The Ramsauer–Townsend Effect in the Presence of the Minimal Length and Maximal Momentum

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The scattering cross-section of electrons in noble gas atoms exhibits the minimum value at electron energies of approximately 1 eV. This is the Ramsauer–Townsend effect. In this letter, we study the Ramsauer–Townsend effect in the presence of both the minimal observable length and the maximal momentum (originating from doubly special theories) through the generalized uncertainty principle.

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1. Introduction

Various approaches to quantum gravity such as string theory and loop quantum gravity as well as black hole physics, in contradiction with the Heisenberg uncertainty principle which in principle agrees with the measurement of highly accurate results for a particles' positions or momenta, predict the minimum measurable length of the order of the Planck length, $\ell_p = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35}$ m. In the presence of this minimal observable length, the standard Heisenberg uncertainty principle attains an important modification leading to the so-called generalized uncertainty principle (GUP). As a result, corresponding commutation relations between position and momenta are generalized, too [1].

In recent years a lot of attention has been attracted to extend the fundamental problems of physics in the GUP framework [2–13]. Since in the GUP framework one cannot probe distances smaller than the minimum measurable length at a finite time, we expect it modifies the Hamiltonian of physical systems, too (see [5] for instance). Recently, a GUP is proposed by Ali et al. which is consistent with the existence of the minimal measurable length and the maximal measurable momentum [14, 15]. The existence of the maximal particles' momentum is a consequence of doubly special relativity theories (see for instance [16]). These natural cutoffs have their origin in the very nature of spacetime manifold at the Planck scale.

In this work we will follow the procedure in Ref. [9], but we are going to address the effect of minimal length and maximal momentum on the Ramsauer–Townsend effect.

The Ramsauer–Townsend effect can be observed as long as the scattering does not become inelastic by excitation of the first excited state of the atom. This condition is best fulfilled by the closed shell noble gas atoms. Physically, the Ramsauer–Townsend effect may be thought of as a diffraction of the electron around the rare-gas atom, in which the wave function inside the atom is distorted in just such a way that it fits on smoothly to an undistorted wave function outside. The effect is analogous to the perfect transmission found at particular energies in one-dimensional scattering from a square well. The one-dimensional treatment of scattering from a square well and also three-dimensional treatment using the partial waves analysis can be found in Ref. [17].

In a recent work we have addressed the quantum gravity effects, through existence of just the minimal measurable length encoded in the generalized uncertainty principle, on the scattering amplitude in the Ramsauer–Townsend effect [18].

Here we generalize that work to the one-dimensional treatment of the scattering from a square well in the presence of both the minimal observable length and also a *maximal observable momentum*. The existence of the maximal momentum for scattered particles, brings new additional correction on the wavelength of scattered particles leading to a new condition for resonance in the Fabry–Perot interferometer. In this respect, we address the condition for interference in the Fabry–Perot interferometer in the presence of minimal observable length and the maximal observable momentum. We note that modification to the transmission rate due to the existence of the minimal length and the maximal momentum studied here, becomes important at or above the Planck energy. Although these modifications are too small to be measurable at present, we speculate on the possibility of

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extracting measurable predictions in the future. Any experimental evidence of these predictions may provide a direct test of underlying quantum gravity scenario.

2. The generalized uncertainty principle

The following GUP which was proposed by Ali et al. [14, 15] is consistent with black hole physics and string theory and ensures space and momentum commutation relations ($[X_i, X_j] = 0$ and $[P_i, P_j] = 0$) separately

$$[X_i, P_j] = \mathrm{i}\hbar\left[\delta_{ij} - \alpha\left(P\delta_{ij} + \frac{P_i P_j}{P}\right) + \alpha^2\left(P^2\delta_{ij} + 3P_iP_j\right)\right], \tag{1}$$

where $\alpha = \alpha_0/M_{\mathrm{Pl}}c = \alpha_0\ell_{\mathrm{Pl}}/\hbar$, M_{Pl} is the Planck mass, ℓ_{Pl} is the Planck length $\approx 10^{-35}$ m, and $M_{\mathrm{Pl}}c^2$ is the Planck energy $\approx 10^{19}$ GeV. It is normally assumed that $\alpha_0 \approx 1$. Using the above commutation relations, we can obtain the generalized uncertainty relation in one-dimension up to the second order of the GUP parameter [14, 15]:

$$\Delta X \Delta P \geq \frac{\hbar}{2}\left[1 - 2\alpha\langle P\rangle + 4\alpha^2\langle P^2\rangle\right] \geq \frac{\hbar}{2}\left[1 + \left(\frac{\alpha}{\sqrt{\langle P^2\rangle}} + 4\alpha^2\right)\Delta P^2 + 4\alpha^2\langle P\rangle^2 - 2\alpha\sqrt{\langle P^2\rangle}\right]. \tag{2}$$

