An Analytical Study of Nonlinear Vibrations of Buckled Euler–Bernoulli Beams

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The current research deals with a way of using a new kind of periodic solutions called He's max-min approach for the nonlinear vibration of axially loaded Euler–Bernoulli beams. By applying this technique, the beam's natural frequencies and mode shapes can be easily obtained and a rapidly convergent sequence is obtained during the solution. The effect of vibration amplitude on the non-linear frequency and buckling load is discussed. To verify the results some comparisons are presented between max-min approach results and the exact ones to show the accuracy of this new approach. It has been discovered that the max-min approach does not necessitate small perturbation and is also suitably precise to both linear and nonlinear problems in physics and engineering.

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1. Introduction

Investigating on the dynamic response of beams is one of the most important parts in the design process of structures. Many researchers have addressed the nonlinear vibration behavior of beams, both experimentally and theoretically. Burgreen [1] considered the free vibrations of a simply supported buckled beam theoretically and experimentally. He found out that the natural frequencies of buckled beams depend on the amplitude of vibration. Moon [2] and Holmes and Moon [3] used a single-mode approximation to investigate chaotic motions of buckled beams under external harmonic excitations. Abu–Rayan et al. [4] continue the study on the nonlinear dynamics of a simply supported buckled beam using a single-mode approximation to a principal parametric resonance. Ramu et al. [5] used a single-mode approximation to study the chaotic motion of a simply supported buckled beam. Reynolds and Dowell [6, 7] used multi-mode Galerkin discretization to analyze the chaotic motion of a simply supported buckled beam under a harmonic excitation. They used Melnikov theory in their analysis. Lestari and Hanagud [8] used a single-mode approximation to study the nonlinear vibrations of buckled beams with elastic end constraints. They considered the beam to be subjected simultaneously to axial and lateral loads without first statically buckling the beam. The nonlinear vibration of beams and distributed and continuous systems are governed by linear and nonlinear partial differential equations in space and time. Solving nonlinear partial differential equations analytically is very difficult.

It is very common to simplify the equations of motion by introducing various assumptions which allow for the derivation of manageable governing equations. Some of the simplifying assumptions include neglecting axial inertia [9] and assuming linear curvature [10]. The partial-differential equations are discrete to nonlinear ordinary-differential equations by using the Galerkin approach and then we can apply the direct techniques to solve them analytically in time domain. In recent years, many approximate analytical methods have been proposed for studying nonlinear vibration equations of beams and shells and etc. such as homotopy perturbation [11], energy balance [12, 13], variational approach [14, 15], max-min approach (MMA) [16], iteration perturbation method [17] and other analytical and numerical methods [18–25].

The Adomian decomposition method (ADM) was applied by Lai et al. [26] to obtain an analytical solution for nonlinear vibration of the Euler–Bernoulli beam with different supporting conditions. Naguleswaran [27] developed the work on the changes of cross-section of an Euler–Bernoulli beam resting on elastic end supports. Pirbodaghi et al. [28] presented an analytical expression for geometrically free vibration of the Euler–Bernoulli beam by using homotopy analysis method (HAM). They point out that the amplitude of the vibration has a great effect on the nonlinear frequency and buckling load of the beams. Liu et al. [29] applied He's variational iteration method to assess an analytical solution for an Euler–Bernoulli beam with different supporting conditions. Bayat et al. [30, 31] applied energy balance method and variational approach method to obtain the natural frequency of the nonlinear equation of the Euler–Bernoulli beam.

In this paper we used the Galerkin method for discretization to obtain an ordinary nonlinear differential equation from the governing nonlinear partial differential equation. It was then assumed that only fundamental mode was excited. Finally, max-min approach is compared with other researcher's results. The max-min approach results are accurate and only one iteration leads to high accuracy of solutions for whole domain.

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2. Description of the problem

Consider a straight Euler–Bernoulli beam of length L, a cross-sectional area A, the mass per unit length of the beam m, a moment of inertia I, and a modulus of elasticity E that is subjected to an axial force of magnitude P as shown in Fig. 1.

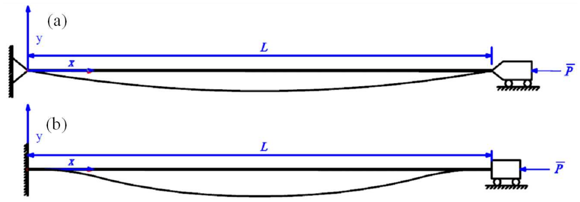

Fig. 1. A schematic of an Euler–Bernoulli beam subjected to an axial load: (a) simply supported beam, (b) clamped-clamped beam.

