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The Symmetry of Current of the Coherent State from the View of $O(3)$ σ-Model

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In this paper, we apply the reduced density trajectory, ϕ-mapping topological current theory and Ginzburg–Landau model to study the current of the coherent state. We give the new expression of the current of the coherent state. Based on this expression, the symmetry of the coherence is studied. We find that the current of the coherent state corresponds to the supercurrent of two-condensate system. The partial wave functions of the coherence carry new charges and their interaction is mediated by new U(1) gauge potential.

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1. Introduction

The idea is inspired by the quantum trajectory description of decoherence [1]. The trajectory was first proposed by Bohm when he made a suggested interpretation of the quantum theory for hidden variables [2]. The theory is known as the de Broglie–Bohm (BB) interpretation of quantum mechanics. In the theory, all particles have well-defined trajectories. The motions of the particles are governed by the wave functions that satisfy the Schrödinger equation. Therefore the BB quantum theory of motion is a suitable tool with which one studies coherence and decoherence [3, 4]. The one reason of the decoherence is that the open quantum system interacts with the environment. Unfortunately, it is very difficult to deal with decoherence problem using this quantum trajectory approach because the environment usually involves large number of degrees of the freedom. To overcome this drawback, we assume the environment to be the Markovian environment and describe the whole system by a Markovian master equation. This equation introduces two contributions: the time-evolution of the coherent state and the quenching factor leading to decoherence. The quenching factor accounts for physical properties of the environment and its interaction with the coherent system. Combining the trajectory theory with reduced density matrix theory yields a new trajectory called reduced quantum trajectory [1]. The advantage of this reduced quantum trajectory is that the environment effects are described by a time-dependent damping factor when these trajectories are applied to the study of an open quantum system. The reduced quantum trajectory then describes in detail the evolution of the coherent state. These provide insight in understanding decoherence.

Recently, the discovery of the high critical temperature of MgB_2 has inspired a wide interest in the charged two-condensate superconductors [5–7]. The two charged condensates in the superconductor are tightly bound fermion pairs, or some other charged bosonic fields such as electronic or protonic Cooper pairs in metallic hydrogen under certain condition [8]. The charged two-condensate wave functions correspond to the order parameters of the two different parts of the Fermi surface. They are coupled because of their electromagnetic interaction. The system is described by the Ginzburg–Landau model with two flavors of Cooper pairs [9–11]. In Ref. [9], the authors showed that the charged-condensate Ginzburg–Landau model can be mapped onto a version of the nonlinear $O(3)$ σ-model and found that this system possesses a hidden $O(3)$ symmetry. There is a stable knot solution in the superconductor. This provides us with a new way to investigate the coherent quantum system.

The topology and geometry play an important role in physics and mathematics and a great deal of works have been done in the topology and geometry [12–17]. Especially, the vorticity of the vortex in condensate meter and topology of the physical system have been studied by applying the ϕ-mapping topological current theory [18–21].

In this paper, we present the relation between the current of coherent state and the supercurrent of the two-gap condensate system. The paper is organized as follows: in Sect. 2, the ϕ-mapping topological current theory in reduced density trajectory is given. The current of the coherent state is also presented. In Sect. 3, the new expression of the current is derived. We find that this current is similar to the supercurrent of the charged two-condensate system. In Sect. 4, the symmetry and the topological properties of the current of the coherent state are studied based on Faddeev's $O(3)$ nonlinear σ-model. Finally, we make a conclusion.

2. ϕ-Mapping topological current theory in reduced density trajectory and the current of the coherent state

We give a brief review of the reduced quantum trajectory approach as presented in [1]. We start with the

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calculation of the reduced density matrix. The total density matrix of the system is given by

$$\widehat{\rho} = |\psi_t\rangle \times \langle_t\psi|, \tag{1}$$

where the subscript t denotes the time-dependence of the wave function. We take the environment degrees of freedom to be $\boldsymbol{r}_i$ $(i = 1, \ldots, N)$. The system's reduced density matrix is then given by tracing the total density matrix $\widehat{\rho}$ over the environment degrees of freedom, resulting in

$$\widetilde{\rho}_t(\boldsymbol{r}, \boldsymbol{r}') = \int \langle \boldsymbol{r}, r_1, \ldots \boldsymbol{r}_N|\psi_t\rangle \times \langle_t\psi|\boldsymbol{r}', \boldsymbol{r}'_1, \ldots \boldsymbol{r}'_N\rangle\, \mathrm{d}\boldsymbol{r}_1 \ldots \mathrm{d}\boldsymbol{r}_N. \tag{2}$$

Next the system reduced quantum density current can be derived as follows:

$$\widetilde{\boldsymbol{J}}_t = \frac{\hbar}{m}\mathrm{Im}(\nabla_r \widetilde{\rho}_t(\boldsymbol{r}, \boldsymbol{r}'))_{r=r'}, \tag{3}$$

where $\widetilde{\boldsymbol{J}}_t$ satisfies the continuity equation, which is given as

$$\partial_t \widetilde{\rho}_t + \nabla \widetilde{\boldsymbol{J}}_t = 0, \tag{4}$$

where $\widetilde{\rho}_t$ is the diagonal element of the reduced density matrix, which provides the measured intensity. We now define the Bohmian-like velocity using (3) and (4):

$$\boldsymbol{V} = \frac{\widetilde{\boldsymbol{J}}_t}{\widetilde{\rho}_t}. \tag{5}$$