The above inequality implies both the minimum length and the maximum momentum at the same time, namely [14, 15]:

$$\begin{cases} \Delta X \geq (\Delta X)_{\min} \approx \alpha_0\ell_{\mathrm{Pl}}, \\ \Delta P \leq (\Delta P)_{\max} \approx \frac{M_{\mathrm{Pl}}c}{\alpha_0}. \end{cases} \tag{3}$$

We note that while with lower bound for position fluctuations, one can rightfully claim that there is the minimum measurable distance, the way from an upper bound of momentum fluctuations to the maximum measurable momentum is not so clear. In fact, existence of an upper bound for momentum fluctuations just means that momentum measurements cannot be arbitrarily imprecise, but it says nothing about the measured momentum or momentum expectation values. We can also rewrite the position and momentum operators in terms of new variables [13]:

$$\begin{cases} X_i = x_i, \\ P_i = p_i\left(1 - \alpha p + 2\alpha^2 p^2\right), \end{cases} \tag{4}$$

where x_i and p_i obey the usual commutation relations $[x_i, p_j] = \mathrm{i}\hbar\delta_{ij}$. It is straightforward to check that with this definition, Eq. (1) is satisfied up to $\mathcal{O}(\alpha^2)$. Therefore, we can interpret p_i and P_i as follows: p_i is the momentum operator at low energies ($p_i = -\mathrm{i}\hbar\partial/\partial x_i$) and P_i is the momentum operator at high energies. Moreover, p is the magnitude of the p_i vector ($p^2 = \sum_{ij}^3 p_i p_j$).

To study the effects of this kind of GUP on the quantum mechanical systems, let us consider the following general Hamiltonian:

$$H = \frac{P^2}{2m} + V(x), \tag{5}$$

which using Eq. (4) can be written as

$$H = H_0 + \alpha H_1 + \alpha^2 H_2 + \mathcal{O}(\alpha^3), \tag{6}$$

where $H_0 = \frac{p^2}{2m} + V(x)$ and

$$H_1 = \frac{-p^3}{m}, \quad H_2 = \frac{5p^4}{m}. \tag{7}$$

In the quantum domain, this Hamiltonian results in the following generalized Schrödinger equation in the quasi-position representation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} - \mathrm{i}\alpha\frac{\hbar^3}{m}\frac{\partial^3\psi(x)}{\partial x^3} + 5\alpha^2\frac{\hbar^4}{m}\frac{\partial^4\psi(x)}{\partial x^4} + V(x)\psi(x) = E\psi(x), \tag{8}$$

where the second and third terms are due to the generalized commutation relation (1). This equation is a 4th-order differential equation which in principle admits 4 independent solutions. Therefore, solving this equation in x space and separating the physical solutions is not an easy task. A transformation to momentum space may help to overcome this difficulty in some cases.

3. Ramsauer–Townsend effect with GUP

For simplicity we restrict ourselves to a one-dimensional problem. We choose the following geometry of the quantum well:

$$V(x) = \begin{cases} -V_0, & 0 < x < a, \\ 0 & \text{elsewhere}, \end{cases} \tag{9}$$

where V_0 is a positive constant and $E > 0$. The geometry of the problem is shown in Fig. 1 (we note that this problem can be treated with more realistic potentials such as the Woods–Saxon potential to have more reliable results but the calculations become very complicated). The eigenfunctions of a particle in this potential well in the presence of both minimal length and maximal momentum satisfy the generalized Schrödinger Eq. (8).

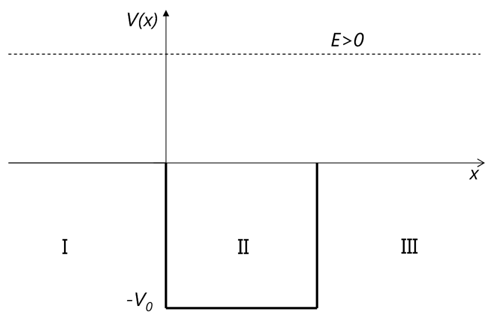

Fig. 1. The geometry of a quantum well.

