Quasi-Particle Localization by Disorder in $\nu = 5/2$ Fractional Quantum Hall State and Its Potential Application

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We consider the filling factor 5/2 fractional quantum Hall state of spin-polarized fermions in a dirty (mobility $\mu_b < 10$ m^2 V^{-1} s^{-1}) bi-layer quantum well system.We show that the system undergoes a quantum phase transition from the effective two-component state to an effective single-component state, at fixed charge imbalance regulatory parameter Δ_c and constant layer separation, as the inter-layer tunneling strength Δ_{SAS} is increased. At finite and constant Δ_{SAS}, a transition from the latter state to the former state is also possible upon increasing the parameter Δ_c. We identify the order parameter to describe quantum phase transition as a pseudo-spin component and calculate this with the aid of the Matsubara propagators in the finite-temperature formalism. Our treatment is able to show that, at low temperature(< 0.1 K) and low value of charge imbalance regulatory parameter, there is a competition between the disorder and the inter-layer tunneling strength in the sample. The competition leads to the quasi-particle localization at low tunneling strength.

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1. Introduction

We visualize a bilayer system as the one consisting of two parallel (quasi-)two-dimensional electron gas (2DEG) systems of width w separated from one another by a tunneling barrier of thickness d ($d \geq w/2$). The barrier height and thickness can be adjusted such that electrons are either localized in separate layers or delocalized between the two layers. We consider spin-less fermions for the filling fraction $\nu = 1+1+\frac{1}{2} = 5/2$ confined in this planar geometry. The well and the barrier materials, respectively, are assumed to be GaAs and AlGaAs. Since the kinetic energy of a system in the fractional quantum Hall states (FQHS) is totally quenched, the Hamiltonian comprises of the electron correlation (captured by the "so-called" Haldane pseudo-potentials (HPP) [1]), the inter-layer tunneling (of strength Δ_{SAS}), and the charge imbalance regulatory (Δ_c) terms [2–7]. The Hamiltonian we consider is in the symmetric-antisymmetric basis [2–5] (S–AS) and expressed in units of ($e^2/\varepsilon l_B$) where the magnetic length $l_B = \sqrt{\hbar/eB}$ is the length unit and ε is the permittivity of the system material. The recent experiments [7] have achieved bilayer fractional quantum Hall (FQH) systems where both the inter-layer and charge imbalance tunneling terms can be controlled by changing system parameters such as gate voltages.

The second Landau level (SLL) ($n = 1$) ($2 < \nu < 4$) electron correlations are different than those in the lowest Landau level (LLL) ($n = 0$) (filling fraction $\nu = \frac{1}{2}$) as was shown by Gossard et al. [8] in their benchmark discovery of $\nu = 5/2$ even denominator FQHS. In fact, the ground state corresponding to the former is described by the non-Abelian Moore–Read Pfaffian (Pf) [9] where two particles of different types are not averse to being located at the same point in real space. The ground state corresponding to the latter, on the other hand, is known to be an Abelian Halperin 331 bilayer one [3, 4]. The multiple zeros in the Halperin state, as in the Laughlin state, ensure that the amplitude of the states, for the states together to be close, always tends towards zero; in the real space two particles of different types avoid being located at the same point. The reasons for considering a bilayer quantum well system for $\nu = 5/2$ and not $\nu = \frac{1}{2}$, in this communication, are (i) the effective electron–electron interaction in the $\nu = 5/2$ case is weaker (and therefore easier to deal with analytically compared to that in the $\nu = \frac{1}{2}$ case) due to the spin-polarized electron (inhabiting the lowest LL) induced screening, and (ii) though the single-layer $\nu = 5/2$ SLL FQHS, with an activation gap of order 100–500 mK, has been observed by a number of experimentalists [10–16] in samples with high mobility ($> 10^3$ m^2 V^{-1} s^{-1}) at $T < 100$ mK, the $\nu = \frac{1}{2}$ LLL FQHS has never been observed experimentally [15]. It may be pointed out that whereas the odd-denominator incompressible FQH states (e.g. 1/3, 1/5, 7/3) are somewhat robust and the even denominator (e.g. 1/2, 1/4, 5/2) FQH states are comparatively fragile [17], nevertheless the experimental $\nu = 5/2$ FQHE is always among the strongest observed FQH states in the latter category.

