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Galois Properties of the Eigenproblem of the Hexagonal Magnetic Heisenberg Ring

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We analyse the number field-theoretic properties of solutions of the eigenproblem of the Heisenberg Hamiltonian for the magnetic hexagon with the single-node spin 1/2 and isotropic exchange interactions. It follows that eigenenergies and eigenstates are expressible within an extension of the prime field $\mathbb{Q}$ of rationals of degree 2^3 and 2^4, respectively. In quantum information setting, each real extension of rank 2 represents an arithmetic qubit. We demonstrate in detail some actions of the Galois group on the eigenproblem.

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1. Introduction

We present in this paper some *arithmetic* properties of solutions of the eigenproblem for the isotropic Heisenberg Hamiltonian [1, 2] of a magnetic ring with $N = 6$ nodes, each with the spin 1/2, with isotropic exchange interaction between nearest neighbours. It is well known that principles of quantum mechanics locate these solutions in the field $\mathbb{C}$ of complex numbers, i.e. a topologically complete and algebraically closed field. However, a simple experience points out that the entries of the Hamiltonian matrix in the basis of all 2^6 magnetic configurations are integers, which implies that both eigenvalues and eigenstates (the latter in a form of density matrices) can be expressed over a *finite* extension of the prime field $\mathbb{Q}$ of rationals [3].

The aim of the present paper is a detailed exposition of these extensions, discussion of some resulting computational simplifications (in particular a Galois-modified Fourier transformation implies that the matrices of some Hermitian operators become non-Hermitian because the basis is not orthonormal), and a proposal of a natural interpretation of quadratic extensions (sufficient in the considered case for a complete diagonalization) in terms of a qubit in quantum information theory [4], characterized fully by its arithmetic features. In particular, digits $\nu = \pm 1$ of the qubits proposed by us carry spectral quantum numbers.

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2. Preliminaries

The Hilbert space of quantum states for the Heisenberg ring is given by the N-th tensor power of the single-node two-dimensional Hilbert space $h := \mathrm{lc}_{\mathbb{C}}\{|0\rangle, |1\rangle\}$, where $|1\rangle$ (respectively $|0\rangle$) means that there is (respectively there is no) spin deviation at the node, and $\mathrm{lc}_{\mathbb{C}}\{\ldots\}$ denotes the linear closure of the set $\{\ldots\}$ over the field $\mathbb{C}$, that is

$$\mathcal{H} = h^{\otimes N}. \quad (1)$$

The Heisenberg Hamiltonian is defined as follows:

$$\hat{H} = \sum_{j \in \mathbb{Z}_N} (\tau_j - \mathrm{id}_{\mathcal{H}}), \quad (2)$$

where τ_j denotes the transposition of j-th and $(j+1)$-th (mod N) factors in the tensor power $h^{\otimes N}$.

There is the canonical decomposition of the space $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_{r=0}^{N} \mathcal{H}_r, \quad (3)$$

where $\mathcal{H}_r$ denotes the subspace generated by the configurations with exactly r deviations. Such a subspace corresponds to the classical configuration space Q_r [5–9] with given number r of spin deviations. The subspace $\mathcal{H}_r$ is $\hat{H}$-invariant. Hence the Hamiltonian can be decomposed as a direct sum

$$\hat{H} = \bigoplus_{r} \hat{H}_r. \quad (4)$$

The Heisenberg Hamiltonian commutes with the operator S^- of creation of a spin deviation, defined by

$$S^-|A\rangle = \sum_{j\notin A} |\{j\}\cup A\rangle, \tag{5}$$

where A is a subset (the set of nodes with spin deviation) of the set of N nodes and $|A\rangle$ is the corresponding quantum state [10–12].

We define subspaces $\mathcal{H}_{r,r'}$ of $\mathcal{H}_r$ for $r' \le r$ as follows. $\mathcal{H}_{r',r'}$ is the orthogonal complement of the image of S^- in $\mathcal{H}_{r'}$ and $\mathcal{H}_{r,r'} = (S^-)^{r-r'}\mathcal{H}_{r',r'}$ for $r' < r$. One can see that

$$\mathcal{H}_r = \bigoplus_{r'=0}^{r} \mathcal{H}_{r,r'}. \tag{6}$$

Clearly, r and r' determine the total spin quantum numbers $M = 3 - r$ and $S = 3 - r'$, respectively.

The magnetic configuration space Q_2 for two spin deviations consists of three orbits of the translation symmetry group $\mathbb{Z}_6$, labelled by relative configuration vectors $\boldsymbol{t} = (1,5),\ (2,4),\ (3,3)$ where the first coordinate t_1 of $\boldsymbol{t}$ is the distance between two spin deviations and the second $t_2 = N - t_1$. The cardinality $N_{\boldsymbol{t}}$ of an orbit equals 6 for $\boldsymbol{t} = (1,5)$ or $\boldsymbol{t} = (2,4)$, and 3 for $\boldsymbol{t} = (3,3)$. The first two orbits are generic and the third is twofold rarefied.

