the lower layer. A transition would not take place along the same path because the model is based on the perfect delay convention. When the trajectory runs along a given equilibrium surface, singularity is encountered where one layer of the surface ends, and a new layer begins. A dynamic system is maintained on each layer for as long as possible, and it leaves a given layer at the last moment. This indicates that an economy can cyclically jump between the layers, and the resulting trajectory forms a loop, as demonstrated by the arrows.

In the model, cyclic movement along the hysteresis loop involves chaotic dynamics. Changes in the system occur clockwise. Locally stable and locally unstable behaviors are found on every layer. Stable behaviors are maintained over a given period of time, after which they disappear suddenly. Chaotic dynamics appears after each jump, and the system runs along the part of the chaotic attractor that resembles a period-halving bifurcation sequence. This is followed by a period of stability, but at this point, the system occupies a different layer. Global dynamics is chaotic, but stability is possible at the local level.

Hysteresis is reinforced with an increase in the technological gap. A significant role in this process is played by production factors, mainly labor, which are carriers of system memory. They contribute elements of the old system to a transformed economy, thus decreasing its flexibility, they transfer and reinforce technological backwardness and become a source of the hysteresis effect. Labor plays a key role in explaining the transformation process. It is a carrier of old habits epitomized by the “they pretend they are paying us, we pretend we are working” saying and demanding attitudes. The model thus shows that the implementation of the institutional foundations of a market economy in former socialist countries in itself does not guarantee the success of reforms. The above indicates that the transformation period can cover several generations of entrepreneurs and consumers. The model seems to predict that, because the performed transformations expand the modeled time horizon.

Numerical explorations of model (5), (6) shed new light onto a macroeconomic problem which is neglected by mainstream economics, namely the macroeconomic costs of reform complexity. An intuitive understanding of this cost category follows from business theory [14]. The global financial crisis prompted a broader analysis of the complexity of economic processes and the accompanying problems [15]. The economy under transformation is at the risk of falling victim to the trade-off between complexity and instability, which accounts for the fact that the profits generated by the reforms can, over a long period of time, be lower than the costs of complexity. It is a new quality-based position in the transformation balance. Future research is required to investigate the relevant measurement methods. The above poses a challenge for economic policy which should attempt to simplify economic life already today.

### 3. Complexity catastrophe in Simonovits' model of socialist economy

The complexity catastrophe concept was introduced into the realm of science by Kauffman [16]. A simple system consisting of several segments and several correlations between them has a weak adaptive ability since the number of reached states is much lower than the number of situations which should be faced. However, the space in which the system can evolve has not only a lower but also a higher limit. The increase in complexity beyond a given limit reduces selective pressure, consequently, the adaptive ability of such a system decreases sharply. This phenomenon is referred to as the complexity catastrophe. In such a situation, the desired changes in selected parts of the system bring about unwanted results in other segments.

Simonovits' model of a socialist economy has the form of a two-dimensional piecewise linear map [17–19]:

$$\begin{aligned} e_t &= f e_{t-1} + \sigma_s s(e_{t-1}, a_{t-1}) - i(e_{t-1}) - \varepsilon_0, \\ \varepsilon_0 &= (1 - f)k^*, \end{aligned} \quad (14)$$

$$a_t = \beta_i i(e_{t-1}) - \beta_0, \quad \beta_0 = \beta + b^*, \quad (15)$$

where

$$s(e_{t-1}, a_{t-1}) = \begin{cases} s^l \\ s_t^p = \sigma - \sigma_e e_{t-1} - \sigma_a a_{t-1} \\ s^u \end{cases}, \quad (16)$$

$$i(e_{t-1}) = \begin{cases} i^l \\ i_t^p = i + i_e e_{t-1} \\ i^u \end{cases}. \quad (17)$$

The state variables are  $e$  — internal tension,  $a$  — external tension. The remaining symbols represent the parameters. Parameter space is 15-dimensional. The values of parameters have been estimated by means of econometric methods based on Hungarian economic data [20, 21].