Therefore, we can define a new trajectory associated with the reduced density matrix

$$\boldsymbol{V} = \frac{\hbar}{m}\frac{\mathrm{Im}(\nabla_r \widetilde{\rho}_t(\boldsymbol{r}, \boldsymbol{r}'))}{\mathrm{Re}(\widetilde{\rho}_t(\boldsymbol{r}, \boldsymbol{r}'))}_{r=r'}, \tag{6}$$

which is called reduced quantum trajectory. The disadvantage of this definition of velocity is it is difficult to give the detailed information at $\widetilde{\rho}_t = 0$, or at the zero points of the wave functions. These zero points are the singularity of the velocity. Next, we will illustrate the exact expression of the velocity field and its topology at zero point of wave functions based on ϕ-mapping topological current theory. To do this, we must consider the BB quantum mechanics ansatz of the wave function

$$\langle r|\psi_t\rangle = R_t(r)\,\mathrm{e}^{\mathrm{i}S_t(r)/\hbar}, \tag{7}$$

from the topological viewpoint, the wave function $\langle r|\psi_t\rangle$ is the section of the complex linear bundle, i.e. a section of 2-dimensional real vector bundle. We can then write this ansatz as

$$\langle r|\psi_t\rangle = \phi^1 + \mathrm{i}\phi^2. \tag{8}$$

Defining the unit vector of this ansatz yields

$$n^1 = \frac{\phi^1}{\|\langle r|\psi_t\rangle\|}, \quad n^2 = \frac{\phi^2}{\|\langle r|\psi_t\rangle\|}. \tag{9}$$

It is obvious that the unit vector satisfies the condition

$$n^a n^a = 1, \quad a = 1, 2. \tag{10}$$

Using this unit vector and (6), we write the velocity as

$$\boldsymbol{V}_i = \frac{\hbar}{m}\epsilon_{ab} n^a \partial_i n^b. \tag{11}$$

In traditional quantum mechanics, the curl of the velocity vanishes at zero points of the wave functions. However, the curl of the velocity must be modified along trajectories because $\boldsymbol{\nabla} \times \boldsymbol{V}$ need not vanish at nodal points of the wave function [18]. The curl of the velocity is

$$\boldsymbol{\nabla} \times \boldsymbol{V} = \frac{\hbar}{m}\left(\epsilon^{ijk}\epsilon_{ab}\partial_j n^a \partial_k n^b\right)\boldsymbol{e}_i. \tag{12}$$

Using Eqs. (9), the curl of the velocity can further be written as

$$\boldsymbol{\nabla} \times \boldsymbol{V} = \frac{\hbar}{m}\boldsymbol{e}_i \epsilon^{ijk}\epsilon_{ab}\frac{\partial}{\partial\phi^c}\frac{\partial}{\partial\phi^a}(\ln\|\phi\|)\partial_j\phi^c\partial_k\phi^b. \tag{13}$$

Defining the vector Jacobian of $\boldsymbol{\phi}$ by

$$\boldsymbol{e}_i \epsilon^{ijk}\partial_j\phi^c\partial_k\phi^b = \epsilon^{cb}\boldsymbol{D}\left(\frac{\phi}{x}\right), \tag{14}$$

and using the well-known result from the Green function theory in ϕ-space, we find that

$$\frac{\partial}{\partial\phi^a}\frac{\partial}{\partial\phi^a}\ln\|\phi\| = 2\pi\delta^2(\phi). \tag{15}$$

Finally, the curl of the velocity is

$$\boldsymbol{\nabla} \times \boldsymbol{V} = \frac{\hbar}{m}2\pi\delta^2(\phi)\boldsymbol{D}\left(\frac{\phi}{x}\right), \tag{16}$$

where $\boldsymbol{D}\left(\frac{\phi}{x}\right)$ is the vector Jacobian of $\boldsymbol{\phi}$ and satisfies $\epsilon^{ijk}\partial_j\phi^c\partial_k\phi^b = \epsilon^{cb}D^i\left(\frac{\phi}{x}\right)$. From this, we learn that the trajectory is at the zero point of the wave function. We consider, in general, a vector field $\boldsymbol{\phi}$ on the smooth manifold Σ; a zero point p is a singular point of $\boldsymbol{\phi}$ if $\boldsymbol{\phi}_p = 0$. Consider a closed curve $\gamma \in \Sigma$ encircling but never touching p. In completing one turn along γ, the vector field $\boldsymbol{\phi}$ will turn around itself a certain number of times. By appropriately assigning signs to the direction of the turn, the algebraic sum of turns is called index of the curve. It is well known that the sum of all the indices of a chosen vector field $\boldsymbol{\phi}$ on a compact differentiable manifold Σ equals the Euler–Poincaré characteristic of Σ that describes the topological properties of singular points. In application here, all nodal points form the zero-line of wave function and the zero-line of wave function is just the locations of trajectories in de Broglie–Bohm quantum mechanics. The zero points can be denoted by z_l^i, where l represent the ℓ isolated zero points on Σ. We assume that $u = (u_1, u_2)$ are the coordinates, so that $\delta^2(\phi)$ can be expanded at the zero point

$$\delta^2(\phi) = \sum_{l=1}^{\ell} C_l\delta^2\left(x^i - z_l^i\right), \tag{17}$$

where C_l are positive coefficients. The winding number of the l-th trajectory is

$$W(\phi, z_i) = C_l\int_\Sigma \delta^2\left(x^i - z_l^i\right)D\left(\frac{\phi}{x}\right)\mathrm{d}^2x$$

$$= C_l D\left(\frac{\phi}{u}\right)_{z_l}. \tag{18}$$

Here, $D\left(\frac{\phi}{u}\right)$ is

$$D\left(\frac{\phi}{u}\right) = \frac{1}{2}\epsilon^{jk}\epsilon_{ab}\frac{\partial}{\partial u^j}\phi^a\frac{\partial}{\partial u^k}\phi^b. \tag{19}$$

If we let

$$|W_l| = |W(\phi, z_l)| = \beta_l, \tag{20}$$

where β_l is the Hopf index of ϕ-mapping on Σ, with the interpretation that the function $\boldsymbol{\phi}$ covers the corresponding region in ϕ-space β_l times when a point covers the neighborhood of the zero point z_l^i once. Furthermore, $\delta^2(\phi)$ can be expressed as

$$\delta^2(\phi) = \sum_{l=1}^{\ell}\frac{\beta_l}{\left|D\left(\frac{\phi}{u}\right)\right|_{z_l}}\delta^2(x^i - z_l^i). \tag{21}$$

