Perturbation to Noether Symmetry and Noether Adiabatic Invariants of Discrete Difference Variational Hamilton Systems

M.-J. Zhang[a,b,*] and J.-H. Fang[c]

[a]Purple Mountain Observatory, Chinese Academy of Sciences, Nanjing 210008, China

[b]Key Laboratory of Space Object and Debris Observation, PMO, CAS, Nanjing 210008, China

[c]College of Science, China University of Petroleum, Qingdao 266555, China

(Received April 15, 2011; in final form March 2, 2012)

The perturbation to the Noether symmetry and the Noether adiabatic invariants of discrete difference variational Hamilton systems are investigated. The discrete the Noether exact invariant induced directly by the the Noether symmetry of the system without perturbation is given. The concept of discrete high-order adiabatic invariant is presented, the criterion of the perturbation to the Noether symmetry is established, and the discrete the Noether adiabatic invariant induced directly by the perturbation to the Noether symmetry is obtained. Lastly, an example is discussed to illustrate the application of the results.

PACS: 02.20.Sv, 11.30.−j, 45.05.+x

1. Introduction

The research on symmetries and conserved quantities of a mechanical system possesses important mathematical and physical significance, and is also an important development direction in analytical mechanics. In Refs. [1, 2] symmetries and conserved quantities of constrained mechanical systems have been thoroughly investigated. Moreover, symmetries and conserved quantities of more general systems (optimal control problems) are also studied in depth [3–5]. Recently, researches on symmetries and conserved quantities have been extended to a general time scale (including, as particular cases, both discrete and continuous settings). Levi et al. [6–8] first extended the Lie symmetry to discrete systems; Dorodnitsyn [9] adapted the Noether's theory to discrete Lagrangian system; Torres et al. [10–12] studied the Noether symmetry theorems for an arbitrary time scale; Shi et al. [13, 14] investigated Lie symmetry and the Noether conserved quantities (or exact invariants) of discrete non-conservative mechanical systems and discrete difference variational Hamilton systems without perturbations; based upon the property of the discrete models entirely inheriting the symmetry of the continuous systems, Fu et al. [15] presented the Hojman conserved quantities and Lie symmetries of discrete mechanico-electrical coupling systems by the Lie groups of transformations of continuous systems.

As we know, even a tiny disturbance acting on the mechanical systems, that we can call a perturbation, may influence the original symmetries and conserved quantities of mechanical systems. Pioneered in this area, Burgers [16] proposed adiabatic invariants for a special kind of the Hamilton systems. Perturbation to symmetries and adiabatic invariants play a very important role in the research on the quasi-integrability of mechanical systems. A classical adiabatic invariant is a certain physical quantity that changes more slowly than some slowly-varying parameter of the system [17]. In fact, the parameter varying very slowly is equivalent to the action of a small disturbance. At present, studies in this field have become very active, and many important results have been obtained [18–23]. But all these studies have focused on perturbation to symmetries and adiabatic invariants of the continuous mechanical systems. Recently, Zhang et al. [24] presented the concept of discrete high-order adiabatic invariant, and studied the perturbation to the the Noether symmetry and the Noether adiabatic invariant of the general discrete holonomic system. Wang and Zhu [25] further discussed perturbation to symmetry and adiabatic invariants of general discrete holonomic dynamical systems on a uniform lattice. However, perturbation to symmetries and adiabatic invariants of the discrete Hamilton systems has never been studied so far.

In this paper, based on the concept of discrete high-order adiabatic invariant, the perturbation to the Noether symmetry and the Noether adiabatic invariants of discrete difference variational Hamilton systems are studied. The discrete the Noether exact invariant induced directly by the the Noether symmetry of the system without perturbation is given. The criterion of the perturbation to the Noether symmetry is established, and the discrete the Noether adiabatic invariant induced directly by the perturbation to the Noether symmetry is obtained. Meanwhile, an example is discussed to illustrate the application of the results.

2. the Noether symmetry and discrete the Noether exact invariant

For brevity of notation, we consider a one-dimensional discrete difference variational Hamil-

* corresponding author; e-mail: mjzhang@pmo.ac.cn

ton system. The Hamiltonian of the system is $H_{\mathrm{d}}(t_k, t_{k+1}, q_k, q_{k+1}, p_k, p_{k+1})$ ($k = 0, 1, \ldots, N-1$), t_k, q_k, and p_k are discrete time, discrete generalized coordinates, and discrete generalized momenta, respectively.

The discrete equations of motion of the system can be written as [14]:

$$\tfrac{1}{2}(p_{k-1} - p_{k+1}) - D_3 H_{\mathrm{d}}(\varphi_k)(t_{k+1} - t_k) - D_4 H_{\mathrm{d}}(\varphi_{k-1})(t_k - t_{k-1}) = 0, \tag{1}$$

$$\tfrac{1}{2}(q_{k+1} - q_{k-1}) - D_5 H_{\mathrm{d}}(\varphi_k)(t_{k+1} - t_k) - D_6 H_{\mathrm{d}}(\varphi_{k-1})(t_k - t_{k-1}) = 0, \tag{2}$$

$$H_{\mathrm{d}}(\varphi_k) - H_{\mathrm{d}}(\varphi_{k-1}) - D_1 H_{\mathrm{d}}(\varphi_k)(t_{k+1} - t_k) - D_2 H_{\mathrm{d}}(\varphi_{k-1})(t_k - t_{k-1}) = 0, \tag{3}$$

where D_j is the partial derivative of the discrete function with respect to the argument j and $\varphi_k = (t_k, t_{k+1}, q_k, q_{k+1}, p_k, p_{k+1})$ represents the discrete sequence.