In the $r = 3$ case we have four orbits labelled by relative configuration vectors $\boldsymbol{t} = (1,1,4),\ (1,2,3),\ (1,3,2),\ (2,2,2)$ with cardinality $N_{\boldsymbol{t}} = 6,6,6,2$, respectively.

The above decomposition induces the following choice of the orbit basis:

$$|\boldsymbol{t}, j\rangle, \tag{7}$$

where $\boldsymbol{t}$ is the label of an orbit and j is the number of an element within the orbit, with $j \in \tilde{6}$ for $\boldsymbol{t} = (1,5),\ (2,5)$ and $j \in \tilde{3}$ for $\boldsymbol{t} = (3,3)$ in $r = 2$ case, with $\tilde{n}$ denoting the set $\{1,\ldots,n\}$ for natural n. Further $j \in \tilde{6}$ for $\boldsymbol{t} = (1,1,4), (1,2,3), (1,3,2)$, and $j \in \tilde{2}$ for $\boldsymbol{t} = (2,2,2)$ in $r = 3$ case.

We use the Fourier-like transformation to a new basis in the form

$$|k, \boldsymbol{t}\rangle = \sum_{j=1}^{N_t} \omega^{-kj} |\boldsymbol{t}, j\rangle, \tag{8}$$

where k denotes the quasimomentum (k has the meaning of the label of the irreducible representation of the translation group, $k = 0, \pm1, \pm2, 3$) of the wavelet and $\omega = \mathrm{e}^{2\pi \mathrm{i}/6}$. The relative position vectors run as follows: for $r = 2$ and k odd, $\boldsymbol{t} = (1,5),\ (2,5)$, and for k even one has to add $\boldsymbol{t} = (3,3)$. similarly for $r = 3$ and k not divisible by 3, $\boldsymbol{t} = (1,1,4),\ (1,2,3),\ (1,3,2)$, and for k divisible by 3 one has to add $\boldsymbol{t} = (2,2,2)$. This basis is unnormalized, which we motivate by minimisation of the order of necessary field extension. We refer to $|k, \boldsymbol{t}\rangle$ as to the Galois wavelet, since its coordinates are located in the cyclotomic field $\mathbb{Q}(\omega)$.

The Gram–Schmidt matrix in the Galois basis is diag[6, 6] for odd k and diag[6, 6, 3] for even k in $r = 2$ case. Further diag[6, 6, 6] for k not divisible by 3, and diag[6, 6, 6, 2] elsewhere in $r = 3$ case.

Translation operators commute with Hamiltonian and preserve the total spin quantum numbers $r,\ r'$. Hence we have the following mutually consistent decompositions:

$$\mathcal{H} = \bigoplus_{r,r',k} \mathcal{H}^k_{r,r'}, \quad \hat{H} = \bigoplus_{r,r',k} \hat{H}^k_{r,r'}. \tag{9}$$

One has

$$\dim \mathcal{H}^k_{r,2} = \begin{cases} 1 & \text{for odd } k, \\ 2 & \text{for even k} \end{cases} \tag{10}$$

for $r = 2,3,4$ and

$$\dim \mathcal{H}^k_{3,3} = \begin{cases} 1 & \text{for } k = 0, \pm1, \\ 0 & \text{for } k = \pm2, \\ 2 & \text{for } k = 3. \end{cases} \tag{11}$$

Basing on these results, we find that the spaces $\mathcal{H}^k_{r,2}$ for $r = 2,3$ and k even, as well as $\mathcal{H}^3_{3,3}$, relevant in the diagonalization, are all two-dimensional. They can be therefore interpreted within quantum information setting as *qubits*. Moreover, we point out here that these spaces are also distinguished by their specific arithmetic properties determined by various finite Galois extensions.

3. Hamiltonian and creation operator of spin deviation in the Galois wavelets

The secular matrices H^k_r in the basis of the Galois wavelets read

$$H^0_0 = [0], \quad H^k_1 = \left[-2 + \bar{\xi} + \xi\right], \tag{12}$$

$$H^k_2 = C^{3;k,2}_{3;k,2} \begin{bmatrix} -2 & 1+\xi & 0 \\ 1+\bar{\xi} & -4 & 1+\xi \\ 0 & 2(1+\bar{\xi}) & -4 \end{bmatrix}, \tag{13}$$

$$H^k_3 = C^{4;k,3}_{4;k,3} \begin{bmatrix} -2 & 1 & \bar{\xi} & 0 \\ 1 & -4 & 1+\bar{\xi}^2 & \bar{\xi} \\ \xi & 1+\xi^2 & -4 & 1 \\ 0 & 3\xi & 3 & -6 \end{bmatrix}, \tag{14}$$

where $\xi = \omega^k$ and $C^{i;k,m}_{j;k,n}$ denotes the operation of the deletion of the i-th row of the subsequent matrix for $k \ne 0 \mod m$ and j-th column for $k \ne 0 \mod n$. This operation is a consequence of various ranges of relative position vector $\boldsymbol{t}$ for quasimomenta k with different divisibility properties.