The equation of motion including the effects of mid-plane stretching is given by

$$m\frac{\partial^2 w'}{\partial t'^2}+EI\frac{\partial^4 w'}{\partial x'^4}+\bar{P}\frac{\partial^2 w'}{\partial x'^2}-\frac{EA}{2L}\frac{\partial^2 w'}{\partial x'^2}\times\int_0^L\left(\frac{\partial^2 w'}{\partial x'}\right)^2\mathrm{d}x'=0. \tag{2.1}$$

For convenience, the following non-dimensional variables are used:

$$x=x'/L,\quad w=w'/\rho,\quad t=t'(EI/ml^4)^{1/2},\qquad P=\bar{P}L^2/EI, \tag{2.2}$$

where $\rho=(I/A)^{1/2}$ is the radius of gyration of the cross-section. As a result Eq. (2.1) can be written as follows:

$$\frac{\partial^2 w}{\partial t^2}+\frac{\partial^4 w}{\partial x^4}+P\frac{\partial^2 w}{\partial x^2}-\frac{1}{2}\frac{\partial^2 w}{\partial x^2}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x=0. \tag{2.3}$$

Assuming $w(x,t)=W(t)\varphi(x)$ where $\varphi(x)$ is the first eigenmode of the beam [32] and applying the Galerkin method, the equation of motion is obtained as follows:

$$\frac{\mathrm{d}^2W(t)}{\mathrm{d}t^2}+(\alpha_1+P\alpha_2)W(t)+\alpha_3W^3(t)=0. \tag{2.4}$$

Equation (2.3) is the differential equation of motion governing the nonlinear vibration of Euler–Bernoulli beams. The center of the beam is subjected to the following initial conditions:

$$W(0)=\varDelta,\quad \frac{\mathrm{d}W(0)}{\mathrm{d}t}=0, \tag{2.5}$$

where $\varDelta$ denotes the non-dimensional maximum amplitude of oscillation and α_1, α_2 and α_3 are as follows:

$$\alpha_1=\int_0^1\varphi''''\varphi\,\mathrm{d}x\Big/\int_0^1\varphi^2\,\mathrm{d}x, \tag{2.6a}$$

$$\alpha_2=\int_0^1\varphi''\varphi\,\mathrm{d}x\Big/\int_0^1\varphi^2\,\mathrm{d}x, \tag{2.6b}$$

$$\alpha_3=-\frac{1}{2}\int_0^1\left(\varphi''\int_0^1\varphi'^2\,\mathrm{d}x\right)\varphi\,\mathrm{d}x\Big/\int_0^1\varphi^2\,\mathrm{d}x. \tag{2.6c}$$

Post-buckling load-deflection relation for the problem can be obtained from Eq. (2.4) as

$$P=\left(-\alpha_1-\alpha_3W^2\right)/\alpha_2. \tag{2.7}$$

Neglecting the contribution of W in Eq. (2.7), the buckling load can be determined as

$$P_c=-\alpha_1/\alpha_2. \tag{2.8}$$

3. Basic idea of max-min approach

We consider a generalized nonlinear oscillator in the form [33]:

$$\ddot{W}+Wf(W)=0,\quad W(0)=\varDelta,\quad \dot{W}(0)=0, \tag{3.1}$$

where $f(W)$ is a non-negative function of W. According to the idea of the max-min method, we choose a trial-function in the form

$$W(t)=\varDelta\cos(\omega t), \tag{3.2}$$

where ω is the unknown frequency to be determined further.

Observe that the square of frequency, ω^2, is never less than that in the solution

$$W_1(t)=\varDelta\cos\left(\sqrt{f_{\min}}t\right), \tag{3.3}$$

of the following linear oscillator:

$$\ddot{W}+Wf_{\min}=0,\quad W(0)=\varDelta,\quad \dot{W}(0)=0, \tag{3.4}$$

where $f_{\min}$ is the minimum value of the function $f(W)$.

In addition, ω^2 never exceeds the square of frequency of the solution

$$W_1(t)=\varDelta\cos\left(\sqrt{f_{\max}}t\right), \tag{3.5}$$

of the following oscillator:

$$\ddot{W}+Wf_{\max}=0,\quad W(0)=\varDelta,\quad \dot{W}(0)=0, \tag{3.6}$$

where $f_{\max}$ is the maximum value of the function $f(W)$.