We introduce the infinitesimal transformations with respect to discrete time t_k, discrete generalized coordinates q_k, and discrete generalized momenta p_k as

$$t_k^* = t_k + \Delta t_k = t_k + \varepsilon \tau_k^0(t_k, q_k, p_k),$$
$$q_k^* = q_k + \Delta q_k = q_k + \varepsilon \xi_k^0(t_k, q_k, p_k),$$
$$p_k^* = p_k + \Delta p_k = q_k + \varepsilon \eta_k^0(t_k, q_k, p_k), \tag{4}$$

where ε is an infinitesimal group parameter, τ_k^0, ξ_k^0 and η_k^0 are the discrete infinitesimal generators. The vector field of generators is

$$X_{0,\mathrm{d}}^{(0)} = \tau_k^0 \frac{\partial}{\partial t_k} + \xi_k^0 \frac{\partial}{\partial q_k} + \eta_k^0 \frac{\partial}{\partial p_k}, \tag{5}$$

which can be prolonged to the two-point scheme

$$X_{0,\mathrm{d}}^{(1)} = X_{0,\mathrm{d}}^{(0)} + \tau_{k+1}^0 \frac{\partial}{\partial t_{k+1}} + \xi_{k+1}^0 \frac{\partial}{\partial q_{k+1}} + \eta_{k+1}^0 \frac{\partial}{\partial p_{k+1}}. \tag{6}$$

The recursive and derivative operators of discrete systems for any discrete variable or function are represented as

$$R_{\pm} f(z_k) = f(z_{k\pm 1}), \tag{7}$$

$$D_{\mathrm{d}} f(z_k) = \frac{R_+ f(z_k) - f(z_k)}{t_{k+1} - t_k}. \tag{8}$$

The the Noether symmetry is an invariance of the Hamiltonian action functional under the infinitesimal transformations. Then the requirement of the the Noether symmetry of the discrete difference variational Hamilton system gives

$$H_{\mathrm{d}}(\varphi_k) D_{\mathrm{d}}(\tau_k^0) + X_{0,\mathrm{d}}^{(1)}[H_{\mathrm{d}}(\varphi_k)] - \tfrac{1}{2}(p_{k+1} + p_k) D_{\mathrm{d}}(\xi_k^0) + \tfrac{1}{2}(q_{k+1} + q_k) D_{\mathrm{d}}(\eta_k^0) + D_{\mathrm{d}}(G_{Nk}^0) = 0, \tag{9}$$

where $G_{Nk}^0 = G_{Nk}^0(t_k, q_k, p_k)$ is a discrete gauge function.

Criterion 1. *If a discrete gauge function G_{Nk}^0 exists such that the infinitesimal generators τ_k^0, ξ_k^0 and η_k^0 satisfy Eq. (9), the invariance is the Noether symmetry of the discrete difference variational Hamilton system (1), (2) and (3).*

Equation (9) is called the discrete the Noether identity of the discrete difference variational Hamilton system (1), (2) and (3).

Theorem 1 [14]. *For the discrete difference variational Hamilton system (1), (2) and (3), if the infinitesimal generators τ_k^0, ξ_k^0 and η_k^0 of the the Noether symmetry and discrete gauge function G_{Nk}^0 satisfy the discrete the Noether identity (9), the the Noether symmetry of the system can induce the discrete the Noether exact invariant*

$$\begin{aligned} I_{\mathrm{N0,d}} &= \tau_k^0 (t_k - t_{k-1}) D_2[R_- H_{\mathrm{d}}(\varphi_k)] \\ &+ \xi_k^0 (t_k - t_{k-1}) D_4[R_- H_{\mathrm{d}}(\varphi_k)] \\ &+ \eta_k^0 (t_k - t_{k-1}) D_6[R_- H_{\mathrm{d}}(\varphi_k)] + \tau_k^0 R_- H_{\mathrm{d}}(\varphi_k) \\ &- \tfrac{1}{2}(p_{k-1} + p_k)\xi_k^0 + \tfrac{1}{2}(q_{k-1} + q_k)\eta_k^0 + G_{Nk}^0 = \text{const.} \end{aligned} \tag{10}$$

3. Perturbation to the Noether symmetry and discrete the Noether adiabatic invariant

Small force acting on the discrete difference variational Hamilton system may result in a small change in its symmetries, i.e. perturbation to symmetries. The conserved quantity associated with the symmetries, under a corresponding change, is an adiabatic invariant. In analytical mechanics, we study perturbation to symmetries and adiabatic invariants of mechanical systems based on the concept of high-order adiabatic invariant. According to the concept of adiabatic invariant [17], for the discrete difference variational Hamilton system, we have

Definition. *If $L_{z,\mathrm{d}}(t_k, t_{k+1}, q_k, q_{k+1}, p_k, p_{k+1}, \varepsilon)$ is a physical quantity including ε in which the highest power is z in a discrete difference variational Hamilton system, and its derivative with respect to discrete time t_k is directly proportional to ε^{z+1}, $I_{z,\mathrm{d}}$ is called a z-th order adiabatic invariant of the discrete difference variational Hamilton system.* **Definition.** *If $L_{z,\mathrm{d}}(t_k, t_{k+1}, q_k, q_{k+1}, p_k, p_{k+1}, \varepsilon)$ is a physical quantity including ε in which the highest power is z in a discrete difference variational Hamilton system, and its derivative with respect to discrete time t_k is directly proportional to ε^{z+1}, $I_{z,\mathrm{d}}$ is called a z-th order adiabatic invariant of the discrete difference variational Hamilton system.*