So we need to find the solutions in three regions which are indicated in Fig. 1. To proceed further, we can rewrite Eq. (8) in these regions separately

$$q^2\psi(x)_{\mathrm{II}} + \mathrm{d}^2\psi(x)_{\mathrm{II}} + \mathrm{i}\ell_p\mathrm{d}^3\psi(x)_{\mathrm{II}} - \frac{5}{2}\ell_p^2\mathrm{d}^4\psi(x)_{\mathrm{II}} = 0, \tag{10}$$

for $0 < x < a$, and

$$k^2\psi(x)_{\mathrm{I,III}} + \mathrm{d}^2\psi(x)_{\mathrm{I,III}} + \mathrm{i}\ell_p\mathrm{d}^3\psi(x)_{\mathrm{I,III}} - \frac{5}{2}\ell_p^2\mathrm{d}^4\psi(x)_{\mathrm{I,III}} = 0 \tag{11}$$

for regions I and III. By definition $\mathrm{d}^n \equiv \frac{\mathrm{d}^n}{\mathrm{d}x^n}$, $k = \sqrt{\frac{2mE}{\hbar^2}}$, $q = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$, and $\ell_p = 2\alpha\hbar$. Let us note that the above equations are fourth-order differential equations which in general admit four independent solutions. However, some solutions would be unphysical which should be removed upon imposing the boundary conditions. Assuming solutions of the form $\psi_{\mathrm{I,II,III}} = \mathrm{e}^{mx}$ yields [14]:

$$m^2 + k^2 + \mathrm{i}\ell_p m^3 - \frac{5}{2}\ell_p^2 m^4 = 0, \quad \text{in I, III}, \tag{12}$$

$$m^2 + q^2 + \mathrm{i}\ell_p m^3 - \frac{5}{2}\ell_p^2 m^4 = 0, \quad \text{in II} \tag{13}$$

with the following solution sets to leading order in α, each consisting of four values of m:

$$\begin{aligned} &\mathrm{I,\ III}\ :\ m = \{\mathrm{i}k', -\mathrm{i}k'', \pm\mathrm{i}/\ell_p\}, \\ &\mathrm{II}\ :\ m = \{\mathrm{i}q', -\mathrm{i}q'', \pm\mathrm{i}/\ell_p\}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} k' &= k\big(1 + k\ell_p/2 - 5k^2\ell_p^2/4\big), \\ k'' &= k\big(1 - k\ell_p/2 - 5k^2\ell_p^2/4\big), \\ q' &= q\big(1 + q\ell_p/2 - 5q^2\ell_p^2/4\big), \\ q'' &= q\big(1 - q\ell_p/2 - 5q^2\ell_p^2/4\big). \end{aligned} \tag{15}$$

The wave functions in the I, II and III regions are

$$\psi_{\mathrm{I}} = \mathrm{e}^{\mathrm{i}k'x} + A\mathrm{e}^{-\mathrm{i}k''x} + B\mathrm{e}^{\mathrm{i}\frac{x}{\ell_p}}, \tag{16}$$

$$\psi_{\mathrm{II}} = F\mathrm{e}^{\mathrm{i}q'x} + G\mathrm{e}^{-\mathrm{i}q''x} + H\mathrm{e}^{\mathrm{i}\frac{x}{\ell_p}} + L\mathrm{e}^{-\mathrm{i}\frac{x}{\ell_p}}, \tag{17}$$

$$\psi_{\mathrm{III}} = C\mathrm{e}^{\mathrm{i}k'x} + D\mathrm{e}^{-\mathrm{i}\frac{x}{\ell_p}}, \tag{18}$$

respectively, where based on physical grounds we have omitted the left-mover component from ψ_{III} and the exponentially growing terms from both ψ_{I} and ψ_{III} [9]. In comparison to the case that there is just the minimal observable length, we see the appearance of an oscillatory term here with characteristic wavelength $2\pi\ell_p$ and momentum $1/4\alpha = \hbar/4\ell_p\alpha_0$. This is due to existence of the maximal momentum in this case. Now, the boundary conditions consist of eight equations as follows:

$$\mathrm{d}^n\psi_{\mathrm{I}}|_{x=0} = \mathrm{d}^n\psi_{\mathrm{II}}|_{x=0}, \quad n = 0,1,2,3, \tag{19}$$

$$\mathrm{d}^n\psi_{\mathrm{II}}|_{x=a} = \mathrm{d}^n\psi_{\mathrm{III}}|_{x=0}, \quad n = 0,1,2,3. \tag{20}$$

By setting $k' = k''$ and $q' = q''$ in the above equation, the solutions are similar to the solutions of the ordinary quantum mechanics but with modified wave number. Now the boundary conditions are the continuity of the wave function and its first derivative at the boundaries. The resulting equations can be solved analytically to obtain the coefficients A, B, C, and D. For our purposes, the solution for A is as follows:

$$A = \frac{(k'^2 - q'^2)\sin(q'a)}{(k'^2 + q'^2)\sin(q'a) + 2\mathrm{i}k'q'\cos(q'a)}. \tag{21}$$