Hence, it follows that

$$\frac{f_{\min}}{1}<\omega^2<\frac{f_{\max}}{1}. \tag{3.7}$$

According to the Chentian interpolation [33], we obtain

$$\omega^2=\frac{mf_{\min}+nf_{\max}}{m+n}, \tag{3.8}$$

or

$$\omega^2=\frac{f_{\min}+kf_{\max}}{1+k}, \tag{3.9}$$

where m and n are weighting factors, $k=n/m$. So the solution of Eq. (3.1) can be expressed as

$$W(t)=\varDelta\cos\sqrt{\frac{f_{\min}+kf_{\max}}{1+k}}t. \tag{3.10}$$

The value of k can be approximately determined by various approximate methods [34, 35]. Among others, hereby we use the residual method [34]. Substituting Eq. (3.10) into Eq. (3.1) results in the following residual:

$$R(t;k)=-\omega^2A\cos(\omega t)+[A\cos(\omega t)]\,f\left(A\cos(\omega t)\right), \tag{3.11}$$

where $\omega=\sqrt{\frac{f_{\min}+kf_{\max}}{1+k}}$.

If, by chance, Eq. (3.10) is the exact solution, then $R(t;k)$ is vanishing completely. Since our approach is only an approximation to the exact solution, we set

$$\int_0^T R(t;k)\cos\sqrt{\frac{f_{\min}+kf_{\max}}{1+k}}t\,\mathrm{d}t=0, \tag{3.12}$$

where $T=2\pi/\omega$. Solving the above equation, we can easily obtain

$$k = \frac{f_{\max} - f_{\min}}{1 - \sqrt{\frac{A}{\pi}\int_0^{\pi} \cos^2(x) f\,(A\cos x)\,\mathrm{d}x}}. \tag{3.13}$$

Substituting the above equation into Eq. (3.10), we obtain the approximate solution of Eq. (3.1).

4. Applications

We can re-write Eq. (2.4) in the following form:

$$\ddot{W} + (\alpha_1 + P\alpha_2)W + \alpha_3 W^3 = 0. \tag{4.1}$$

We choose a trial-function in the form

$$W(t) = \Delta\cos(\omega t), \tag{4.2}$$

where ω is the frequency to be determined.

By using the trial-function, the maximum and minimum values of ω will be

$$\omega_{\min} = \frac{\alpha_1 + P\alpha_2}{1},$$

$$\omega_{\max} = \frac{\alpha_1 + P\alpha_2 + \alpha_3\Delta}{1}. \tag{4.3}$$

So we can write

$$\frac{\alpha_1 + P\alpha_2}{1} < \omega^2 < \frac{\alpha_1 + P\alpha_2 + \alpha_3\Delta}{1}. \tag{4.4}$$

According to the Chengtian inequality [34], we have

$$\omega^2 = \frac{m\,(\alpha_1 + P\alpha_2) + n\,(\alpha_1 + P\alpha_2 + \alpha_3\Delta)}{m+n}$$

$$= \alpha_1 + P\alpha_2 + k\alpha_3\Delta \tag{4.5}$$

where m and n are weighting factors, $k = n/m + n$. Therefore the frequency can be approximated as

$$\omega = \sqrt{(\alpha_1 + p\alpha_2) + k\alpha_3\Delta^2}. \tag{4.6}$$

Its approximate solution reads

$$W(t) = \Delta\cos\sqrt{(\alpha_1 + p\alpha_2) + k\alpha_3\Delta^2}t. \tag{4.7}$$

In view of the approximate solution, Eq. (4.6), we re--write Eq. (4.1) in the form

$$\ddot{W} + (\alpha_1 + P\alpha_2 + k\alpha_3\Delta)W = (k\alpha_3\Delta)W - \alpha_3 W^3. \tag{4.8}$$

If by any chance Eq. (4.6) is the exact solution, then the right side of Eq. (4.8) vanishes completely. Considering our approach which is just an approximation one, we set

$$\int_0^{T/4} \left[(k\alpha_3\Delta)\,W - \alpha_3 W^3\right]\cos\omega t\,\mathrm{d}t = 0, \tag{4.9}$$

where $T = 2\pi/\omega$. Solving the above equation, we can easily obtain

$$k = \frac{3}{4}. \tag{4.10}$$

Finally the frequency is obtained as

$$\omega = \frac{1}{2}\sqrt{4\,(\alpha_1 + p\alpha_2) + 3\alpha_3\Delta^2}. \tag{4.11}$$

Hence, the approximate solution can be readily obtained

$$W(t) = \Delta\cos\left(\frac{1}{2}\sqrt{4\,(\alpha_1 + p\alpha_2) + 3\alpha_3\Delta^2 t}\right). \tag{4.12}$$