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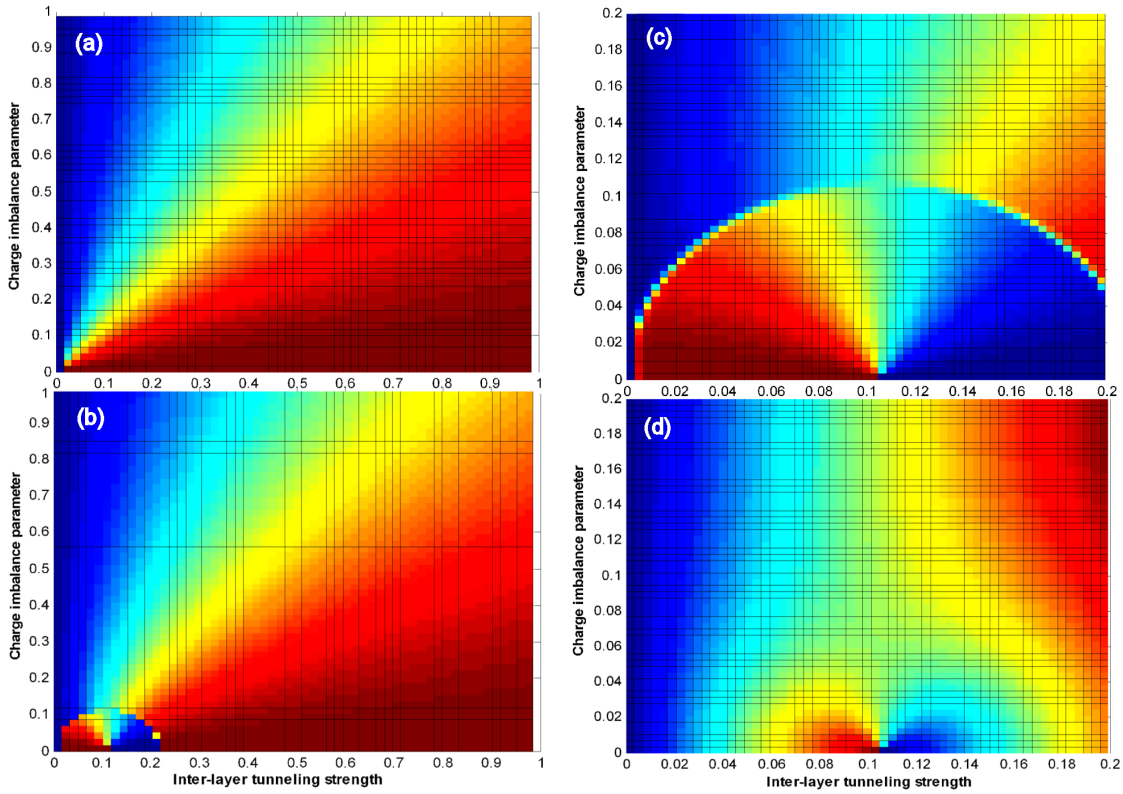

Fig. 1. The contour plots of the order parameter of QPT (Γ) as a function of the inter-layer tunneling strength ($\Delta_{\rm SAS}$) and the charge imbalance regulatory parameter ($\Delta_{\rm c}$). The color scale of the plots spans from 0 to 1 with the deep blue (cold) scaling up to 0.3, the lighter shade (neutral) and green up to 0.55, the yellow (warm) up to 0.65, and the red (hot/burning) beyond. We have shown that at finite chemical potential (temperature (T) $\approx$ 50 mK in Fig. 1a,b, and c and $T \approx 1$ K in Fig. 1d), the system under consideration undergoes zero-order QPT at fixed $\Delta_{\rm c}$ and the constant layer separation as the inter-layer tunneling strength $\Delta_{\rm SAS}$ is increased from the effective two-component state (bilayer FQHS) to eventually an effective single-component state (single layer FQHS). The cold (deep blue) region in these contour plots correspond to the former state whereas the hot (red) region to the latter. The quantum critical region (lighter blue shade, green, and yellow) is sandwiched between the regions corresponding to bi-layer FQHS and the single-layer FQHS. Part (a) corresponds to a clean sample (mobility $\mu_{\rm b} \approx 10^3$ m^2 V^{-1} s^{-1}) and the rest to dirty samples. As shown in the almost semicircular region in part (c), the dirty system behaves in an intriguing fashion. If the parameter $\Delta_{\rm c}$ is kept fixed at a value less than 0.1, upon increasing $\Delta_{\rm SAS}$ from zero, one encounters a crossover from the bilayer FQHS to the single-layer one at $\Delta_{1,\rm SAS} \approx 0.005$. This is followed by an entry to quantum critical state close to $\Delta_{2,\rm SAS} = 0.1$ and the (quantum phase) transition to the bi-layer state beyond. The possible explanation of this transition is the "quasi-particle localization by disorder at very low temperature". Upon increasing $\Delta_{\rm SAS}$ beyond $\Delta_{3,\rm SAS} = 0.2$ there is a crossover to the single-layer state once again as the (increased) tunneling starts dominating now over the disorder. We find that the manifestation of the localization (by disorder) is very sensitive to the increase in temperature. For example, as could be seen in part (d) which corresponds to a temperature of 1 K, there is a softened trace of this phenomenon at the inter-layer tunneling strength 0.1. We have assumed above the optimum tunneling strengths to be one order of magnitude less than the Coulomb energy ($e^2/\varepsilon l_{\rm B}$) $\approx$ 0.01 eV and two order of magnitude greater compared to the thermal energy ($k_{\rm B}T$) for the 50–100 mK range.