So the reflection coefficient is given by

$$R = |A|^2 = \frac{(k'^2 - q'^2)^2\sin^2(q'a)}{(k'^2 + q'^2)^2\sin^2(q'a) + 4k'^2q'^2\cos^2(q'a)}. \tag{22}$$

By using Eq. (15) we can write the reflection coefficient in terms of the physical wave numbers as

$$R = \left[Q\sin^2\left(q\left(1 - \frac{\ell_p}{2}q - \frac{5\ell_p^2}{4}q^2\right)a\right)\right] \Big/ \left[P\sin^2\left(q\left(1 - \frac{\ell_p}{2}q - \frac{5\ell_p^2}{4}q^2\right)a\right) + Z\cos^2\left(q\left(1 - \frac{\ell_p}{2}q - \frac{5\ell_p^2}{4}q^2\right)a\right)\right], \tag{23}$$

where by definition

$$Q \equiv \left[\left(k\left(1 - \frac{\ell_p}{2}k - \frac{5\ell_p^2}{4}k^2\right)\right)^2 - \left(q\left(1 - \frac{\ell_p}{2}q - \frac{5\ell_p^2}{4}q^2\right)\right)^2\right]^2,$$

$$P \equiv \left[\left(k\left(1 - \frac{\ell_p}{2}k - \frac{5\ell_p^2}{4}k^2\right)\right)^2 + \left(q\left(1 - \frac{\ell_p}{2}q - \frac{5\ell_p^2}{4}q^2\right)\right)^2\right]^2,$$

$$Z \equiv \left[k\left(1 - \frac{\ell_p}{2}k - \frac{5\ell_p^2}{4}k^2\right)q\left(1 - \frac{\ell_p}{2}q - \frac{5\ell_p^2}{4}q^2\right)\right]^2.$$

For the special case where $\sin\big(q\big(1-\frac{\ell_p}{2}q-\frac{5\ell_p^2}{4}q^2\big)a\big) = 0$, there is no reflection that is $R = 0$ and therefore we will have maximum transmission. This is the Ramsauer–Townsend effect. In this case

$$q\left(1 - \frac{\ell_p}{2}q - \frac{5\ell_p^2}{4}q^2\right) = \frac{n\pi}{a}. \tag{24}$$

In ordinary quantum mechanics this effect occurs at those wave numbers that satisfy the condition $q_{\mathrm{ord}} = \frac{n\pi}{a}$. This feature shows that there is a shift ($\Delta_q = q - q_{\mathrm{ord}}$) in the wave number of the transmission resonance and this shift itself is wave number dependent. So, up to the first order in the GUP parameter we have

$$\Delta_q \simeq \frac{\ell_p}{2}\left(\frac{n\pi}{a}\right)^2 + \frac{5\ell_p^2}{4}\left(\frac{n\pi}{a}\right)^3. \tag{25}$$

We also note that in ordinary quantum mechanical description, the condition for resonance is $\lambda_{\mathrm{ord}} = \frac{2\pi}{q} = \frac{2a}{n}$ which is the same condition as in the Fabry–Perot interferometer. In the presence of the minimal length and maximal momentum, this condition modifies as follows

$$\lambda' = \frac{2\pi}{q'} \simeq \frac{2\pi}{q}\left(1 + \frac{\ell_p}{2}q + \frac{5\ell_p^2}{4}q^2\right)$$
$$= \lambda_{\mathrm{ord}}\left(1 + \frac{\ell_p}{2}q + \frac{5\ell_p^2}{4}q^2\right). \tag{26}$$

Therefore, in the presence of the minimal length and maximal momentum, the condition for interference in the Fabry–Perot interferometer will change. Amazingly, this change is itself wavelength dependent.

4. Conclusion

The scattering cross-section of electrons in noble gas atoms exhibits the minimum value at electron energies of approximately 1 eV, an effect of which is called the Ramsauer–Townsend effect. We studied the Ramsauer–Townsend effect in the presence of minimal length and maximal momentum in the framework of a newly proposed generalized uncertainty principle. We have shown that in the presence of the minimal length and maximal momentum there is a shift ($\Delta_q = q - q_{\mathrm{ord}}$) in the wave number of the transmission resonance and this shift itself is wavenumber dependent. This shift also affects the resonance wavelength in the Fabry–Perot interferometer in such a way that this change is itself wavelength dependent. If in future experiments one finds a similar shift in the Fabry–Perot interferometer resonance wavelength, it will be a footprint of quantum gravity effect.

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