Notice that H^k_2, H^k_3 are not Hermitian which is a consequence of the non-orthonormality of the Galois–Fourier transformation basis.

The creation of spin deviation in the basis of the Galois wavelets has the form

$$S^{-k}_r : \mathcal{H}^k_r \longrightarrow \mathcal{H}^k_{r+1}, \tag{15}$$

$$S^{-k}_1 = C^{3;k,2} \begin{bmatrix} 1+\bar{\xi} \\ 1+\bar{\xi}^2 \\ 1+\bar{\xi}^3 \end{bmatrix},$$

$$S_2^{-k} = C_{3;k,2}^{4;k,3} \begin{bmatrix} 1+\overline{\xi} & 1 & 0 \\ 1 & \overline{\xi} & 2 \\ 1 & \xi^2 & 2\overline{\xi} \\ 0 & 3 & 0 \end{bmatrix}. \quad (16)$$

It is worth to observe that the matrix elements of Hamiltonians H_r^k and operators of creation of spin deviation belong to the cyclotomic field of $\mathbb{Q}(\omega)$. Similarly the matrices of projection operators

$$P_{r,r'}^k : \mathcal{H}_r^k \longrightarrow \mathcal{H}_{r,r'}^k \quad (17)$$

have elements belonging also to $\mathbb{Q}(\omega)$. The eigenenergies $E_{r'}^k$ for the case $\mathcal{H}_{r,r'}^k$ being one-dimensional are rationals ($\mathbb{Q}(\omega + \overline{\omega}) = \mathbb{Q}$ since $\omega + \overline{\omega} = 1$). In the case when $\mathcal{H}_{r,r'}^k$ is a qubit, the eigenergies generate a quadratic extension of rationals: for $(k = 0, r' = 2)$, $(k = \pm 2, r' = 2)$ and $(k = 3, r' = 3)$ the eigenergies are

$$E_{2,\pm 1}^0 = -5 \pm \sqrt{5} \in \mathbb{Q}(\sqrt{5}), \quad (18)$$

$$E_{2,\pm 1}^2 = -\frac{7}{2} \pm \frac{\sqrt{17}}{2} \in \mathbb{Q}(\sqrt{17}),$$

$$\text{and} \quad E_{3,\pm 1}^3 = -5 \pm \sqrt{13} \in \mathbb{Q}(\sqrt{13}),$$

respectively.

4. Density matrices and Galois action

Equations (10, 11) point out that all subspaces relevant in the diagonalization are either one- or two--dimensional. Clearly, projection operators $P_{r,r'}^k$ for one-dimensional case, i.e. when the rank of $P_{r,r'}^k$ is one, are identical with density matrices $\varrho_{r,r'}^k$ of the corresponding eigenstates, that is

$$\varrho_{r,r'}^k = P_{r,r'}^k. \quad (19)$$

In the other case, i.e. when $\mathcal{H}_{r,r'}^k$ constitute a qubit, the density matrices for the two eigenstates are

$$\varrho_{r,r',\nu}^k = \nu \frac{H_r^k P_{r,r'}^k - E_{r',-\nu}^k P_{r,r'}^k}{\sqrt{\Delta_{r'}^k}}, \quad (20)$$

where $\nu = \pm 1$ are digits of the qubit. All eigenvalues of the Hamiltonian are elements of the field

$$\boldsymbol{H}_{\mathrm{E}} = \mathbb{Q}(\sqrt{5}, \sqrt{17}, \sqrt{13}), \quad (21)$$

which we refer in the sequel to as the real (or energetic) Heisenberg number field. This real field is not sufficient to describe wavelets so we propose to define the complex Heisenberg number field

$$\boldsymbol{H} = \mathbb{Q}(\mathrm{i}\sqrt{3}, \sqrt{5}, \sqrt{17}, \sqrt{13}). \quad (22)$$