Let us define

$$\eta_l = \operatorname{sgn} D\left(\frac{\phi}{u}\right)_{z_l} = \frac{D\left(\frac{\phi}{u}\right)}{\left|D\left(\frac{\phi}{u}\right)\right|_{z_l}} = \pm 1, \tag{22}$$

which is called the Brouwer degree of the map $x \to \phi(x)$. Finally, the vorticity of the velocity at the zero points on Σ is

$$\Gamma = \int_{\Sigma}(\boldsymbol{\nabla}\times\boldsymbol{V})\cdot \mathrm{d}\boldsymbol{S} = \frac{h}{m}\sum_l \beta_l\eta_l = \frac{h}{m}W, \tag{23}$$

where W is the winding number of the zero points of the trajectories on Σ. The zero points on the plane can be seen as the topological solutions of the equation $\delta^2(\phi)$ and can be written as

$$\phi^1(x^\mu) = 0, \quad \phi^2(x^\mu) = 0, \tag{24}$$

where $\mu = 1, 2, 3\ldots$

Considering a quantum system in the double-slit experiment, the system is described by the coherent state of a particle and the state of the environment. The coherent state of a particle is

$$|\Psi_t\rangle = c_1|\psi_{1,t}\rangle + c_2|\psi_{2,t}\rangle, \tag{25}$$

where the coefficients c_a satisfy the condition

$$|c_1|^2 + |c_2|^2 = 1. \tag{26}$$

We assume that the environment states are subject to the elastic system–environment scattering conditions [1], then only the environment state will evolve with time. The environment state associated with each partial wave is denoted by $|H_\alpha\rangle$. The initial state of the environment states can be given by

$$|H_1\rangle = |H_2\rangle = |H_0\rangle. \tag{27}$$

Using BB quantum mechanics ansatz, the coherent state can be described without considering the interaction between coherence states and the environment

$$\Psi_t(r) = \langle r|\Psi_t\rangle. \tag{28}$$

The density matrix associated with coherent state is

$$\rho_t(\boldsymbol{r},\boldsymbol{r}') = \Psi_t(\boldsymbol{r},\boldsymbol{r}')[\Psi_t(\boldsymbol{r},\boldsymbol{r}')]^*. \tag{29}$$

The diagonal element of this density matrix is the measured intensity. We write it as

$$\rho_t = |c_1|^2|\psi_{1,t}|^2 + |c_2|^2|\psi_{2,t}|^2 + 2|c_1||c_2||\psi_{1,t}|_t|\psi_{2,t}|\cos\delta_t, \tag{30}$$

where δ_t is the time-dependent phase shift between the partial waves. Similarly, the partial wave function $\psi_{i,t}$ can be written as

$$\psi_{i,t} = \langle r|\psi_{i,t}\rangle. \tag{31}$$

In addition to writing the measured intensity for $\Psi_t(r)$, we define the measured intensity of the partial wave function $\rho_t^{(i)}$ as

$$\rho_t^{(i)} = \psi_{i,t}^*\psi_{i,t}, \quad i = 1, 2, \ldots \tag{32}$$

The partial wave function can also be expressed as

$$\psi_{i,t} = \phi_{i,t}^1 + \mathrm{i}\phi_{i,t}^2. \tag{33}$$

Recalling (9), the unit vector $\boldsymbol{n}_{(i)}$ of the partial wave function $\psi_{i,t}$ is defined by

$$n_{(i)}^1 = \frac{\phi_{i,t}^1}{\|\psi_{i,t}\|}, \quad n_{(i)}^2 = \frac{\phi_{i,t}^2}{\|\psi_{i,t}\|}. \tag{34}$$

The general initial coherent states get entangled with the environment states when the environment is considered. The initial entangled state is

$$|\Psi\rangle = |\Psi_0\rangle \otimes |H_0\rangle, \tag{35}$$

where $|\Psi_0\rangle$ is the wave function $|\Psi_t\rangle$ at time $t = 0$. The time-dependence of the entangled state is

$$|\Psi_t\rangle = c_1|\psi_{1,t}\rangle \otimes |H_{1,t}\rangle + c_2|\psi_{2,t}\rangle \otimes |H_{2,t}\rangle, \tag{36}$$

where $|H_{i,t}\rangle$ is the time-dependent environment. Then we obtain the measured intensity of the entangled state by tracing the full density matrix over the environment state

$$\widetilde{\rho}_t = \sum_{a=1}^{2}\langle H_{a,t}|\widehat{\rho}|H_{a,t}\rangle. \tag{37}$$

Substituting (36) and (1) into (37), one obtains the measured intensity by tracing the total density matrix over the environmental degrees of freedom

$$\widetilde{\rho}_t = \left(1 + |a_t|^2\right)\sum_{i=1}^{2}|c_i|^2\psi_{i,t}^*\psi_{i,t} + 2a_t c_1 c_2^*\psi_{1,t}\psi_{2,t}^* + \text{c.c.} \tag{38}$$

This equation means that the interaction between the coherence state and the environment is the reason of the decoherence. The coefficient $a_t = \langle H_{2,t}|H_{1,t}\rangle$ is called the damping factor and indicates the degree of coherence. The cross terms $c_1c_2^*\psi_{1,t}\psi_{2,t}^*$ and its conjugate complex in (38) disappear; when $a_t = 0$, the coherent state suffers a total loss of coherence. If one introduces the coherence time τ, then this damping factor can be written as $a_t = \mathrm{e}^{-t/\tau}$. By using (6) and (38), the current is given by

$$\boldsymbol{J} = \widetilde{\rho}_t \boldsymbol{V} = \frac{\mathrm{i}\left(1+|a_t|^2\right)\hbar}{2m} \sum_{i=1}^{2} |c_i|^2 \left(\psi_{i,t}^* \nabla \psi_{i,t} - \psi_{i,t} \nabla \psi_{i,t}^*\right) + \frac{\mathrm{i}\hbar}{m} |a_t| c_1 c_2^* \left(\psi_{2,t}^* \nabla \psi_{1,t} - \psi_{1,t} \nabla \psi_{2,t}^*\right) + \text{c.c.} \tag{39}$$