The topological protection in non-Abelian FQHS, such as the one for $\nu = 5/2$, hinges on the energy gap separating the many-body degenerate ground states from the low-lying excited states; larger the size of this activation gap compared to the temperature, more robust is the topological protection. This gap also leads to the incompressibility of the quantum Hall state and the quantization of the Hall resistance. Unfortunately, the excitation gaps for the expected non-Abelian FQHS are typically very small compared, for example, with the $\nu = 1/3$ FQHS. In the highest mobility samples available, the (experimentally measured) activation energy gap $\Delta_{\rm exp} \approx 500$ mK [18]. Indeed, there appears to be a close connection between the excitation gap and the mobility ($\mu_{\rm b}$). The enhancement of the 2D mobility invariably leads to larger measured excitation gaps. In particular, an empirical relation [19–21], $\Delta_{\rm exp} = \Delta_{5/2} - I^*$, may be written out where $\Delta_{5/2} \approx 0.005$–$0.010$ ($e^2/\varepsilon l_{\rm B}$) is the $\nu = 5/2$ intrinsic energy gap (i.e. the gap that would be observed in the absence of disorder) and I^* is the level

broadening arising from impurity and disorder scattering. Our analysis below, though starts with a microscopic basis, ultimately leans on this empirical relation to obtain the plots in Figs. 1 and 2. In fact, comprehending the effect of disorder on the observed QHE is of fundamental importance since impurity scattering can have a significant and drastic effect on correlated electron behavior as will be shown below. In this communication, we show that for a sample where disorder level may not be very low (mobility $\mu_{\rm b} < 10^2$ m^2 V^{-1} s^{-1}), at low temperature (< 0.1 K) and low value of $\Delta_{\rm c}$, there is an intriguing competition between the disorder, leading to the quasi-particle localization, and the inter-layer tunneling strength $\Delta_{\rm SAS}$. The point we wish to make is that a sample, *with a somewhat higher disorder level*, may possibly find potential application in performing memory function and gating operation by simply maneuvering the gate voltages.

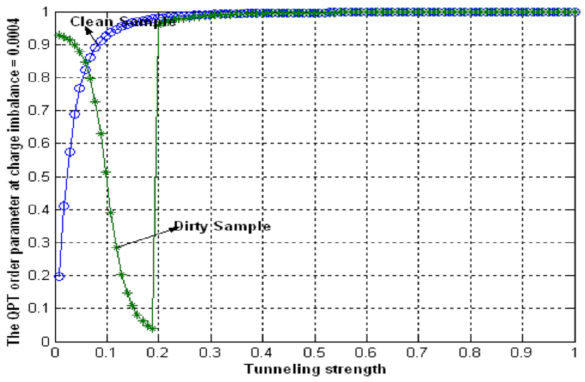

Fig. 2. The 2D plots of Γ as a function of $\Delta_{\rm SAS}$ for clean and dirty samples for the charge imbalance parameter equal to 0.0004. The graph reveals a range ($0 < \Delta_{\rm SAS} \leq 0.2$) of negative susceptibility ($\partial\Gamma/\partial\Delta_{\rm SAS} < 0$), for a dirty sample. For a clean sample, where the quasi-particle localization by disorder will not be possible, the susceptibility is never negative.

The paper is organized as follows: in Sect. 2 we present the Hamiltonian for a LL in the symmetric-antisymmetric basis (SAS) including the electron correlations captured by the Haldane pseudo-potentials. We also outline that it is possible to construct something like a Landau–Fermi liquid theory description directly in terms of quasi-particle excitations at the filling fraction $\nu = 5/2$. In Sect. 3 we present the calculation of energy gap (Δ) and quantum phase transition (QPT) order parameter (Γ) in finite temperature formalism with the inclusion of disorder. We show with the aid of these results that when $(d/l_{\rm B})$ is held fixed at a value corresponding to the moderately correlated and a dirty (mobility $\mu_{\rm b} < 10^2$ m^2 V^{-1} s^{-1}) system we have the possibility of a QPT and the quasi-particle localization by disorder upon tinkering with the inter-layer tunneling strength and the charge imbalance regulatory parameter. We conclude the paper with a brief discussion of the results obtained.

2. Hamiltonian for a LL in the symmetric-antisymmetric basis

The Hamiltonian for the system, in the SAS basis (or equivalently, in the right and the left layer basis), apart from the electron correlations also contains the terms accounting for the inter-layer tunneling and the charge imbalance for SLL [2–5]. Under the independent Landau level (LL) (indexed by $n = 0, 1, 2, \ldots$) assumption, expressed in units of $(e^2/\varepsilon\, l_{\rm B})$, the Hamiltonian for a LL in the SAS basis is given by

$$\begin{aligned} H^{(n)} = \frac{1}{2}\sum_{m} \big[& -\Delta_{\rm SAS} d^{\dagger}_{m,n,{\rm S}} d_{m,n,{\rm S}} \\ & + \Delta_{\rm SAS} d^{\dagger}_{m,n,{\rm AS}} d_{m,n,{\rm AS}} + \Delta_{\rm c} d^{\dagger}_{m,n,{\rm S}} d_{m,n,{\rm AS}} \\ & + \Delta_{\rm c} d^{\dagger}_{m,n,{\rm AS}} d_{m,n,{\rm S}} \big] \\ & + \frac{1}{2}\sum_{m,m',m'',m''',\sigma,\sigma'} V^{(n)}_{\{m\}} d^{\dagger}_{m,n,\sigma} d^{\dagger}_{m',n,\sigma'} d_{m'',n,\sigma'} d_{m''',n,\sigma} \\ & \times \delta(m + m' - m'' - m'''), \end{aligned} \tag{1}$$