The complexity catastrophe is a basic dynamic change existing in the system (14)–(17). For market participants, the above implies sudden and unforeseeable changes in economic complexity. The source of complexity catastrophes are border-collision bifurcations which result from Hicksian nonlinearities [22]. Border-collision bifurcations are related to closed invariant sets which come into contact with the border of a region defined in the map. Figure 2 presents complexity catastrophes in bifurcation diagrams. In Fig. 2a, at least ten border-collision bifurcations can be identified for the growing bifurcation parameter: fixed point attractor → chaotic attractor, chaotic attractor → period 5, chaotic attractor → period 4, period 4 → period 12, period 12 → chaotic attractor, period 5 → period 15, period 9 → chaotic attractor, period 8 → chaotic attractor, chaotic attractor → period 18, period 20 → period 5. Some bifurcations can be seen only when specific fragments of the image are enlarged. The change in the largest Lyapunov exponent ( $\lambda_1$ ) in the bifurcation diagram in Fig. 2b confirms the above conclusions concerning dynamic changes

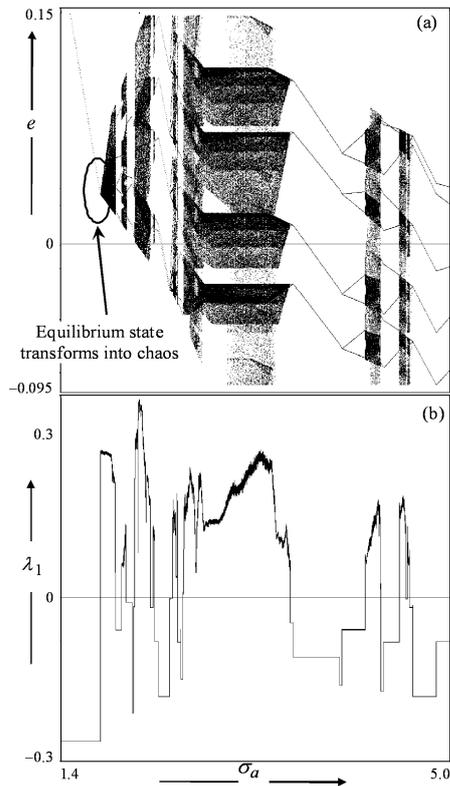


Fig. 2. Examples of complexity catastrophes in Simonovits' model: (a) a bifurcation diagram of internal tension; (b) the largest Lyapunov exponent.

in the system. Similar results were reported by Nusse and Yorke in another piecewise linear map [23].

#### 4. Progressing complexity in economic systems

Market structures are subject to the effects of a new economic law which will be referred to as progressing complexity [24]. According to this law, most microeconomic systems naturally head for a state known as the edge of chaos. The dynamics of complex market structures is shaped by two forces. The effects of their activity are spread over various periods, although both forces have the same source. The powers that guarantee equilibrium over a short period of time are at the same time the source of complex behavior in the long run. A short period is a time span during which the structural parameters of the model are constant. A long period is a time span during which parameters can change. The first force is effective over a short period, and it can lead markets towards states of stable equilibrium. The second force remains in effect over a long period, and it makes the system drift towards a distinct non-equilibrium state referred to as the edge of chaos.