Nonlinear to linear frequency ratio is

$$\frac{\omega_{\mathrm{NL}}}{\omega_{\mathrm{L}}} = \frac{1}{2}\frac{\sqrt{4\,(\alpha_1 + p\alpha_2) + 3\alpha_3\Delta^2}}{\sqrt{\alpha_1 + p\alpha_2}}. \tag{4.13}$$

5. Results and discussions

To illustrate and verify the results obtained by the MMA, some comparisons with the published data and the exact solutions are presented. The exact frequency ω_{Exact} for a dynamic system governed by Eq. (2.4) can be derived, as shown in Eq. (5.1), as follows:

$$\omega_{\mathrm{Exact}} = (2\pi/4\sqrt{2}\Delta)\int_0^{\pi/2} \mathrm{d}t\ \sin(t)/ \tag{5.1}$$

$$\sqrt{\Delta^2\sin^2(t)\,(\Delta^2\alpha_3\cos^2(t) + 2p\alpha_2 + 2\alpha_1 + \Delta^2\alpha_3)}.$$

To obtain numerical solution we must specify the parameter $\beta = \alpha_3/(p\alpha_2 + \alpha_1)$. This parameter depends on the type of structure and boundary condition considered.

The comparison of nonlinear to linear frequency ratio $(\omega_{\mathrm{NL}}/\omega_{\mathrm{L}})$ with those reported by Azrar et al. [36] and the exact one are tabulated in Tables I and II. The maximum relative error of the analytical approaches is 2.004109% for the first order analytical approximations as it is shown in Tables I and II.

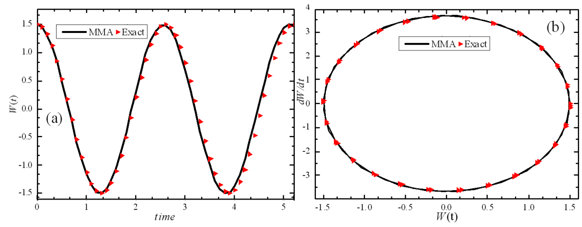

Fig. 2. Comparison of the approximate and exact solutions for simply supported beam with $\Delta = 1.5$, $\alpha_1 = 2$, $\alpha_2 = 0$, $\alpha_3 = 6$: (a) time history response, (b) phase curve.

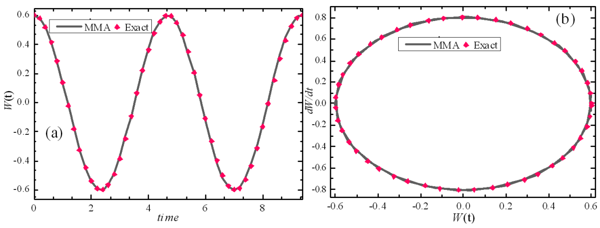

Fig. 3. Comparison of the approximate and exact solutions for simply supported beam with $\Delta = 0.6$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 3$: (a) time history response, (b) phase curve.

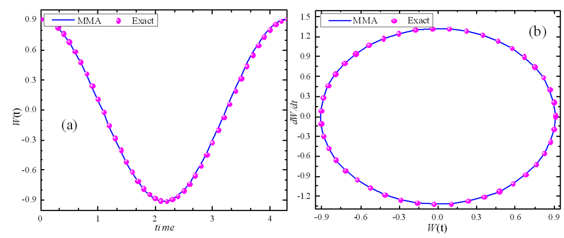

Fig. 4. Comparison of the approximate and exact solutions for clamped-clamped beam with $\Delta = 0.909$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 1.814$: (a) time history response, (b) phase curve.

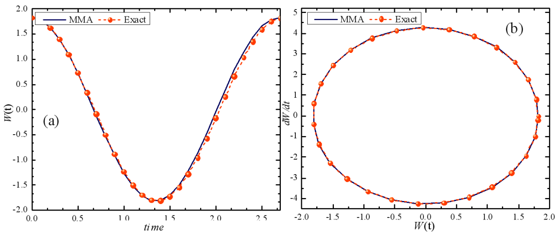

Fig. 5. Comparison of the approximate and exact solutions for clamped-clamped beam with $\Delta = 1.818$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 1.814$: (a) time history response, (b) phase curve.