where $\sigma = ({\rm S}, {\rm AS})$. The operators $d_{m,{\rm S}}$ and $d_{m,{\rm AS}}$ destroy an electron in the S and AS superposition states, respectively; the index m corresponds to the relative angular momentum between two electrons in a LL (with index n). As we shall see below, the terms involving the parameter $\Delta_{\rm SAS}$ in (1) correspond to the tunneling of carriers between the two parallel (quasi-)two-dimensional electron gas systems under consideration whereas those involving the parameter $\Delta_{\rm c}$ to the carrier imbalance between the systems. The HPP functions $V^{(n)}_m$ [1, 20] (which serve as a complete set of basis functions due to angular momentum conservation) for electrons confined to a LL, in the planar geometry, are written as $V^{(n)}_m = \int_0^\infty {\rm d}k\, k[L_n(k^2/2)]^2 L_m(k^2)\exp(-k^2)V(k)$ where $L_n(x)$ are the Laguerre polynomials, and $V(k)$ is the Fourier transform of the interaction potential $V(r)$. We have $V(k) = (\frac{1}{2\pi})\int {\rm d}^2 r \exp({\rm i}k\cdot r)V(r) = \int_0^\infty {\rm d}r\, rJ(k\cdot r)V(r)$. We consider two finite thickness potential models [2–5, 18], viz. (i) the infinite square-well (SQ) potential which is appropriate for 2D GaAs quantum well structures, and (ii) the Fang–Howard (FH) variational potential for a heterostructure. In the dimensionless form, the former HPPs are

$$\begin{aligned} V^{({\rm SQ})}_m = & \int {\rm d}x[L_n(x^2/2)]^2 L_m(x^2)\exp(-x^2) \\ & \times\big\{3xd/l_{\rm B} + 8\pi^2/(xd/l_{\rm B}) \\ & - \big[32\pi^4\big(1 - {\rm e}^{-x(d/l_{\rm B})}\big)(x^4(d/l_{\rm B})^4 \\ & + 4\pi^2 x^2 (d/l_{\rm B})^2)^{-1}\big]\big\}[x^2(d/l_{\rm B})^2 + 4\pi^2]^{-1} \end{aligned} \tag{2}$$

and the latter ones are

$$\begin{aligned} V^{({\rm FH})}_m = & \int {\rm d}x[L_n(x^2/2)]^2 L_m(x^2)\exp(-x^2) \\ & \times (9/8)[(24 + 9(xd/l_{\rm B}) \\ & + (xd/l_{\rm B})^2)(3 + xd/l_{\rm B})^{-3}]. \end{aligned} \tag{3}$$

Including the finite thickness effect (FTE) it was shown [2–5, 18] that in the LLL ($n = 0$) (and the SLL ($n = 1$)) FTE corresponds to softening all of the pseudo-potentials in a "trivial" way for $(d/l_{\rm B}) < 1$ as well as $(d/l_{\rm B}) \geq 1$ (see also Table).

TABLE

The values of the HPP are summarized here considering (a) the infinite SQ potential, and (b) the FH variational potential in Eqs. (2) and (3). Since $V_1^{(n)} > V_3^{(n)} > V_5^{(n)} > \ldots$ for a given $(d/l_{\rm B})$ does not get violated, the conclusion is that finite thickness effect (FTE) corresponds to softening of all of the pseudo-potentials in a rather "trivial" way for all $(d/l_{\rm B})$. We have taken the limits of integration in Eqs. (2) and (3) as 0.001 and 10.

$d/l_{\rm B}=$	0.2	0.5	0.6	0.8	1.0	1.5
			(a)			
V_1^0	11.2091	1.7505	1.2156	0.6838	0.4376	0.1945
V_3^0	6.7849	1.1043	0.7669	0.4314	0.2761	0.1227
V_5^0	5.5785	0.8709	0.6048	0.3402	0.2177	0.0968
V_1^1	10.9555	1.6372	1.1370	0.6396	0.4093	0.1819
V_3^1	7.9503	1.2432	0.8634	0.4857	0.3108	0.1382
V_5^1	5.9444	0.9256	0.6428	0.3616	0.2314	0.1029
			(b)			
V_1^0	10.4314	1.6441	1.1418	0.6423	0.4110	0.1827
V_3^0	6.6646	1.0777	0.7484	0.4210	0.2694	0.1198
V_5^0	5.4421	0.8580	0.5959	0.3352	0.2145	0.0954
V_1^1	9.9730	1.5300	1.0625	0.5977	0.3825	0.1700
V_3^1	7.4312	1.1763	0.8169	0.4595	0.2941	0.1307
V_5^1	5.7489	0.9038	0.6277	0.3531	0.2260	0.1004