3. The current as a supercurrent in two-condensate system

From Eq. (39), the current is seen to be expressed as a sum of two contributions: the first term, which does not include the cross term of the partial wave functions, will be denoted by $\boldsymbol{J}_1$:

$$\boldsymbol{J}_1 = \frac{\mathrm{i}\left(1+|a_t|^2\right)\hbar}{2m} \sum_{i=1}^{2} |c_i|^2 \left(\psi_{i,t}^* \nabla \psi_{i,t} - \psi_{i,t} \nabla \psi_{i,t}^*\right), \tag{40}$$

and the second term, which includes the cross term which indicates the coherent effects, will be written by $\boldsymbol{J}_2$:

$$\boldsymbol{J}_2 = \frac{\mathrm{i}\hbar}{m} |a_t| c_1 c_2^* \left(\psi_{2,t}^* \nabla \psi_{1,t} - \psi_{1,t} \nabla \psi_{2,t}^*\right) + \frac{\mathrm{i}\hbar}{m} |a_t| c_1^* c_2 \left(\psi_{1,t}^* \nabla \psi_{2,t} - \psi_{2,t} \nabla \psi_{1,t}^*\right). \tag{41}$$

In terms of the partial measured intensity of the partial wave function $\rho_t^{(i)}$, $\boldsymbol{J}_1$ is

$$\boldsymbol{J}_1 = \frac{\mathrm{i}\left(1+|a_t|^2\right)\hbar}{2m} \left[|c_1|^2 \left(\psi_{1,t}^* \psi_{1,t}\right) \times \frac{\left(\psi_{1,t}^* \nabla \psi_{1,t} - \psi_{1,t} \nabla \psi_{1,t}^*\right)}{\left(\psi_{1,t}^* \psi_{1,t}\right)} + |c_2|^2 \left(\psi_{2,t}^* \psi_{2,t}\right) \times \frac{\left(\psi_{2,t}^* \nabla \psi_{2,t} - \psi_{2,t} \nabla \psi_{2,t}^*\right)}{\left(\psi_{2,t}^* \psi_{2,t}\right)} \right]. \tag{42}$$

In a similar manner, $\boldsymbol{J}_2$ is also rewritten as

$$\boldsymbol{J}_2 = \frac{\mathrm{i}\hbar}{m} |a_t| \left[\left(c_1 c_2^* \psi_{1,t} \psi_{2,t}^* \frac{\psi_{1,t}^* \nabla \psi_{1,t}}{\psi_{1,t}^* \psi_{1,t}} - c_1^* c_2 \psi_{1,t}^* \psi_{2,t} \frac{\psi_{1,t} \nabla \psi_{1,t}^*}{\psi_{1,t}^* \psi_{1,t}} \right) \right] + \frac{\mathrm{i}\hbar}{m} |a_t| \left[\left(c_1^* c_2 \psi_{2,t} \psi_{1,t}^* \frac{\psi_{2,t}^* \nabla \psi_{2,t}}{\psi_{2,t}^* \psi_{2,t}} - c_1 c_2^* \psi_{2,t}^* \psi_{1,t} \frac{\psi_{2,t} \nabla \psi_{2,t}^*}{\psi_{2,t}^* \psi_{2,t}} \right) \right]. \tag{43}$$

Let us define the complex variable $\Lambda = c_1 c_2^* \psi_{1,t} \psi_{2,t}^*$; then $\Lambda^* = c_1^* c_2 \psi_{1,t}^* \psi_{2,t}$, the current $\boldsymbol{J}_2$ can be rewritten as

$$\boldsymbol{J}_2 = \frac{\mathrm{i}\hbar}{m} |a_t| \left(\Lambda \frac{\psi_{1,t}^* \nabla \psi_{1,t}}{\psi_{1,t}^* \psi_{1,t}} - \Lambda^* \frac{\psi_{1,t} \nabla \psi_{1,t}^*}{\psi_{1,t}^* \psi_{1,t}} \right) + \frac{\mathrm{i}\hbar}{m} |a_t| \left(\Lambda^* \frac{\psi_{2,t}^* \nabla \psi_{2,t}}{\psi_{2,t}^* \psi_{2,t}} - \Lambda \frac{\psi_{2,t} \nabla \psi_{2,t}^*}{\psi_{2,t}^* \psi_{2,t}} \right). \tag{44}$$

It is convenient to write $\Lambda = \Lambda_1 + \mathrm{i}\Lambda_2$ and $\Lambda^* = \Lambda_1 - \mathrm{i}\Lambda_2$, where Λ_1 and Λ_2 are real numbers. Substituting Λ_1 and Λ_2 into (44), one obtains

$$\boldsymbol{J}_2 = \frac{\mathrm{i}\hbar}{m} |a_t| \left[\Lambda_1 \left(\frac{\psi_{1,t}^* \nabla \psi_{1,t}}{\psi_{1,t}^* \psi_{1,t}} - \frac{\psi_{1,t} \nabla \psi_{1,t}^*}{\psi_{1,t}^* \psi_{1,t}} \right) + \mathrm{i}\Lambda_2 \left(\frac{\psi_{1,t}^* \nabla \psi_{1,t}}{\psi_{1,t}^* \psi_{1,t}} + \frac{\psi_{1,t} \nabla \psi_{1,t}^*}{\psi_{1,t}^* \psi_{1,t}} \right) \right] + \frac{\mathrm{i}\hbar}{m} |a_t| \left[\Lambda_1 \left(\frac{\psi_{2,t}^* \nabla \psi_{2,t}}{\psi_{2,t}^* \psi_{2,t}} - \frac{\psi_{2,t} \nabla \psi_{2,t}^*}{\psi_{2,t}^* \psi_{2,t}} \right) - \mathrm{i}\Lambda_2 \left(\frac{\psi_{2,t}^* \nabla \psi_{2,t}}{\psi_{2,t}^* \psi_{2,t}} + \frac{\psi_{2,t} \nabla \psi_{2,t}^*}{\psi_{2,t}^* \psi_{2,t}} \right) \right]. \tag{45}$$