Let us consider the Cournot–Puu duopoly model [25–27]. The basic version of this model takes on the following form:

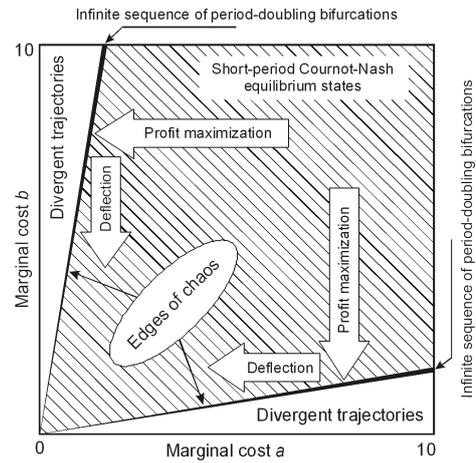


Fig. 3. Dynamics of the Cournot–Puu duopoly in the parameter space.

$$x_{t+1} = \sqrt{\frac{y_t}{a}} - y_t, \tag{18}$$

$$y_{t+1} = \sqrt{\frac{x_t}{b}} - x_t, \tag{19}$$

where  $x$  and  $y$  represent production volumes of the first and second entrepreneur, respectively, whereas marginal costs of duopolists are marked as  $a$  and  $b$ .

The dynamics of a duopoly in parameter space is presented in Fig. 3. The shaded area defines points of stable equilibrium within a short period of time which are determined by the points of intersection of reaction curves [28]. The stability area is bound by two half-lines which determine the beginning of two edges of chaos in the parameter space. The duopoly is deprived of stability when those half-lines are exceeded. White areas represent divergent trajectories. The model loses its logical coherence at those marginal cost values. Two edges of chaos, marked in black, are located between the areas of stability and divergence, and an infinite sequence of period-doubling bifurcations which lead to chaos exists in each edge. The pairs of parameters which are responsible for equilibrium have a 82.77% share of parameter space, whereas the pairs of chaotic parameters account for only 0.15% of that space. It could seem that stability is the predominant state, and thus conventional economic theories have been confirmed. This conclusion is, however, false.

Businesses aim to maximize their profits in both the short and the long term. In the long term, profit maximization requires technical and organizational development which is manifested by a decrease in marginal cost. In the diagram, this is represented by every entrepreneur moving towards one edge of chaos, i.e. a state of growing complexity [29]. The first producer will aim for edge

$$b = 3 + 2\sqrt{2}, \tag{20}$$

and the second entrepreneur — for edge

$$b = 3 - 2\sqrt{2}, \quad (21)$$

as shown by white arrows. The resultant duopoly will be much more complex if we assume that the distribution of forces on the market will change in such a way that every producer will be able to maintain his competitive advantage over a certain period of time. The resulting dynamics will, however, always feature those two basic movements.

The analyzed dynamical system is characterized by a certain degree of global stability [30]. It is an intelligent system which is oriented towards long-term survival. In this sense, the system can be compared to a living organism. Businesses differ in their marginal costs, which implies that they generate different profits. Let us refer to a producer who generates higher profits as an efficient manufacturer, and his competitor as an inefficient producer. As of the moment the efficient producer achieves the edge of chaos, his long-term profit decreases, and the long-term profit of the inefficient producer begins to grow. This leads to role reversal, and in the diagram, the market bounces off the edge of chaos (Fig. 3). The system displays a certain type of globally rational behavior which contributes to its survival. Interesting changes can be observed when both economic entities resort to reasoning based on a traditional understanding of rational behavior. Clearly, profit maximization does not guarantee success in every situation. Further work is needed to explore the problem in greater detail.

## 5. Conclusions

When an economic system approaches the edge of chaos, its further development can follow one of three paths. Firstly, the system can be trapped in a trade-off relationship between complexity and instability, which means that economic policy aiming to lower complexity increases instability and vice versa. This is exemplified by a chaotic hysteresis model. Secondly, some economic systems are designed in such a way that once they reach the edge of chaos, they are not transformed into complex adaptive systems, but they cross an upper critical level of complexity determined by the complexity catastrophe. This leads to a sudden disappearance of a system's adaptive capabilities and its disintegration. An example of the above is the socialist economy model. Thirdly, the investigated object can be a complex adaptive system. In this case, its emergent properties will be revealed over time, which entails the emergence of ordered collective phenomena. The Cournot–Puu duopoly model is an example of a complex adaptive system.

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