We employ the finite-temperature formalism involving the Matsubara propagators to deal with the Hamiltonian in (1). We identify the order parameter $\mathit{\Gamma} = \left[\sum_m V_m^{(1)}(\langle n_{m,\rm S}^{(1)}\rangle - \langle n_{m,\rm AS}^{(1)}\rangle)/\sum_m V_m^{(1)}\right]$, where $\langle n_{m,\sigma}^{(1)}\rangle = \langle d_{m,l,\sigma}^\dagger d_{m,l,\sigma}\rangle$, $\sigma = {\rm S/AS}$ is a thermal average, as the one to describe the QPT and calculate the order parameter with the aid of these propagators. We shall now explain below how this identification is possible. The real single spin-1/2 operator $S_{\rm real}$ is represented in terms of the Pauli matrices. The basis states here are eigenstates of $S_{\rm real}^z$ i.e. $|\uparrow\rangle$, and $|\downarrow\rangle$. This operator in the second quantized language can be written as $S_{\rm real}^i = \sum_{\mu,\mu'} d_\mu^\dagger S_{{\rm real}\mu,\mu'}^i d_{\mu'}$ where $d_\mu^\dagger$ creates a particle in the state $|\mu\rangle$. This immediately gives $S_{\rm real}^x = \frac{1}{2}(d_\uparrow^\dagger d_\downarrow + d_\downarrow^\dagger d_\uparrow)$, $S_{\rm real}^y = \frac{1}{2{\rm i}}(d_\uparrow^\dagger d_\downarrow - d_\downarrow^\dagger d_\uparrow)$, and $S_{\rm real}^z = \frac{1}{2}(d_\uparrow^\dagger d_\uparrow - d_\downarrow^\dagger d_\downarrow)$. The spin-reversal operators are $S_{\rm real}^+ = d_\uparrow^\dagger d_\downarrow$ and $S_{\rm real}^- = d_\downarrow^\dagger d_\uparrow$. We now consider the terms $T_{\rm t} = [-(\mathit{\Delta}_{\rm SAS}/2)\sum_m (d_{m,\rm S}^\dagger d_{m,\rm S} - d_{m,AS}^\dagger d_{m,\rm AS})]$ and $T_{\rm b} = [(\mathit{\Delta}_{\rm c}/2)\sum_m (d_{m,\rm S}^\dagger d_{m,\rm AS} + d_{m,AS}^\dagger d_{m,\rm S})]$ in (1). The comparison of $S_{\rm real}^z$ and $S_{\rm real}^x$ with the terms $T_{\rm t}$ and $T_{\rm b}$, respectively, shows that whereas, for a given m, $T_{\rm t}$ may be represented by the pseudo-spin operator [6] component $S_m^z = \frac{1}{2}(d_{m,\rm S}^\dagger d_{m,\rm S} - d_{m,AS}^\dagger d_{m,\rm AS})$, $T_{\rm b}$ may be represented by the operator $S_m^x = \frac{1}{2}(S_m^+ + S_m^-) = \frac{1}{2}(d_{m,\rm S}^\dagger d_{m,\rm AS} + d_{m,AS}^\dagger d_{m,\rm S})$. Also, whereas the effect of averaging S_m^z over the pseudo-spin states, viz. $|{\rm S}\rangle$ and $|{\rm AS}\rangle$ states [2–5], is to yield eventually the number of electrons in the S–AS states, the effect of S_m^x on these states is to "switch" the states. We note that in the bilayer problem with the total number of carriers $(2N_A A)$, where N_A is the areal density and A is the area of each of the layers, the total number of particles in each layer is $(N_A A)$ when the electron density is balanced in each layer. It is not difficult to see that the average of S_m^x over S–AS states is zero (non-zero) when there is "balance" (imbalance). Thus, the operator $T_{\rm b}$ represented by the pseudo-spin operators S_m^x indeed accounts for the charge imbalance. To seek further confirmation of this and to extract physical meaning for the operator $T_{\rm t}$ from a different perspective, we go over to the layer basis using the transformations [2–5] given by $|{\rm S}\rangle = (1/\sqrt{2})(|{\rm R}\rangle + |{\rm L}\rangle)$, and $|{\rm AS}\rangle = (1/\sqrt{2})(|{\rm R}\rangle - |{\rm L}\rangle)$ where $|R\rangle$ and $|L\rangle$, respectively, correspond to the right and the left layer basis states. Here the operators $d_{m,R}$ and $d_{m,L}$, respectively, destroy an electron in the right and left quantum well and $d_{m,\rm S} = (d_{m,R} + d_{m,L})/\sqrt{2}$ and $d_{m,\rm AS} = (d_{m,R} - d_{m,L})/\sqrt{2}$. In terms of $(d_{m,R}, d_{m,L})$, we have $T_{\rm t} = -(\mathit{\Delta}_{\rm SAS}/2)\sum_m (d_{m,\rm R}^\dagger d_{m,L} + d_{m,L}^\dagger d_{m,R})$ and $T_{\rm b} = (\mathit{\Delta}_{\rm c}/2)\sum_m (d_{m,\rm R}^\dagger d_{m,R} - d_{m,L}^\dagger d_{m,L})$.

It is now amply clear that the field $\mathit{\Delta}_{\rm SAS}$ controlled by external gates, acting like an effective magnetic field along the pseudo-spin x direction, alters the spread of the electron density along the direction perpendicular to the two-dimensional plane thereby tuning the interlayer overlap. On the other hand, the field $\mathit{\Delta}_{\rm c}$, acting like an effective magnetic field along the pseudo-spin z direction, energetically favors one pseudo-spin state over another. We shall also see below that the fields $\mathit{\Delta}_{\rm SAS}$ and $\mathit{\Delta}_{\rm c}$ will appear as the tunneling and charge imbalance induced gaps, respectively, in the single-particle excitation spectrum. Written in the layer basis, the term $T_{\rm t}$ indicates the possibility of transition from an effective bilayer state to a practically single-layer one upon increasing $\mathit{\Delta}_{\rm SAS}$ at a constant $\mathit{\Delta}_{\rm c}$. In principle, the possibility of the opposite transition also exists upon increasing $\mathit{\Delta}_{\rm c}$ at a constant $\mathit{\Delta}_{\rm SAS}$. It follows that the average of the pseudo-spin operators $\sum_m S_m^x$ or $\sum_m S_m^z$ may be chosen as the order parameter to investigate the transition here. We have chosen the latter average to be our order parameter.