In term of the relations

$$\nabla \ln\left(\psi_{i,t}^* \psi_{i,t}\right) = \left(\frac{\psi_{i,t}^* \nabla \psi_{i,t}}{\psi_{i,t}^* \psi_{i,t}} + \frac{\psi_{i,t} \nabla \psi_{i,t}^*}{\psi_{i,t}^* \psi_{i,t}} \right), \quad i = 1, 2, \tag{46}$$

finally, $\boldsymbol{J}_2$ can be expressed by

$$\boldsymbol{J}_2 = \frac{\mathrm{i}\hbar}{m} |a_t| \Lambda_1 \left[\left(\frac{\psi_{1,t}^* \nabla \psi_{1,t}}{\psi_{1,t}^* \psi_{1,t}} - \frac{\psi_{1,t} \nabla \psi_{1,t}^*}{\psi_{1,t}^* \psi_{1,t}} \right) + \left(\frac{\psi_{2,t}^* \nabla \psi_{2,t}}{\psi_{2,t}^* \psi_{2,t}} - \frac{\psi_{2,t} \nabla \psi_{2,t}^*}{\psi_{2,t}^* \psi_{2,t}} \right) \right] + \frac{\hbar}{m} |a_t| \Lambda_2 \boldsymbol{\nabla} \left[\ln\left(\frac{\psi_{1,t}^* \psi_{1,t}}{\psi_{2,t}^* \psi_{2,t}} \right) \right]. \tag{47}$$

This formula shows there is a topological reason leading to the decoherence. The new parameter Λ_1 can be used to indicate the coherent degree. This parameter also can be called damping factor, but it is very different from the parameter a_t. The parameter a_t relates to the degrees of the freedom of the environment. But from (30), the parameter Λ_1 relates to the phase shift of the partial wave functions. The parameter Λ_1 is indispensable to give the exact expression (47), which is essential for giving the topological structure of the current. Then the parameter Λ_1 is important to the topological structure of the current, but the parameter a_t has nothing to do with the topological structure. In addition, we find $\hbar\boldsymbol{\nabla}[\ln\left(\frac{\psi_{1,t}^* \psi_{1,t}}{\psi_{2,t}^* \psi_{2,t}}\right)]$ is a vector, then a new U(1) gauge potential is defined by

$$\boldsymbol{A} = \hbar \boldsymbol{\nabla} \left[\ln\left(\frac{\psi_{1,t}^* \psi_{1,t}}{\psi_{2,t}^* \psi_{2,t}} \right) \right]. \tag{48}$$

Therefore, the current $\boldsymbol{J}_2$ is

$$\boldsymbol{J}_2 = \frac{\mathrm{i}\hbar}{m}|a_t|\Lambda_1\left[\left(\frac{\psi_{1,t}^*\nabla\psi_{1,t}}{\psi_{1,t}^*\psi_{1,t}} - \frac{\psi_{1,t}\nabla\psi_{1,t}^*}{\psi_{1,t}^*\psi_{1,t}}\right)\right.$$
$$\left.+\left(\frac{\psi_{2,t}^*\nabla\psi_{2,t}}{\psi_{2,t}^*\psi_{2,t}} - \frac{\psi_{2,t}\nabla\psi_{2,t}^*}{\psi_{2,t}^*\psi_{2,t}}\right)\right]$$
$$+\frac{1}{m}|a_t|\Lambda_2\boldsymbol{A}. \quad (49)$$

We assume the system is in the coherence, that is to say, the damping factor $|a_t| = 1$. The current using the measured intensity of the partial wave function $\rho_t^{(i)}$ is

$$\boldsymbol{J} = \frac{\mathrm{i}\hbar}{m}\left(|c_1|^2\rho_t^{(1)} + \Lambda_1\right)\frac{\psi_{1,t}^*\nabla\psi_{1,t} - \psi_{1,t}\nabla\psi_{1,t}^*}{\psi_{1,t}^*\psi_{1,t}}$$
$$+\frac{\mathrm{i}\hbar}{m}\left(|c_2|^2\rho_t^{(2)} + \Lambda_1\right)\frac{\psi_{2,t}^*\nabla\psi_{2,t} - \psi_{2,t}\nabla\psi_{2,t}^*}{\psi_{2,t}^*\psi_{2,t}}$$
$$+\frac{1}{m}\Lambda_2\boldsymbol{A}. \quad (50)$$

In order to study the current in detail, we define new charges $q_1 = \frac{|c_1|^2\rho_t^{(1)}+\Lambda_1}{\rho_t^{(1)}}$ and $q_2 = \frac{|c_2|^2\rho_t^{(2)}+\Lambda_1}{\rho_t^{(2)}}$. Then the new U(1) gauge potential is given by

$$\widetilde{\boldsymbol{A}} = \frac{\Lambda_2}{4(q_1^2+q_2^2)\left(|\psi_{1,t}|^2 + |\psi_{2,t}|^2\right)}\boldsymbol{A}.$$