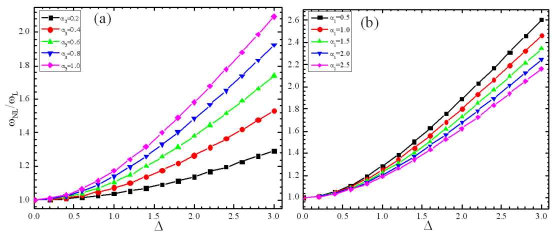

Fig. 6. (a) Influence of α_3 on nonlinear to linear frequency base on Δ for $\alpha_1 = 1$, $\alpha_2 = 0.5$, $p = 2$. (b) Influence of α_1 on nonlinear to linear frequency base on Δ for $\alpha_2 = 1$, $\alpha_3 = 3$, $p = 3$.

Figures 2 to 5 show the comparison of the analytical solution of $W(t)$ based on time and $\frac{\mathrm{d}W(t)}{\mathrm{d}t}$ based on $W(t)$ with the numerical solution for simply supported beam and clamped-clamped beam. The time history diagrams of $W(t)$ start without an observable deviation with $A = 1.5$ and 0.6. The motion of the system is a periodic and the amplitude of vibration is a function of the initial conditions.

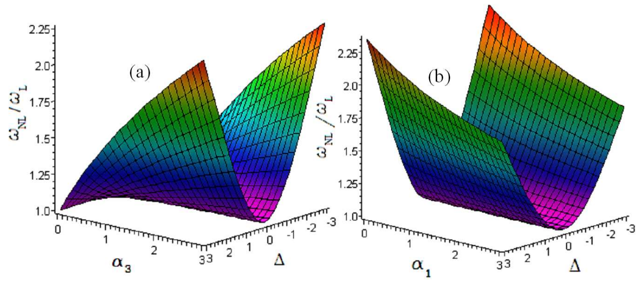

Fig. 7. Sensitivity analysis of nonlinear to linear frequency: (a) with respect to α_3 and Δ, (b) with respect to α_1 and Δ.

The influences of α_3 and α_1 on nonlinear to linear frequency base on Δ are presented in Fig. 6. By increasing α_3 nonlinear to linear frequency is increased and the opposite result is obtained by increasing α_1. The effects of different parameters of α_3, Δ, and α_1, Δ on the nonlinear to linear frequency are studied simultaneously in Fig. 7.

TABLE I

Comparison of nonlinear to linear frequency ratio ($\omega_{\mathrm{NL}}/\omega_{\mathrm{L}}$) for simply supported beam.

Δ	β	Present study (MMA)	Pade approximate P{4,2}[36]	Exact solution	Error [%] $(\omega_{\mathrm{MMA}} - \omega_{\mathrm{ex}})/\omega_{\mathrm{ex}}$
0.2	3	1.0440	1.0439	1.0439	0.0142
0.6	3	1.3454	1.3397	1.3397	0.4224
1	3	1.8028	1.7847	1.7844	1.0287
1.5	3	2.4622	2.4262	2.4254	1.5178
2	3	3.1623	3.1085	3.1071	1.7761
2.5	3	3.8810	3.8099	3.8080	1.9190
3	3	4.6098	4.5217	4.5192	2.0041

It has illustrated that MMA is a very simple method and quickly convergent and valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that the MMA can be potentially used for the analysis of strongly nonlinear oscillation problems accurately.

6. Conclusions

In this paper, the MMA was employed to solve the governing equations of buckled Euler–Bernoulli beams. This approach prepares high accurate analytical solutions, with respective errors of 2.004109% for the considered problem. We showed excellent agreement between the solution given by MMA and the exact one. It was indicated that MMA remains more effective and accurate for solving highly nonlinear oscillators and possesses clear advantages over other periodic solutions which are based on a Fourier series, complicated numerical integration, or traditional perturbation methods (which require the presence of a small parameter). Its excellent accuracy for the whole range of oscillation amplitude values is one of the most significant features of this approach. MMA requires smaller computational effort and only the one iteration leads to accurate solutions.

TABLE II

Comparison of nonlinear to linear frequency ratio ($\omega_{\mathrm{NL}}/\omega_{\mathrm{L}}$) for clamped-clamped beams.

Δ	β	Present study (MMA)	Pade approximate P{4,2}[36]	Exact solution	Error [%] $(\omega_{\mathrm{MMA}}-\omega_{\mathrm{ex}})/\omega_{\mathrm{ex}}$
0.273	1.8142	1.0494	1.0492	1.0492	0.0177
0.545	1.8142	1.1852	1.1831	1.1831	0.1741
0.727	1.8142	1.3112	1.3064	1.3064	0.3688
0.909	1.8142	1.4574	1.4488	1.4488	0.5935
1.818	1.8142	2.3443	2.3114	2.3107	1.4539
3.635	1.8142	4.3569	4.2746	4.2723	1.9791

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