Upon assuming $d/l_{\rm B}$ fixed and greater than unity, in an essentially mean-field approach, we obtain the propagators as $G_{m,1,\sigma}({\rm i}\omega_n) = \sum_{j=\pm} a_\sigma^{(j)}({\rm i}\omega_n - \varepsilon_{m,\sigma}^{(j)} + \mu)^{-1}$, where $a_\sigma^{(\pm)} = \frac{1}{2}(1 \pm (\sigma \mathit{\Delta}_{\rm SAS}/\mathit{\Delta}_0)$ and $\mathit{\Delta}_0 = \sqrt{\mathit{\Delta}_{\rm c}^2 + \mathit{\Delta}_{\rm SAS}^2}$ and $\varepsilon_{m,\sigma}^{(\pm)} = \pm\mathit{\Delta}_0 + V_m^{(1)}\langle n_{m,\sigma}^{(1)}\rangle$. For a given m, since $\sigma = +1$ and -1, respectively, for $\sigma = {\rm S}$ and $\sigma = {\rm AS}$ which will ensure $\langle n_{m,\sigma}^{(1)}\rangle$ being different in the two cases, the poles of $G_{m,1,\sigma}({\rm i}\omega_n)$ correspond to the quasi-particle energies $\varepsilon_{m,\rm S}^{(\pm)} - \mu = V_m^{(1)}\langle n_{m,\rm S}^{(1)}\rangle \pm \mathit{\Delta}_0 - \mu$ and $\varepsilon_{m,\rm AS}^{(\pm)} - \mu = V_m^{(1)}\langle n_{m,\rm AS}^{(1)}\rangle \pm \mathit{\Delta}_0 - \mu$. One can see that the tunneling gap $\mathit{\Delta}_{\rm SAS}$, merging into single "quadrature" gap $\mathit{\Delta}_0$ together with the charge imbalance induced gap $\mathit{\Delta}_{\rm c}$, represents the splitting between the symmetric and the antisymmetric energy eigenvalues. These expressions lead to

the thermal averages $\langle n^{(1)}_{m,\sigma}\rangle$ involved in the expression for Γ. We find $\langle n^{(1)}_{m,\sigma}\rangle = \sum_{j=\pm} a^{(j)}_\sigma [\exp\beta(\varepsilon^{(j)}_{m,\sigma} - \mu) + 1)^{-1}$, where $\beta = (k_{\rm B}T]^{-1}$. These equations, together with the usual equation to determine the chemical potential μ in terms of $(N_A A)$ and $\langle n^{(1)}_{m,\sigma}\rangle$ constitute the set of self-consistent equations to determine $(\langle n^{(1)}_{m,\sigma}\rangle, \mu)$. The equations are self-consistent, for the thermal averages $\langle n^{(1)}_{m,\sigma}\rangle$ involved determine as well as are determined by these equations. We have obtained values of $\langle n^{(n)}_{m,\sigma}\rangle$, for the given number of carrier states $2(N_A A)$ using the values of HPP functions in Table, following a tedious iterative procedure. The coarse-grained approximation made above makes it possible to construct something like a Landau–Fermi liquid theory description directly in terms of low-energy quasi-particles.

3. Effect of disorder inclusion: quasi-particle localization by disorder at very low temperature

At this stage we empirically introduce a self-energy contribution Σ, real and independent of the Matsubara frequencies, which is supposedly linked to the charge disorder in the bulk. One common source for this disorder is the electrostatic potential due to disordered dopant layers. In this situation we expect the incompressible quantum Hall phase to occupy a major fraction of the sample, with the remainder occupied by the puddles of compressible electron liquid. Thus, the incompressible phase forms a network of channels separating puddles of size equal to the distance to the dopants; the width of the channels or strips of course is of the order of the magnetic length. The incompressible strips, within which there is a charge gap, form links which join together at nodes — associated with saddle points of the disorder potential — to form possibly a random network.