Based on the new gauge potential $\widetilde{\boldsymbol{A}}$, the current of the coherent system can be expressed by

$$\boldsymbol{J} = \frac{\mathrm{i}\hbar q_1}{m}\left(\psi_{1,t}^*\nabla\psi_{1,t} - \psi_{1,t}\nabla\psi_{1,t}^*\right)$$
$$+\frac{\mathrm{i}\hbar q_2}{m}\left(\psi_{2,t}^*\nabla\psi_{2,t} - \psi_{2,t}\nabla\psi_{2,t}^*\right)$$
$$+\frac{4\left(q_1^2+q_2^2\right)}{m}\left(|\psi_{1,t}|^2 + |\psi_{2,t}|^2\right)\widetilde{\boldsymbol{A}}. \quad (51)$$

However, we find that the total current can be deduced from the following free energy:

$$F = \left[\frac{1}{2m}\left|\left(\hbar\partial_k + \mathrm{i}\frac{2q_1}{c}\widetilde{\boldsymbol{A}}_k\right)\psi_{1,t}\right|^2\right.$$
$$+\frac{1}{2m}\left|\left(\hbar\partial_k + \mathrm{i}\frac{2q_2}{c}\widetilde{\boldsymbol{A}}_k\right)\psi_{2,t}\right|^2$$
$$\left.+V\left(|\psi_{a,t}|^2\right) + \frac{\widetilde{\boldsymbol{B}}^2}{8\pi}\right], \quad (52)$$

where $\widetilde{\boldsymbol{B}} = \boldsymbol{\nabla}\times\widetilde{\boldsymbol{A}}$ is U(1) gauge field. The potential $V(|\psi_{a,t}|^2)$ is

$$V\left(|\psi_{a,t}|^2\right) = -b_a|\psi_{a,t}|^2 + \frac{c_a}{2}|\psi_{a,t}|^4, \quad a = 1, 2. \quad (53)$$

It is well known that this free energy is called the Ginzburg–Landau free energy, which is used to describe the charged two-condensate Bose system [9]. The total current (51) of quantum coherent system is similar to the supercurrent of the charged two-condensate Bose system. In two-condensate superconductor, the charged two-condensate wave functions, or charged order parameters, can carry the electronic charges. The interaction of charged order parameters is mediated by the electromagnetic potential $\boldsymbol{A}_{\mathrm{e}}$. In this description, we find that the coherent system interacting with the environment is similar to the two-condensate superconductor. The partial wave functions can be seen as the charged order parameters. The partial wave functions are weakly-coupled because they carry the charges q_1 and q_2, which is different from the electronic charge. The interaction of the partial wave functions is mediated by the new U(1) gauge potential $\widetilde{\boldsymbol{A}}$, not the electromagnetic potential.

4. The symmetry of the current and its topology

In this section, we try to study the free energy, symmetry and the topological properties of the current of the coherent state. Let us define the partial wave function as

$$\psi_{a,t} = \sqrt{2m}\rho\xi_a, \quad a = 1, 2, \quad (54)$$

where the complex variable $\xi_a = |\xi_a|\mathrm{e}^{\mathrm{i}\theta}$. The modular ρ is

$$\rho = \frac{1}{2}\left(\frac{|\psi_{1,t}|^2}{m} + \frac{|\psi_{2,t}|^2}{m}\right). \quad (55)$$

By using these new variables, the Ginzburg–Landau-like free energy of the coherent state is given as

$$F = \hbar^2(\partial\rho)^2 + \hbar^2\rho^2\left|\left(\partial_k + \mathrm{i}\frac{2q_1}{\hbar c}\widetilde{\boldsymbol{A}}\right)\xi_1\right|^2$$
$$+\hbar^2\rho^2\left|\left(\partial_k + \mathrm{i}\frac{2q_2}{\hbar c}\widetilde{\boldsymbol{A}}\right)\xi_2\right|$$
$$+V\left(|\psi_{a,t}|^2\right) + \frac{\widetilde{\boldsymbol{B}}^2}{8\pi}. \quad (56)$$

It can be rewritten by

$$F = \hbar^2(\partial\rho)^2 + \hbar^2\rho^2\left(|\partial\xi_1|^2 + |\partial\xi_2|^2\right)$$
$$+V\left(|\psi_{a,t}|^2\right) + \frac{\widetilde{\boldsymbol{B}}^2}{8\pi}$$
$$+\hbar^2\rho^2\left[\mathrm{i}\frac{2q_1}{\hbar c}\left(\widetilde{\boldsymbol{A}}\xi_1\partial\xi_1^* - \widetilde{\boldsymbol{A}}\xi_1^*\partial\xi_1\right) + \frac{4q_1^2}{\hbar^2c^2}|\xi_1|^2\widetilde{\boldsymbol{A}}\right]$$
$$+\hbar^2\rho^2\left[\mathrm{i}\frac{2q_2}{\hbar c}\left(\widetilde{\boldsymbol{A}}\xi_2\partial\xi_2^* - \widetilde{\boldsymbol{A}}\xi_2^*\partial\xi_2\right) + \frac{4q_2^2}{\hbar^2c^2}|\xi_2|^2\widetilde{\boldsymbol{A}}\right]. \quad (57)$$

The supercurrent of the free energy can be derived as

$$\boldsymbol{J} = \mathrm{i}\hbar^2\rho^2\left[\frac{2q_1}{\hbar c}(\xi_1\partial\xi_1^* - \xi_1^*\partial\xi_1)\right.$$

$$+\frac{2q_2}{\hbar c}(\xi_2\partial\xi_2^* - \xi_2^*\partial\xi_2)\bigg]$$
$$+\hbar^2\rho^2\left(\frac{4q_1^2}{\hbar^2c^2}|\xi_1|^2 + \frac{4q_2^2}{\hbar^2c^2}|\xi_2|^2\right)\widetilde{\boldsymbol{A}}. \tag{58}$$