Suppose the discrete difference variational Hamilton system (1), (2) and (3) is perturbed by small quantity $\varepsilon W_{\mathrm{d}}(t_k, t_{k+1}, q_k, q_{k+1}, p_k, p_{k+1}) = \varepsilon W_{\mathrm{d}}(\varphi_k)$, then we have the following total variation of corresponding discrete Hamiltonian action functional:

$$\Delta S_{\mathrm{d}} + \sum_{k=0}^{N-1} \varepsilon W_{\mathrm{d}}(\varphi_k)(t_{k+1} - t_k)\Delta q_k - \sum_{k=0}^{N-1} \varepsilon W_{\mathrm{d}}(\varphi_k)(q_{k+1} - q_k)\Delta t_k = 0, \tag{11}$$

where Δ is a total variation symbol, and

$$S_{\rm d} = \sum_{k=0}^{N-1} \tfrac{1}{2}(p_{k+1}+p_k)(q_{k+1}-q_k) - \sum_{k=0}^{N-1} H_{\rm d}(t_k, t_{k+1}, q_k, q_{k+1}, p_k, p_{k+1})(t_{k+1}-t_k). \tag{12}$$

From (11), using the similar deduction in [14], we obtain the discrete equations of motion of the discrete difference variational Hamilton system perturbed by small quantity $\varepsilon W_{\rm d}(\varphi_k)$:

$$\tfrac{1}{2}(p_{k-1}-p_{k+1}) - D_3 H_{\rm d}(\varphi_k)(t_{k+1}-t_k) - D_4 H_{\rm d}(\varphi_{k-1})(t_k - t_{k-1}) + \varepsilon W_{\rm d}(\varphi_k)(t_{k+1}-t_k) = 0, \tag{13}$$

$$\tfrac{1}{2}(q_{k+1}-q_{k-1}) - D_5 H_{\rm d}(\varphi_k)(t_{k+1}-t_k) - D_6 H_{\rm d}(\varphi_{k-1})(t_k - t_{k-1}) = 0, \tag{14}$$

$$H_{\rm d}(\varphi_k) - H_{\rm d}(\varphi_{k-1}) - D_1 H_{\rm d}(\varphi_k)(t_{k+1}-t_k) - D_2 H_{\rm d}(\varphi_{k-1})(t_k - t_{k-1}) - \varepsilon W_{\rm d}(\varphi_k)(q_{k+1}-q_k) = 0. \tag{15}$$

Because of the action of $\varepsilon W_{\rm d}(\varphi_k)$, the original symmetries and invariants of the system may vary. Assume that the variation is a small perturbation based on the symmetrical transformations of the system without perturbation, and $\tau_k(t_k, q_k, p_k)$, $\xi_k(t_k, q_k, p_k)$ and $\eta_k(t_k, q_k, p_k)$ express the generators of infinitesimal transformations after being perturbed, then

$$\tau_k = \tau_k^0 + \varepsilon\tau_k^1 + \varepsilon^2\tau_k^2 + \ldots,$$
$$\xi_k = \xi_k^0 + \varepsilon\xi_k^1 + \varepsilon^2\xi_k^2 + \ldots,$$
$$\eta_k = \eta_k^0 + \varepsilon\eta_k^1 + \varepsilon^2\eta_k^2 + \ldots \tag{16}$$

The infinitesimal transformations become

$$t_k^* = t_k + \Delta t_k = t_k + \varepsilon\tau_k(t_k, q_k, p_k),$$
$$q_k^* = q_k + \Delta q_k = q_k + \varepsilon\xi_k(t_k, q_k, p_k),$$
$$p_k^* = p_k + \Delta p_k = p_k + \varepsilon\eta_k(t_k, q_k, p_k). \tag{17}$$

According to the the Noether symmetry theory, the the Noether identity of the discrete difference variational Hamilton system perturbed by small quantity $\varepsilon W_{\rm d}(\varphi_k)$ becomes

$$H_{\rm d}(\varphi_k)D_{\rm d}(\tau_k) + X_{\rm d}^{(1)}[H_{\rm d}(\varphi_k)] - \tfrac{1}{2}(p_{k+1}+p_k)D_{\rm d}(\xi_k) + \tfrac{1}{2}(q_{k+1}+q_k)D_{\rm d}(\eta_k) - \varepsilon W_{\rm d}(\varphi_k)[\xi_k - D_{\rm d}(q_k)\tau_k] + D_{\rm d}(G_{Nk}) = 0, \tag{18}$$

where

$$X_{\rm d}^{(1)} = \tau_k\frac{\partial}{\partial t_k} + \xi_k\frac{\partial}{\partial q_k} + \eta_k\frac{\partial}{\partial p_k} + \tau_{k+1}\frac{\partial}{\partial t_{k+1}} + \xi_{k+1}\frac{\partial}{\partial q_{k+1}} + \eta_{k+1}\frac{\partial}{\partial p_{k+1}}, \tag{19}$$

and $G_{Nk} = G_{Nk}(t_k, q_k, p_k)$ is a gauge function. After being perturbed, the gauge function comes into

$$G_{Nk} = G_{Nk}^0 + \varepsilon G_{Nk}^1 + \varepsilon^2 G_{Nk}^2 + \ldots \tag{20}$$