Out of this intricate scenario we pick up only the key fact, through the structure-less quantity Γ', that the puddles (of quasi-holes or quasi-electrons), which are mobile and dense enough to create only dent (and not complete destruction) in the incompressibility of the quantum Hall fluid, should ultimately render the effective activation gap less than the ideal excitation gap. The reason being the existence of a finite gap implies the overall incompressibility. It is thus imperative that the sign of the quantity Γ' has to be chosen ultimately in such a way so as to ensure a decreased activation gap arising out of the full Matsubara propagator which includes this phenomenologically motivated self energy contribution, of course, through the Dyson equation. We find that the full propagator $\mathrm{G}^{(\mathrm{Full})}_{m,1,\sigma}(\mathrm{i}\omega_n)$, given by the Dyson equation $G^{(\mathrm{Full})}_{m,1,\sigma}(\mathrm{i}\omega_n)^{-1} = G_{m,1,\sigma}(\mathrm{i}\omega_n)^{-1} - \Sigma$, is $G^{(\mathrm{Full})}_{m,1,\sigma}(z) = (z - U^{(1)}_{m,\sigma} - \sigma\Delta_{\rm SAS})/[z^2 - z(2U^{(1)}_{m,\sigma} + \Sigma) + ({U^{(1)}_{m,\sigma}}^2 - \Delta_0^2 + \Sigma(U^{(1)}_{m,\sigma} + \sigma\Delta_{\rm SAS})]$ where $z = \mathrm{i}\omega_n$ and $U^{(1)}_{m,\sigma} = V^{(1)}_m\langle n^{(1)}_{m,\sigma}\rangle - \mu$. We write this propagator as $G^{(\mathrm{Full})}_{m,1,\sigma}(z) = A^{(+)}_\sigma (z - z^+_\sigma)^{-1} + A^{(-)}_\sigma (z - z^-_\sigma)^{-1}$ where the upper and the lower branches and the coherence factors are given by

$$
\begin{aligned}
&z^\pm_\sigma = U^{(1)}_{m,\sigma} + (\Sigma/2) \pm (-\Delta'_0),\\
&\Delta'_0 \equiv [\Delta_0^2 + (\Sigma^2/4) - \sigma\Sigma\Delta_{\rm SAS}]^{\frac{1}{2}},\\
&A^{(\pm)}_\sigma = \{1 \pm [(\sigma\Delta_{\rm SAS} - \Sigma/2)/\Delta'_0]\}/2\\
&\quad = (1 \pm -2\partial\Delta'_0/\partial\Sigma)/2. \qquad (4)
\end{aligned}
$$

We notice that for a given m, the effective S–AS gap $G^{(1)}_{m,\rm eff}$ between the lower branches of the symmetric and the antisymmetric energy states is $[V^{(1)}_m(\langle n^{(1)}_{m,\rm S}\rangle - \langle n^{(1)}_{m,\rm AS}\rangle) - \delta\Delta]$ where $\delta\Delta = 2\Gamma'(\Delta_{\rm SAS}/\Delta')$, and $\Delta' = \sqrt{\Delta_{\rm c}^2 + (\Delta_{\rm SAS} + \Gamma'/2)^2} + \sqrt{\Delta_{\rm c}^2 + (\Delta_{\rm SAS} - \Gamma'/2)^2}$. We have assumed the empirical self-energy to be negative, i.e. $\Sigma = -\Gamma'$, where Γ' (> 0) is the empirical level broadening arising from impurity and disorder scattering. Upon writing the drift mobility $\mu_{\rm b} = e\tau/m$, with τ the zero field Drude scattering time, we find that $\Gamma' = \hbar/(2\tau)$, indicating $\Gamma' = \hbar e/(2m\mu_{\rm b})$ in this simple picture. This classical Drude model neglects electron–electron interactions, heating of the electrons, non-equilibrium dynamics, and quantum transport effects, and is thus restricted to near-zero fields, moderate carrier densities and temperature ranges. If any of these conditions are violated, nonlinear and non-Ohmic conduction arises. As is found experimentally, increasing the drift mobility should steadily enhance the operational excitation gap. We notice that $\delta\Delta$, being inversely proportional to $\mu_{\rm b}$, will precisely ensure this situation.

As for the order parameter of QPT, viz. $\Gamma = \sum_m V^{(1)}_m\{\langle n^{(1)}_{m,\rm S}\rangle - \langle n^{(1)}_{m,\rm AS}\rangle\}/\sum_m V^{(1)}_m$ we obtain, for the term within the curly brackets, in the moderately correlated regime

$$
\begin{aligned}
&\frac{1}{2}[A_1(\exp(-(\beta\Delta_1 + \ln g)) + 1)^{-1}\\
&\quad + A_2(\exp(\beta\Delta_1 + \ln g) + 1)^{-1}\\
&\quad - A_3(\exp(-(\beta\Delta_2 + \ln g)) + 1)^{-1}\\
&\quad - A_4(\exp(\beta\Delta_2 + \ln g) + 1)^{-1}],\\
&A_1 = \frac{1}{2}[1 + (\Delta_{\rm SAS} - \frac{\Gamma'}{2})/\Delta_1)],\\
&A_2 = \frac{1}{2}[1 - (\Delta_{\rm SAS} - \frac{\Gamma'}{2})/\Delta_1)],\\
&A_3 = \frac{1}{2}[1 - (\Delta_{\rm SAS} + \frac{\Gamma'}{2})/\Delta_2)],\\
&A_4 = \frac{1}{2}[1 + (\Delta_{\rm SAS} + \frac{\Gamma'}{2})/\Delta_2)]. \qquad (5)
\end{aligned}
$$

where $g = \exp(-\beta(\mu + \Gamma'/2))$, $\Delta_1 = \sqrt{\Delta_{\rm c}^2 + (\Delta_{\rm SAS} - \Gamma'/2)^2}$, and $\Delta_2 = \sqrt{\Delta_{\rm c}^2 + (\Delta_{\rm SAS} + \Gamma'/2)^2}$. Let us note that in the limiting case g close to unity (zero chemical potential and infinite mobility $\mu_{\rm b}$), the order parameter is $\Gamma \approx (\Delta_{\rm SAS}/\Delta_0)\tanh(\beta\Delta_0/2)$. Upon comparing this result with its counterpart above we notice that, whereas the latter is valid for the ideal case, the former is more realistic as it corresponds to the finite mobility (cf. $\Gamma' = \hbar e/(2m\mu_{\rm b})$) and the finite chemical potential. We now wish to see how does

the finite mobility (or disorder inclusion) affect the aforementioned QPT.