Let $\Delta = \frac{4q_1^2}{\hbar^2c^2}|\xi_1|^2 + \frac{4q_2^2}{\hbar^2c^2}|\xi_2|^2$, the new supercurrent $\widetilde{\boldsymbol{J}}$ is given as

$$\widetilde{\boldsymbol{J}} = \frac{\boldsymbol{J}}{\hbar^2\rho^2\Delta} = \mathrm{i}\left[\frac{2q_1}{\hbar c\Delta}(\xi_1\partial\xi_1^* - \xi_1^*\partial\xi_1)\right.$$
$$\left.+\frac{2q_2}{\hbar c\Delta}(\xi_2\partial\xi_2^* - \xi_2^*\partial\xi_2)\right] + \widetilde{\boldsymbol{A}}. \tag{59}$$

To find the symmetry of the coherent system, a new complex variable $\widetilde{\xi}_a$ is defined by

$$\widetilde{\xi}_a = \sqrt{\frac{2q_a}{\hbar\Delta Qc}}\xi_a, \tag{60}$$

where the real number Q guarantees that the new partial wave functions satisfy

$$\left|\widetilde{\xi}_1\right|^2 + \left|\widetilde{\xi}_2\right|^2 = 1. \tag{61}$$

In terms of the new complex variable, the supercurrent $\widetilde{\boldsymbol{J}}$ is

$$\widetilde{\boldsymbol{J}} = \mathrm{i}Q\left[\left(\widetilde{\xi}_1\partial\widetilde{\xi}_1^* - \widetilde{\xi}_1^*\partial\widetilde{\xi}_1\right) + \left(\widetilde{\xi}_2\partial\widetilde{\xi}_2^* - \widetilde{\xi}_2^*\partial\widetilde{\xi}_2\right)\right] + \widetilde{\boldsymbol{A}}. \tag{62}$$

Next we define a gauge invariant unit vector $\widetilde{\boldsymbol{n}}$:

$$\widetilde{\boldsymbol{n}} = \left(\overline{\widetilde{\xi}}, \boldsymbol{\sigma}\widetilde{\xi}\right), \tag{63}$$

where $\overline{\widetilde{\xi}} = (\widetilde{\xi}_1^*, \widetilde{\xi}_2^*)$ and σ are the Pauli matrices. It is obvious that the unit vector satisfies

$$\widetilde{\boldsymbol{n}} \bullet \widetilde{\boldsymbol{n}} = 1.$$

Then a new vector $\boldsymbol{C}$ can be defined by

$$\boldsymbol{C} = Q\frac{\boldsymbol{j}}{2} + \widetilde{\boldsymbol{A}}, \tag{64}$$

where $\boldsymbol{j} = \mathrm{i}[(\widetilde{\xi}_1\partial\widetilde{\xi}_1^* - \widetilde{\xi}_1^*\partial\widetilde{\xi}_1) + (\widetilde{\xi}_2\partial\widetilde{\xi}_2^* - \widetilde{\xi}_2^*\partial\widetilde{\xi}_2)]$. We add and subtract from (56) a term $\frac{1}{4}\hbar^2\rho^2Q^2\Delta^2\boldsymbol{j}^2$, the two charged free energy of the coherent state can be expressed with these new variables

$$F = \hbar^2(\partial\rho)^2 + \frac{\hbar^2\rho^2Q^2\Delta^2}{4}(\partial\widetilde{\boldsymbol{n}})^2$$
$$+\frac{1}{8\pi}\left[(\partial_iC_j - \partial_jC_i) - \frac{Q}{4}\widetilde{\boldsymbol{n}}\cdot\partial_i\widetilde{\boldsymbol{n}}\times\partial_j\widetilde{\boldsymbol{n}}\right]$$
$$+\hbar^2\rho^2\Delta\boldsymbol{C}^2 + V$$
$$+\hbar^2\rho^2\left[\left(1-\frac{2q_1}{\hbar c}\right)|\partial\xi_1|^2 + \left(1-\frac{2q_2}{\hbar c}\right)|\partial\xi_2|^2\right]. \tag{65}$$

Considering the London limit, we have $\partial\rho = 0$, and the free energy is given by

$$F = \frac{\hbar^2\rho^2Q^2\Delta^2}{4}(\partial\widetilde{\boldsymbol{n}})^2 + \frac{1}{8\pi}\left[(\partial_iC_j - \partial_jC_i)\right.$$
$$\left.-\frac{Q}{4}\widetilde{\boldsymbol{n}}\cdot\partial_i\widetilde{\boldsymbol{n}}\times\partial_j\widetilde{\boldsymbol{n}}\right] + \hbar^2\rho^2\Delta\boldsymbol{C}^2 + V$$
$$+\hbar^2\rho^2\left[\left(1-\frac{2q_1}{\hbar c}\right)|\partial\xi_1|^2 + \left(1-\frac{2q_2}{\hbar c}\right)|\partial\xi_2|^2\right]. \tag{66}$$

Finally, we find there is a stable knotted solution in coherent system, which is described by the Skyme–Faddeev–Niemi action

$$F_0 = \frac{\rho^2\hbar^2Q^2\Delta^2}{4}(\partial\widetilde{\boldsymbol{n}})^2 + \frac{Q}{32\pi}\widetilde{\boldsymbol{n}}\cdot\partial_i\widetilde{\boldsymbol{n}}\times\partial_j\widetilde{\boldsymbol{n}}. \tag{67}$$

The knotted solution displays a $O(3)$ symmetry in the free energy. The knotted solution is just the nontrivial map

$$\widetilde{\boldsymbol{n}} : S^3 \to S^2. \tag{68}$$

The boundary condition of this knotted solution is

$$\widetilde{\boldsymbol{n}}(x) \to \widetilde{\boldsymbol{n}}_0, \quad \boldsymbol{x} \to \infty, \tag{69}$$

where $\boldsymbol{n}_0$ is the constant vector in spatial direction. The knotted solution has an important relation to the current of the coherent state. It is convenient to write the current as