The Galois group of the extension $\boldsymbol{H}/\mathbb{Q}$ is isomorphic to $C_2^{\times 4}$ where $C_2 = \{\pm 1\}$ is the multiplicative two-elements group. The Galois group is generated by four Coxeter reflections:

$$g_q = (-1, 1, 1, 1), \ g_2^0 = (1, -1, 1, 1),$$

$$g_2^2 = (1, 1, -1, 1), \ g_3^3 = (1, 1, 1, -1), \quad (23)$$

and acts (by definition) on the Heisenberg field $\boldsymbol{H}$. Further we define additive actions θ (Θ respectively) on linear combinations of the vectors of orbit basis $|\boldsymbol{t}, j\rangle$ (of endomorphism of the form $|\boldsymbol{t}, j\rangle\langle \boldsymbol{t}', j'|$ respectively) over $\boldsymbol{H}$ in the following way [11]:

$$\theta_g \left(\sum_{\boldsymbol{t},j} a_{\boldsymbol{t},j} |\boldsymbol{t}, j\rangle \right) = \sum_{\boldsymbol{t},j} (g a_{\boldsymbol{t},j}) |\boldsymbol{t}, j\rangle \quad (24)$$

$$\left(\Theta_g \left(\sum_{\boldsymbol{t},j;\boldsymbol{t}',j'} a_{\boldsymbol{t},j;\boldsymbol{t}',j'} |\boldsymbol{t}, j\rangle\langle \boldsymbol{t}', j'| \right) = \sum_{\boldsymbol{t},j;\boldsymbol{t}',j'} (g a_{\boldsymbol{t},j;\boldsymbol{t}',j'}) |\boldsymbol{t}, j\rangle\langle \boldsymbol{t}', j'| \right) \text{ respectively.}$$

The g_q reverses the sign of the quantum number k of quasimomentum, and $g_{r'}^k$ — the digit $\nu = \pm 1$ which labels the eigenstates in the qubit $\mathcal{H}_{r,r'}^k$. The Galois actions preserve the vector of relative positions and quantum numbers r, r'. Let $g = (\epsilon_q, \epsilon_2^0, \epsilon_2^2, \epsilon_3^3)$ where $\epsilon_q, \epsilon_{r'}^k = \pm 1$. Then the Galois action on wavelets reads

$$\theta_g |\boldsymbol{t}, k\rangle = |\boldsymbol{t}, \epsilon_q k\rangle. \quad (25)$$

Further if $P_{r,r'}^k$ has rank one the action does not change eigenenergy and changes density matrix by reversing of quasimomentum

$$g E_{r'}^k = E_{r'}^k, \quad \Theta_g \varrho_{r'}^k = \varrho_{r'}^{\epsilon_q k}. \quad (26)$$

If $\mathcal{H}_{r,r'}^k$ is a qubit these actions change eigenenergies and density matrices by means of the Coxeter reflections

$$g E_{r',\nu}^k = E_{r',\epsilon_{r'}^{|k|}\nu}^k, \quad \Theta_g \varrho_{r',\nu}^k = \varrho_{r',\epsilon_{r'}^{|k|}\nu}^{\epsilon_q k}. \quad (27)$$

5. Conclusions

We have presented the exact solutions of the eigenproblem of the isotropic Heisenberg Hamiltonian for a hexagonal magnetic ring. This solution is expressed within a minimal extension of the prime field $\mathbb{Q}$ of rationals, referred to as the Heisenberg number field $\boldsymbol{H} = \mathbb{Q}(\mathrm{i}\sqrt{3}, \sqrt{5}, \sqrt{17}, \sqrt{13})$. The extension is of the order $2^4 = 16$, and the corresponding Galois group is recognised as the fourth Carthesian power of the Coxeter group C_2. The first factor of the Cartesian power correponds to reflection of quasimomenta (k to $-k$), whereas the other three factors are related to exchange of digits of arithmetic qubits, associated with two-dimensional secular problems, with hallmarks given by square roots of prime integers 5, 17, and 13 resulting from initial integer Hamiltonian matrices.

We have demonstrated that exploitation of the arithmetic structure of the eigenproblem yields precisely defined quantum spaces (qubits) in terms of algebraic numbers (e.g. $\mathrm{i}\sqrt{3}$, $\sqrt{5}$, $\sqrt{13}$, $\sqrt{17}$; cf. Eqs. (18, 22)), and determines transformations between such qubits in terms of appropriate Galois groups. We believe that in particular such a form of selection of the eigenproblem will be convenient in disussion of entanglement in finite rings [13].

It is worth to mention, however, that the formalism heavily depends on arithmetic properties of the integer N (cf. e.g. [12] for $N = 5$) which vary so rapidly and irregularly that nowadays one cannot expect an easy extension of such considerations neither to larger nor smaller systems. The situation resemblies somehow the Hofstadter butterfly in fractional quantum Hall effect (cf. e.g. [14]) or Shor factorisation and quantum computing [15, 4] where arithmetic properties of integers play crucial role.

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