Substituting (16) into (19), we have

$$X_{\rm d}^{(1)} = \varepsilon^m X_{\rm m,d}^{(1)} \quad (m = 0, 1, \ldots, z), \tag{21}$$

where

$$X_{\rm m,d}^{(1)} = \tau_k^m\frac{\partial}{\partial t_k} + \xi_k^m\frac{\partial}{\partial q_k} + \eta_k^m\frac{\partial}{\partial p_k} + \tau_{k+1}^m\frac{\partial}{\partial t_{k+1}} + \xi_{k+1}^m\frac{\partial}{\partial q_{k+1}} + \eta_{k+1}^m\frac{\partial}{\partial p_{k+1}}. \tag{22}$$

Substituting (16) into (18), noticing (19)–(22), and making the coefficients of ε^m equal, we obtain

$$H_{\rm d}(\varphi_k)D_{\rm d}(\tau_k^m) + X_{\rm m,d}^{(1)}[H_{\rm d}(\varphi_k)] - \tfrac{1}{2}(p_{k+1}+p_k)D_{\rm d}(\xi_k^m) + \tfrac{1}{2}(q_{k+1}+q_k)D_{\rm d}(\eta_k^m) - W_{\rm d}(\varphi_k)\big[\xi_k^{m-1} - D_{\rm d}(q_k)\tau_k^{m-1}\big] + D_{\rm d}(G_{Nk}^m) = 0, \tag{23}$$

when $m = 0$, we note that $\tau_k^{m-1} = \xi_k^{m-1} = \eta_k^{m-1} = 0$ holds, then Eq. (23) turns into Eq. (9).

Criterion 2. *For the discrete difference variational Hamilton system (1), (2) and (3) perturbed by small quantity $\varepsilon W_{\rm d}(\varphi_k)$, if the discrete gauge function G_{Nk}^m exists such that the infinitesimal generators τ_k^m, ξ_k^m and η_k^m satisfy (23), the corresponding variety of the the Noether symmetry is called the perturbation to the the Noether symmetry.*

Equation (23) is the determining equation of the perturbation to the the Noether symmetry of the discrete difference variational Hamilton system (1), (2) and (3).

Theorem 2. *For the discrete difference variational Hamilton system (1), (2) and (3) perturbed by small quantity $\varepsilon W_{\rm d}(\varphi_k)$, if the infinitesimal generators τ_k^m, ξ_k^m and η_k^m, and discrete gauge function G_{Nk}^m satisfy the determining of Eq. (23), the perturbation to the the Noether symmetry of the system can induce a z-th order discrete adiabatic invariant*

$$I_{\rm Nz,d} = \varepsilon^m\Big\{\tau_k^m(t_k - t_{k-1})D_2[R_- H_{\rm d}(\varphi_k)] + \xi_k^m(t_k - t_{k-1})D_4[R_- H_{\rm d}(\varphi_k)] + \eta_k^m(t_k - t_{k-1})D_6[R_- H_{\rm d}(\varphi_k)] + \tau_k^m R_- H_{\rm d}(\varphi_k) - \tfrac{1}{2}(p_{k-1}+p_k)\xi_k^m + \tfrac{1}{2}(q_{k-1}+q_k)\eta_k^m + G_{Nk}^m\Big\} \quad (m = 0, 1, \ldots, z). \tag{24}$$

Proof: Using representations (7), (8) and (22), the expansion of (23) gives

$$H_{\rm d}(\varphi_k)D_{\rm d}(\tau_k^m) + X_{\rm m,d}^{(1)}[H_{\rm d}(\varphi_k)] - \tfrac{1}{2}(p_{k+1}+p_k)D_{\rm d}(\xi_k^m) + \tfrac{1}{2}(q_{k+1}+q_k)D_{\rm d}(\eta_k^m) - W_{\rm d}(\varphi_k)\big[\xi_k^{m-1} - D_{\rm d}(q_k)\tau_k^{m-1}\big] + D_{\rm d}(G_{Nk}^m)$$
$$= H_{\rm d}(\varphi_k)\frac{\tau_{k+1}^m - \tau_k^m}{t_{k+1}-t_k} + \tau_k^m\frac{\partial H_{\rm d}(\varphi_k)}{\partial t_k} + \tau_{k+1}^m\frac{\partial H_{\rm d}(\varphi_k)}{\partial t_{k+1}}$$