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2, \tag{70}$$

where

$$\boldsymbol{J}_1 = \frac{\hbar q_1\rho_t^{(1)}}{\mathrm{i}m}\frac{\psi_{1,t}^*\nabla\psi_{1,t} - \psi_{1,t}\nabla\psi_{1,t}^*}{\psi_{1,t}^*\psi_{1,t}}$$
$$+\frac{4q_1^2}{m}|\psi_{1,t}|^2\widetilde{\boldsymbol{A}} \tag{71}$$

and

$$\boldsymbol{J}_2 = \frac{\hbar q_2\rho_t^{(2)}}{\mathrm{i}m}\frac{\psi_{2,t}^*\nabla\psi_{2,t} - \psi_{2,t}\nabla\psi_{2,t}^*}{\psi_{2,t}^*\psi_{2,t}}$$
$$+\frac{4q_2^2}{m}|\psi_{2,t}|^2\widetilde{\boldsymbol{A}}. \tag{72}$$

Recalling the unit vector $\boldsymbol{n}_{(i)}$, these components can be derived as

$$\boldsymbol{J}_1 = \frac{\hbar q_1\rho_t^{(1)}}{m}\epsilon_{ab}n_{(1)}^a\partial_in_{(1)}^b + \frac{4q_1^2}{m}|\psi_{1,t}|^2\widetilde{\boldsymbol{A}} \tag{73}$$

and

$$\boldsymbol{J}_2 = \frac{\hbar q_2\rho_t^{(2)}}{m}\epsilon_{ab}n_{(2)}^a\partial_in_{(2)}^b + \frac{4q_2^2}{m}|\psi_{2,t}|^2\widetilde{\boldsymbol{A}}. \tag{74}$$

By making use of ϕ-mapping topological current theory, the vorticity of the current is given as

$$\varGamma = \int_{\Sigma_i}(\nabla\times\boldsymbol{J})\cdot\mathrm{d}\boldsymbol{S}$$
$$= \int_{\Sigma_i}(\nabla\times\boldsymbol{J}_1)\cdot\mathrm{d}\boldsymbol{S} + \int_{\Sigma_i}(\nabla\times\boldsymbol{J}_2)\cdot\mathrm{d}\boldsymbol{S}. \tag{75}$$

Then the curls of the currents $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are calculated as

$$(\nabla \times \boldsymbol{J}_1) = \frac{\hbar q_1 \rho_t^{(1)}}{m} \sum_{l=1}^{\ell} \beta_l^{(1)} \eta_l^{(1)} \delta_{(1)}^2 (x^i - z_l^i) \frac{\mathrm{d}x_{(1)}^i}{\mathrm{d}s} + \frac{4q_1^2}{m} |\psi_{1,t}|^2 \boldsymbol{\nabla} \times \widetilde{\boldsymbol{A}} \tag{76}$$

and

$$(\nabla \times \boldsymbol{J}_2) = \frac{\hbar q_2 \rho_t^{(2)}}{m} \sum_{l=1}^{\ell} \beta_l^{(2)} \eta_l^{(2)} \delta_{(2)}^2 (x^i - z_l^i) \frac{\mathrm{d}x_{(2)}^i}{\mathrm{d}s} + \frac{4q_2^2}{m} |\psi_{2,t}|^2 \nabla \times \widetilde{\boldsymbol{A}}. \tag{77}$$

Furthermore, the vorticity of the current is

$$\Gamma = \frac{\hbar q_1 \rho_t^{(1)}}{m} W_1 + \frac{\hbar q_2 \rho_t^{(2)}}{m} W_2 + \frac{4\left(q_1^2 \rho_t^{(1)} + q_1^2 \rho_t^{(2)}\right)}{m} \int_{\Sigma_i} \left(\nabla \times \widetilde{\boldsymbol{A}}\right) \cdot \mathrm{d}\boldsymbol{S}. \tag{78}$$

It is well known that the property of a supercurrent is the magnetic flux passing through any area bounded by such a current is quantized. The quantization of the flux in the superconductor is

$$\int_{\Sigma_i} (\nabla \times \boldsymbol{A}_E) \cdot \mathrm{d}\boldsymbol{S} = \frac{h}{2e} \widetilde{W},$$

where e is the electronic charge. Similarly, we give the flux quantization of this new U(1) gauge potential $\widetilde{\boldsymbol{A}}$

$$\int_{\Sigma_i} \left(\nabla \times \widetilde{\boldsymbol{A}}\right) \cdot \mathrm{d}\boldsymbol{S} = \frac{h}{q_1 + q_2} \widetilde{W}. \tag{79}$$

Finally, the vorticity of the current is

$$\Gamma = \frac{\hbar q_1 \rho_t^{(1)}}{m} W_1 + \frac{\hbar q_2 \rho_t^{(2)}}{m} W_2 + \frac{4\left(q_1^2 \rho_t^{(1)} + q_1^2 \rho_t^{(2)}\right)}{m} \frac{\hbar}{q_1 + q_2} \widetilde{W}. \tag{80}$$

5. Conclusion

In this paper, the relation between the coherent quantum system and the charged two-condensate system is investigated. The new expression of the current of the coherent state is given based on reduced density trajectory and ϕ-mapping topological current theory. A topological reason leading to the decoherence is found. By defining a new U(1) gauge potential $\widetilde{\boldsymbol{A}}$ and new charges q_1 and q_2, we find that the coherent system can be described by the Ginzburg–Landau-like model with two charged Cooper pairs. The corresponding relation between coherent system and two-gap superconductor is shown as follows: the partial wave functions of the coherence correspond to the charged two-condensate wave functions; the charges q_1 and q_2 correspond to the electronic charges; the new U(1) gauge potential $\widetilde{\boldsymbol{A}}$ corresponds to the electromagnetic potential $\boldsymbol{A}e$. Finally, the hidden $O(3)$ symmetry of the coherent state is found using Faddeev's $O(3)$ nonlinear σ-model and the topological properties of the knot solution are studied based on ϕ-mapping topological current theory.

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