We find using the infinite SQ potential values that, at finite chemical potential and at the temperature ≈ 50 mK, the system undergoes a zero-order QPT at fixed Δ_c and the constant layer separation, as the inter-layer tunneling strength Δ_{SAS} is increased, from the effective two-component state (bilayer FQHS) to an effective single-component state (single layer FQHS); at finite and constant Δ_{SAS} a transition from the latter state to the former state is also possible upon increasing Δ_c. The latter one is regarded to be a non-Abelian FQHS useful for the topological quantum computation. The result remains essentially the same with the FH potential. We have shown these outcomes in Fig. 1 where the order parameter of QPT (Γ) is plotted as a function of the inter-layer tunneling strength (Δ_{SAS}) and the charge imbalance regulatory parameter (Δ_c). The cold (deep blue) region in these contour plots correspond to the former state whereas the hot (red) region to the latter. In the previous theoretical works [2–5], the former state has been linked with the Halperin Abelian 331 FQHS and the latter with the non-Abelian Moore–Read Pfaffian FQHS. It is also reported that theoretically it is not possible for the Pf FQHS to exist in the LLL, and therefore this type of phase transition is unlike for the $\nu = \frac{1}{2}$ case.

The plot in Fig. 1a corresponds to high mobility $\mu_b \approx 10^3$ m^2 V^{-1} s^{-1} (clean sample). We find that in this case the single-layer FQHS is more robust, i.e. it corresponds to greater area on the Δ_{SAS}–Δ_c zone, with the quantum critical region (lighter blue, green, and yellow) sandwiched between the regions corresponding to bilayer FQHS and the single-layer FQHS.

In the regime of lower mobility $\mu_b \approx 10$ m^2 V^{-1} s^{-1} (see Fig. 1b) where disorder is present, the scenario remains almost the same provided the parameter Δ_c is greater than 0.1. If the parameter Δ_c is kept fixed at a value less than 0.1, upon increasing Δ_{SAS} from zero, one encounters a crossover from the bilayer FQHS to the single-layer one at $\Delta_{1,SAS} \approx 0.005$. This is followed by an entry to quantum critical state close to $\Delta_{2,SAS} = 0.1$ and the (quantum phase) transition to the bilayer state beyond. The possible explanation of this transition is the quasi-particle localization by disorder at very low temperature. Upon increasing Δ_{SAS} beyond $\Delta_{3,SAS} = 0.2$ there is a crossover to the single-layer state once again as the (increased) tunneling starts dominating now over the disorder.

Through the blow-up in Fig. 1c (the almost semi-circular region) this intriguing behavior of the system is clearly depicted. We find that this manifestation of the localization (by disorder) is very sensitive to the increase in temperature. For example, as could be seen in Fig. 1d which corresponds to a temperature of 1 K, there is a softened trace of this phenomenon at the inter-layer tunneling strength 0.1.

In Fig. 2 we have shown a graph of QPT order parameter as a function of inter-layer tunneling strength Δ_{SAS} for the charge imbalance parameter equal to 0.0004. The graph reveals a range of negative susceptibility ($\partial\Gamma/\partial\Delta_{SAS}$) for a dirty sample. For a clean sample (mobility $\mu_b \approx 10^3$–10^4 m^2 V^{-1} s^{-1}), where the quasi-particle localization by disorder will not be possible, there is no negative susceptibility.

4. Concluding remarks

The scenario portrayed above is somewhat reminiscent (albeit remotely) of the findings of Ugeda et al. [22] where, atomic vacancies have been artificially generated on a graphite surface and, the formation of local magnetic moments corresponding to the so-called "zero-mode state" with considerable reduction in the carrier mobility has been reported in the neighborhood of the vacant sites. But the similarity ends here as the recently published results [23, 24] indicate the absence of a non-Abelian Pfaffian phase in graphene.

For the spin polarized fermions (filling fraction $\nu = 5/2$) in semi-conductor quantum well system, whereas the numbers, $\Delta_{1,SAS} \approx 0.005$, $\Delta_{2,SAS} = 0.1$, etc. we obtained above may be artifacts of the approximation made, it is quite clear that in the disordered sample, for very low charge imbalance parameter and the temperature, we have a fascinating competition between disorder and the inter-layer tunneling which leads to (three) critical values of Δ_{SAS} characterizing completely reversible smooth crossovers and QPT. The estimation of mobility-lower-limit which could still be a guarantor of the reversibility is an open issue. We notice that a low temperature, close to $T = 0$ K, although suppresses thermally excited quasi-particles but does not preclude disorder induced localized quasi-particles. Such disorder driven scenario may find application in a variety of areas requiring memory functions and gating operations. Indeed, the vitality of the entire field of quantum Hall physics rests upon such possibilities.

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