$$
\begin{aligned}
&+\xi_k^m\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial q_k}+\xi_{k+1}^m\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial q_{k+1}}+\eta_k^m\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial p_k}\\
&+\eta_{k+1}^m\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial p_{k+1}}-\tfrac{1}{2}(p_{k+1}+p_k)\frac{\xi_{k+1}^m-\xi_k^m}{t_{k+1}-t_k}\\
&+\tfrac{1}{2}(q_{k+1}+q_k)\frac{\eta_{k+1}^m-\eta_k^m}{t_{k+1}-t_k}\\
&-W_\mathrm{d}(\varphi_k)\big[\xi_k^{m-1}-D_\mathrm{d}(q_k)\tau_k^{m-1}\big]+D_\mathrm{d}(G_{Nk}^m)\\
&=\tau_k^m\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial t_k}+\tau_k^m\frac{t_k-t_{k-1}}{t_{k+1}-t_k}\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial t_k}\\
&+\tau_k^m\frac{H_\mathrm{d}(\varphi_{k-1})}{t_{k+1}-t_k}-\tau_k^m\frac{H_\mathrm{d}(\varphi_k)}{t_{k+1}-t_k}\\
&+\xi_k^m\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial q_k}+\xi_k^m\frac{t_k-t_{k-1}}{t_{k+1}-t_k}\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial q_k}\\
&-\tfrac{1}{2}\frac{p_{k-1}\xi_k^m}{t_{k+1}-t_k}+\tfrac{1}{2}\frac{p_{k+1}\xi_k^m}{t_{k+1}-t_k}\\
&+\eta_k^m\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial p_k}+\eta_k^m\frac{t_k-t_{k-1}}{t_{k+1}-t_k}\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial p_k}\\
&-\tfrac{1}{2}\frac{q_{k+1}\eta_k^m}{t_{k+1}-t_k}+\tfrac{1}{2}\frac{q_{k-1}\eta_k^m}{t_{k+1}-t_k}\\
&+\tau_{k+1}^m\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial t_{k+1}}-\tau_k^m\frac{t_k-t_{k-1}}{t_{k+1}-t_k}\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial t_k}\\
&+\xi_{k+1}^m\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial q_{k+1}}-\xi_k^m\frac{t_k-t_{k-1}}{t_{k+1}-t_k}\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial q_k}\\
&+\eta_{k+1}^m\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial p_{k+1}}-\eta_k^m\frac{t_k-t_{k-1}}{t_{k+1}-t_k}\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial p_k}\\
&+\tau_{k+1}^m\frac{H_\mathrm{d}(\varphi_k)}{t_{k+1}-t_k}-\tau_k^m\frac{H_\mathrm{d}(\varphi_{k-1})}{t_{k+1}-t_k}\\
&+\tfrac{1}{2}\frac{p_{k-1}\xi_k^m}{t_{k+1}-t_k}-\tfrac{1}{2}\frac{p_{k+1}\xi_{k+1}^m}{t_{k+1}-t_k}-\tfrac{1}{2}\frac{p_k\xi_{k+1}^m}{t_{k+1}-t_k}\\
&+\tfrac{1}{2}\frac{p_k\xi_k^m}{t_{k+1}-t_k}-\tfrac{1}{2}\frac{q_{k-1}\eta_k^m}{t_{k+1}-t_k}+\tfrac{1}{2}\frac{q_{k+1}\eta_{k+1}^m}{t_{k+1}-t_k}\\
&+\tfrac{1}{2}\frac{q_k\eta_{k+1}^m}{t_{k+1}-t_k}-\tfrac{1}{2}\frac{q_k\eta_k^m}{t_{k+1}-t_k}+D_\mathrm{d}(G_{Nk}^m)\\
&-W_\mathrm{d}(\varphi_k)\big[\xi_k^{m-1}-D_\mathrm{d}(q_k)\tau_k^{m-1}\big]\\
&=\tau_k^m\bigg[\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial t_k}+\frac{t_k-t_{k-1}}{t_{k+1}-t_k}\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial t_k}\\
&+\frac{H_\mathrm{d}(\varphi_{k-1})}{t_{k+1}-t_k}-\frac{H_\mathrm{d}(\varphi_k)}{t_{k+1}-t_k}\bigg]\\
&+\xi_k^m\bigg[\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial q_k}+\frac{t_k-t_{k-1}}{t_{k+1}-t_k}\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial q_k}\\
&-\tfrac{1}{2}\frac{p_{k-1}}{t_{k+1}-t_k}+\tfrac{1}{2}\frac{p_{k+1}}{t_{k+1}-t_k}\bigg]\\
&+\eta_k^m\bigg[\frac{\partial H_\mathrm{d}(\varphi_k)}{\partial p_k}+\frac{t_k-t_{k-1}}{t_{k+1}-t_k}\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial p_k}\\
&-\tfrac{1}{2}\frac{q_{k+1}}{t_{k+1}-t_k}+\tfrac{1}{2}\frac{q_{k-1}}{t_{k+1}-t_k}\bigg]\\
&+D_\mathrm{d}\bigg[\tau_k^m(t_k-t_{k-1})\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial t_k}\\
&+\xi_k^m(t_k-t_{k-1})\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial q_k}\\
&+\eta_k^m(t_k-t_{k-1})\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial p_k}\\
&+\tau_k^mH_\mathrm{d}(\varphi_{k-1})-\tfrac{1}{2}(p_{k-1}+p_k)\xi_k^m\\
&+\tfrac{1}{2}(q_{k-1}+q_k)\eta_k^m+G_{Nk}^m\bigg]\\
&-W_\mathrm{d}(\varphi_k)\big[\xi_k^{m-1}-D_\mathrm{d}(q_k)\tau_k^{m-1}\big]=0.
\end{aligned}\tag{25}
$$

Using (13)–(15), from (25), we obtain

$$
\begin{aligned}
&D_\mathrm{d}\bigg\{\varepsilon^m\bigg[\tau_k^m(t_k-t_{k-1})\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial t_k}\\
&+\xi_k^m(t_k-t_{k-1})\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial q_k}\\
&+\eta_k^m(t_k-t_{k-1})\frac{\partial H_\mathrm{d}(\varphi_{k-1})}{\partial p_k}+\tau_k^mH_\mathrm{d}(\varphi_{k-1})\\
&-\tfrac{1}{2}(p_{k-1}+p_k)\xi_k^m+\tfrac{1}{2}(q_{k-1}+q_k)\eta_k^m+G_{Nk}^m\bigg]\bigg\}\\
&=D_\mathrm{d}\bigg\{\varepsilon^m\bigg[\tau_k^m(t_k-t_{k-1})D_2(R_-H_\mathrm{d}(\varphi_k))\\
&+\xi_k^m(t_k-t_{k-1})D_4(R_-H_\mathrm{d}(\varphi_k))\\
&+\eta_k^m(t_k-t_{k-1})D_6(R_-H_\mathrm{d}(\varphi_k))+\tau_k^mR_-H_\mathrm{d}(\varphi_k)\\
&-\tfrac{1}{2}(p_{k-1}+p_k)\xi_k^m+\tfrac{1}{2}(q_{k-1}+q_k)\eta_k^m+G_{Nk}^m\bigg]\bigg\}\\
&=D_\mathrm{d}I_{\mathrm{Nz,d}}\\
&=\varepsilon^m\bigg\{\varepsilon W_\mathrm{d}(\varphi_k)\frac{q_{k+1}-q_k}{t_{k+1}-t_k}\tau_k^m-\varepsilon W_\mathrm{d}(\varphi_k)\xi_k^m\\
&+W_\mathrm{d}(\varphi_k)\big[\xi_k^{m-1}-D_\mathrm{d}(q_k)\tau_k^{m-1}\big]\bigg\}\\
&=\varepsilon^m\bigg\{-\varepsilon W_\mathrm{d}(\varphi_k)[\xi_k^m-D_\mathrm{d}(q_k)\tau_k^m]\\
&+W_\mathrm{d}(\varphi_k)\big[\xi_k^{m-1}-D_\mathrm{d}(q_k)\tau_k^{m-1}\big]\bigg\}\\
&=-\varepsilon^{z+1}W_\mathrm{d}(\varphi_k)[\xi_k^z-D_\mathrm{d}(q_k)\tau_k^z].
\end{aligned}\tag{26}
$$

It shows that $D_\mathrm{d}I_{\mathrm{Nz,d}}$ is directly proportional to ε^{z+1}, so $I_{\mathrm{Nz,d}}$ is a z-th order adiabatic invariant of the discrete difference variational Hamilton system, which can be called the discrete the Noether adiabatic invariant. When $z=0$, which mean that $\varepsilon W_\mathrm{d}(\varphi_k)$ vanish, the system becomes the one without perturbation. Then the discrete the Noether adiabatic invariant (24) turns into the discrete the Noether exact invariant (10).

4. An example

Take the Emden equation to illustrate the application of the above results. The discrete Hamiltonian of the system and its recursive operator are

$$H_{\rm d}(\varphi_k) = \frac{1}{2}\frac{p_{k+1}^2 + p_k^2}{t_{k+1}^2 + t_k^2} + \frac{1}{12}(t_{k+1}^2 q_{k+1}^6 + t_k^2 q_k^6), \quad (27)$$

$$R_- H_{\rm d}(\varphi_k) = H_{\rm d}(\varphi_{k-1}) = \frac{1}{2}\frac{p_k^2 + p_{k-1}^2}{t_k^2 + t_{k-1}^2} + \frac{1}{12}(t_k^2 q_k^6 + t_{k-1}^2 q_{k-1}^6), \quad (28)$$

and the system is perturbed by the small quantity

$$\varepsilon W_{\rm d}(\varphi_k) = -\varepsilon \frac{q_{k+1}p_{k+1} - q_k p_k}{3q_k t_k - q_k t_{k+1} - 2q_{k+1}t_k}. \quad (29)$$

Let us study the perturbation to the the Noether symmetry and the the Noether adiabatic invariant of the discrete difference variational Hamilton system.

We calculate the $D_j H_{\rm d}(\varphi_k)$ $(j = 1, 2, 3, 4, 5, 6)$ as

$$D_1 H_{\rm d}(\varphi_k) = -\frac{t_k(p_{k+1}^2 + p_k^2)}{(t_{k+1}^2 + t_k^2)^2} + \tfrac{1}{6}t_k q_k^6,$$

$$D_2 H_{\rm d}(\varphi_k) = -\frac{t_{k+1}(p_{k+1}^2 + p_k^2)}{(t_{k+1}^2 + t_k^2)^2} + \tfrac{1}{6}t_{k+1} q_{k+1}^6,$$

$$D_3 H_{\rm d}(\varphi_k) = \tfrac{1}{2}t_k^2 q_k^5, \quad D_4 H_{\rm d}(\varphi_k) = \tfrac{1}{2}t_{k+1}^2 q_{k+1}^5,$$

$$D_5 H_{\rm d}(\varphi_k) = \frac{p_k}{t_{k+1}^2 + t_k^2}, \quad D_6 H_{\rm d}(\varphi_k) = \frac{p_{k+1}}{t_{k+1}^2 + t_k^2}. \quad (30)$$

Firstly, we seek the discrete the Noether exact invariant. Suppose that the generators τ_k^0, ξ_k^0 and η_k^0 are linear, i.e.

$$\tau_k^0(t_k, q_k, p_k) = C_1 t_k + C_2 q_k + C_3 p_k + C_4,$$
$$\xi_k^0(t_k, q_k, p_k) = C_5 t_k + C_6 q_k + C_7 p_k + C_8,$$
$$\eta_k^0(t_k, q_k, p_k) = C_9 t_k + C_{10} q_k + C_{11} p_k + C_{12}, \quad (31)$$

where C_1–C_{12} are constants.

Substituting (30) and (31) into the the Noether identity (9) of the system without perturbation, we have

$$\left[\frac{1}{2}\frac{p_{k+1}^2 + p_k^2}{t_{k+1}^2 + t_k^2} + \frac{1}{12}(t_{k+1}^2 q_{k+1}^6 + t_k^2 q_k^6)\right] D_{\rm d}(\tau_k^0) + \left[\tfrac{1}{6}t_k q_k^6 - \frac{t_k(p_{k+1}^2 + p_k^2)}{(t_{k+1}^2 + t_k^2)^2}\right]\tau_k^0 + \left[\tfrac{1}{6}t_{k+1} q_{k+1}^6 - \frac{t_{k+1}(p_{k+1}^2 + p_k^2)}{(t_{k+1}^2 + t_k^2)^2}\right]\tau_{k+1}^0 + \tfrac{1}{2}t_k^2 q_k^5 \xi_k^0 + \tfrac{1}{2}t_{k+1}^2 q_{k+1}^5 \xi_{k+1}^0 + \frac{p_k \eta_k^0}{t_{k+1}^2 + t_k^2} + \frac{p_{k+1}\eta_{k+1}^0}{t_{k+1}^2 + t_k^2} - \tfrac{1}{2}(p_{k+1} + p_k) D_{\rm d}(\xi_k^0) + \tfrac{1}{2}(q_{k+1} + q_k) D_{\rm d}(\eta_k^0) + D_{\rm d}(G_{Nk}^0) = 0. \quad (32)$$

It follows that when

$$\tau_k^0(t_k, q_k, p_k) = 2t_k, \quad \xi_k^0(t_k, q_k, p_k) = -q_k, \quad \eta_k^0(t_k, q_k, p_k) = p_k, \quad (33)$$

the function

$$G_{Nk}^0 = -q_k p_k \quad (34)$$

satisfies the the Noether identity (9). According to Theorem 1, the discrete difference variational Hamilton system has the following discrete the Noether exact invariant:

$$I_{\rm N0,d} = \frac{t_k(p_k^2 + p_{k-1}^2)}{t_k^2 + t_{k-1}^2} - \frac{2t_k^2(t_k - t_{k-1})(p_k^2 + p_{k-1}^2)}{(t_k^2 + t_{k-1}^2)^2} + \frac{(t_k - t_{k-1})p_k^2}{t_k^2 + t_{k-1}^2} + \tfrac{1}{6}t_k t_{k-1}(t_k q_k^6 + t_{k-1} q_{k-1}^6) + \tfrac{1}{2}(p_{k-1}q_k + p_k q_{k-1}) = \text{const}. \quad (35)$$

Secondly, we seek the first order discrete the Noether adiabatic invariant. We also suppose that the generators τ_k^1, ξ_k^1 and η_k^1 are linear, i.e.

$$\tau_k^1(t_k, q_k, p_k) = C_1' t_k + C_2' q_k + C_3' p_k + C_4',$$
$$\xi_k^1(t_k, q_k, p_k) = C_5' t_k + C_6' q_k + C_7' p_k + C_8',$$
$$\eta_k^1(t_k, q_k, p_k) = C_9' t_k + C_{10}' q_k + C_{11}' p_k + C_{12}', \quad (36)$$

where C_1'–C_{12}' are constants.

Substituting (30), (31) and (36) into the determining Eq. (23) of the perturbation to the the Noether symmetry of the system, we have

$$\left[\frac{1}{2}\frac{p_{k+1}^2 + p_k^2}{t_{k+1}^2 + t_k^2} + \frac{1}{12}(t_{k+1}^2 q_{k+1}^6 + t_k^2 q_k^6)\right] D_{\rm d}(\tau_k^1) + \left[\tfrac{1}{6}t_k q_k^6 - \frac{t_k(p_{k+1}^2 + p_k^2)}{(t_{k+1}^2 + t_k^2)^2}\right]\tau_k^1 + \left[\tfrac{1}{6}t_{k+1} q_{k+1}^6 - \frac{t_{k+1}(p_{k+1}^2 + p_k^2)}{(t_{k+1}^2 + t_k^2)^2}\right]\tau_{k+1}^1 + \tfrac{1}{2}t_k^2 q_k^5 \xi_k^1 + \tfrac{1}{2}t_{k+1}^2 q_{k+1}^5 \xi_{k+1}^1 + \frac{p_k \eta_k^1}{t_{k+1}^2 + t_k^2} + \frac{p_{k+1}\eta_{k+1}^1}{t_{k+1}^2 + t_k^2} - \tfrac{1}{2}(p_{k+1} + p_k) D_{\rm d}(\xi_k^1) + \tfrac{1}{2}(q_{k+1} + q_k) D_{\rm d}(\eta_k^1) + \frac{q_{k+1}p_{k+1} - q_k p_k}{3q_k t_k - q_k t_{k+1} - 2q_{k+1}t_k}\left[\xi_k^0 - D_{\rm d}(q_k)\tau_k^0\right] + D_{\rm d}(G_{Nk}^1) = 0. \quad (37)$$

It follows that when

$$\tau_k^1(t_k, q_k, p_k) = 2t_k, \quad \xi_k^1(t_k, q_k, p_k) = -q_k, \quad \eta_k^1(t_k, q_k, p_k) = p_k, \quad (38)$$

the function

$$G_{Nk}^1 = -2q_k p_k \quad (39)$$

satisfies the determining Eq. (23) of the perturbation to the Noether symmetry. According to Theorem 2, the discrete difference variational Hamilton system has the following first order discrete the Noether adiabatic invariant:

$$I_{\mathrm{N1,d}} = \left[\frac{t_k(p_k^2+p_{k-1}^2)}{t_k^2+t_{k-1}^2} - \frac{2t_k^2(t_k-t_{k-1})(p_k^2+p_{k-1}^2)}{(t_k^2+t_{k-1}^2)^2} + \frac{(t_k-t_{k-1})p_k^2}{t_k^2+t_{k-1}^2} + \tfrac{1}{6}t_k t_{k-1}\big(t_k q_k^6 + t_{k-1}q_{k-1}^6\big) + \tfrac{1}{2}(p_{k-1}q_k + p_k q_{k-1})\right](1+\varepsilon) - \varepsilon q_k p_k. \quad (40)$$

Further we can obtain more high-order discrete adiabatic invariants. In addition, it is worth noting that there are many calculations above, which can be also done in an automatic way using a computer algebra system stated in Refs. [26, 27].

5. Conclusion

The perturbation to the the Noether symmetry and the the Noether adiabatic invariants of discrete difference variational systems are studied in this paper. We obtain the discrete the Noether adiabatic invariant induced directly by the perturbation to the Noether symmetry of the system. When $z = 0$, this means that $\varepsilon W_{\mathrm{d}}(\varphi_k)$ vanish, the system becomes the one without perturbations, and the discrete the Noether adiabatic invariant will turn into the discrete the Noether exact invariant (i.e. the so-called discrete the Noether conserved quantity in Ref. [14]) naturally. The results of this paper have important significance for further study on the discrete mechanical systems and symmetrical perturbation theory.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (grant No. 11103086) and Fundamental Research Funds for the Central Universities of China (grant No. 09CX04018A).

References

[1] F.X. Mei, *Symmetries and Conserved Quantities of Constrained Mechanical Systems*, Beijing Institute of Technology Press, Beijing 2004 (in Chinese).

[2] S.K. Luo, Y.F. Zhang, et al., *Advances in the Study of Dynamics of Constrained Mechanics Systems*, Science Press, Beijing 2008 (in Chinese).

[3] D.F.M. Torres, in: *Dynamics, Bifurcations, and Control, Kloster Irsee, 2001, Lecture Notes in Control and Inform. Sci.*, Eds. F. Colonius, L. Grüne, Vol. 273, Springer, Berlin 2002, p. 287.

[4] D.F.M. Torres, *Eur. J. Control.* **8**, 56 (2002).

[5] D.F.M. Torres, *Commun. Pure Appl. Anal.* **3**, 491 (2004).

[6] D. Levi, R. Yamilov, *J. Math. Phys.* **38**, 6648 (1997).

[7] D. Levi, S. Tremblay, P. Winternitz, *J. Phys. A, Math. Gen.* **33**, 8507 (2000).

[8] D. Levi, S. Tremblay, P. Winternitz, *J. Phys. A, Math. Gen.* **34**, 9507 (2001).

[9] V. Dorodnitsyn, *Appl. Numer. Math.* **39**, 307 (2001).

[10] Z. Bartosiewicz, D.F.M. Torres, *J. Math. Anal. Appl.* **342**, 1220 (2008).

[11] N. Martins, D.F.M. Torres, *Appl. Math. Lett.* **23**, 1432 (2010).

[12] Z. Bartosiewicz, N. Martins, D.F.M. Torres, *Eur. J. Control.* **17**, 9 (2011).

[13] S.Y. Shi, J.L. Fu, L.Q. Chen, *Chin. Phys. B* **17**, 385 (2008).

[14] S.Y. Shi, X.H. Huang, X.B. Zhang, L. Jin, *Acta Phys. Sin.* **58**, 3625 (2009) (in Chinese).

[15] J.L. Fu, H. Fu, R.W. Liu, *Phys. Lett. A* **374**, 1812 (2010).

[16] J.M. Burgers, *Ann. Phys.* **357**, 195 (1917).

[17] Y.Y. Zhao, F.X. Mei, *Symmetries and Invariants of Mechanical Systems*, Science Press, Beijing 1999 (in Chinese).

[18] J.L. Fu, L.Q. Chen, *Phys. Lett. A* **324**, 95 (2004).

[19] X.W. Chen, Y.M. Li, Y.H. Zhao, *Phys. Lett. A* **337**, 274 (2005).

[20] M.J. Zhang, J.H. Fang, X.N. Zhang, K. Lu, *Chin. Phys. B* **17**, 1957 (2008).

[21] N. Ding, X.F. Chen, J.H. Fang, C.Z. Liu, *Phys. Lett. A* **373**, 3005 (2009).

[22] M.J. Zhang, J.H. Fang, K. Lu, T. Pang, P. Lin, *Commun. Theor. Phys.* **51**, 961 (2009).

[23] M.J. Zhang, J.H. Fang, K. Lu, *Int. J. Theor. Phys.* **49**, 427 (2010).

[24] M.J. Zhang, J.H. Fang, K. Lu, K.J. Zhang, Y. Li, *Chin. Phys. Lett.* **26**, 120201 (2009).

[25] P. Wang, H.J. Zhu, *Acta Phys. Pol. A* **119**, 298 (2011).

[26] P.D.F. Gouveia, D.F.M. Torres, E.A.M. Rocha, *Control Cybernet.* **35**, 831 (2006).

[27] P.D.F. Gouveia, D.F.M. Torres, *Nonlinear Anal.* **71**, e138 (2009).