Applications of a Generalized Extended $\left(\frac{G'}{G}\right)$-Expansion Method to Find Exact Solutions of Two Nonlinear Schrödinger Equations with Variable Coefficients

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In the present paper, we construct the travelling wave solutions of two nonlinear Schrödinger equations with variable coefficients by using a generalized extended $\left(\frac{G'}{G}\right)$-expansion method, where $G = G(\xi)$ satisfies a second order linear ordinary differential equation. By using this method, new exact solutions involving parameters, expressed by hyperbolic and trigonometric function solutions are obtained. When the parameters are taken as special values, some solitary wave solutions are derived from the hyperbolic function solutions.

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1. Introduction

During the past decades, investigations on the optical fiber communications have become more and more attractive [1], in which the optical solitons have their potential applications in optical fiber transmission systems [2, 3]. As we all know, solitonic structures are seen in many fields of sciences and engineering [4, 5], among which an optical soliton exists in a fiber on the basis of the exact balance between the group velocity dispersion (GVD) and the self-phase modulation (SPM). The propagation of the optical solitons is usually governed by the nonlinear Schrödinger equation (NLSE), which is one of the most important models in modern nonlinear science. Moreover, much attention has been paid to the investigation on the generalized NLSEs with constant coefficients as a kind of ideal models of the much more complicated physical problems [6, 7]. As a matter of fact, in a real fiber there exist some fiber nonuniformities to influence various effects such as the gain or loss, GVD and SPM, etc. Considering the varying dispersion, nonlinearity and gain/loss, we would like to investigate the following two variable coefficients nonlinear Schrödinger equations:

1. The nonlinear Schrödinger equation with the gain and variable coefficients

$$\mathrm{i}u_x + \frac{1}{2}\beta(x)u_{tt} + \alpha(x)u|u|^2 - \mathrm{i}\gamma(x)u = 0\,, \tag{1}$$

where $u(x,t)$ is the complex envelope of the electrical field, x is the normalized propagation distance along the fiber, t is the retarded time and the subscripts denote partial derivatives. All the parameters $\beta(x)$, $\alpha(x)$ and $\gamma(x)$ are real analytic functions of the normalized propagation distance x, which represent the GVD, SPM and distributed gain/loss, respectively. Equation (1) describes the amplification or absorption of pulses propagation in a single mode optical fiber with distributed dispersion and nonlinearity. This equation appears in many fields of physical and engineering sciences, e.g., in plasma physics [8], arterial mechanics [9], long-distance optical communications [10–12]. It describes such situations more realistically than their constant coefficient. In practical applications, Eq. (1) can be used to describe the stable transmission of managed soliton.

2. The higher-order nonlinear Schrödinger equation with variable coefficients

$$u_x = \mathrm{i}\alpha_1(x)u_{tt} + \mathrm{i}\alpha_2(x)u|u|^2 + \alpha_3(x)u_{ttt} + \alpha_4(x)\big(u|u|^2\big)_t + \alpha_5(x)u\big(|u|^2\big)_t + \Gamma(x)u\,, \tag{2}$$

where $\alpha_i(x)$ $(i = 1, 2, \ldots, 5)$ are the distributed parameters, which are real analytic functions of the propagation distance x and t is the related time while $\Gamma(x)$ denotes the amplification or absorption coefficient. Equation (2) has been paid attention by many researchers [13–15] due to its wide range of applications. It describes the femtosecond pulse propagation which can be applied to telecommunication and ultrafast signal-routing systems extensively in the weakly dispersive and nonlinear dielectrics with distributed parameters. The constant coefficients of Eq. (2) has been studied well and the exact solutions (including the dark solitary wave and bright solitary wave) are presented [16, 17]. It is clear that both Eqs. (1) and (2) are very important in the field of mathematical physics. Therefore, it is a significant task to search for explicit solutions of the two equations.

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Up to now, many powerful methods have been established and developed to obtain analytic solutions of NLSEs, such as the inverse scattering method [18], the Hirota bilinear method [19], the Bäcklund transformation [20], the Painlevé expansion [21], the variational iteration method [22], the Adomian decomposition method [23], the tanh function method [24], the Jacobi elliptic function expansion method [25], the exp-function method [26] and so on. More recently, the $\left(\frac{G'}{G}\right)$-expansion method [27–30] has been proposed to obtain traveling wave solutions. This method is firstly proposed by Wang et al. [27] for which the traveling wave solutions of the nonlinear evolution equations are obtained. This method has been extended to solve difference-differential Eqs. [31]. The improved $\left(\frac{G'}{G}\right)$-expansion method has been used in [32–34]. Recently, Guo and Zhou [35] and Zayed and Al-Joudi [36] have obtained the exact traveling wave solutions of some nonlinear partial differential equations (PDEs) using an extended $\left(\frac{G'}{G}\right)$-expansion method.

In the present article, we use the generalized extended $\left(\frac{G'}{G}\right)$-expansion method to derive the exact traveling wave solutions of Eqs. (1) and (2). The generalized extended $\left(\frac{G'}{G}\right)$-expansion method used in this article can be considered as a greater generalization of that used in [35, 36]. This method can be applied to further equations such as difference-differential equations which can be done in forthcoming articles.

2. Description of a generalized extended $\left(\frac{G'}{G}\right)$-expansion method

For a given nonlinear PDEs with independent variables $X = (x, y, z, t)$ and dependent variable u, we consider the PDE

$$F(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0\,, \tag{3}$$

where F is a polynomial in u and its partial derivatives. In order to solve Eq. (3), we use the generalized wave transformation

$$u(X) = u(\xi)\,, \quad \xi = \xi(X)\,. \tag{4}$$

Thus, the solution of Eq. (3) can be expressed by a polynomial in $\left(\frac{G'(\xi)}{G(\xi)}\right)$ as follows:

$$u(X) = a_0(X) + \sum_{i=1}^{M}\left\{a_i(X)\left[\frac{G'(\xi)}{G(\xi)}\right]^i + b_i(X)\left[\frac{G'(\xi)}{G(\xi)}\right]^{i-1}\sqrt{\sigma\left[1 + \frac{1}{\mu}\left(\frac{G'(\xi)}{G(\xi)}\right)^2\right]}\right\}, \tag{5}$$

where $G = G(\xi)$ satisfies the following second order linear ordinary differential equation:

$$G''(\xi) + \mu G(\xi) = 0\,, \tag{6}$$

where μ is a nonzero constant and $\sigma = \pm 1$, while $\xi = \xi(X)$, $a_i(X)$ and $b_i(X)$ are analytic functions of X to be determined later and $' \equiv \frac{\mathrm{d}}{\mathrm{d}\xi}$. To determine $u(X)$ explicitly, we consider the following four steps:

Step 1. Determine the positive integer M by balancing the highest order nonlinear term(s) and the highest order partial derivatives of $u(X)$ in Eq. (3).

Step 2. Substitute (5) along with Eq. (6) into Eq. (3) and collect all terms with the same powers of $\left(\frac{G'}{G}\right)^j$ and $\left(\frac{G'}{G}\right)^j\sqrt{\sigma\left[1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right]}$ together and equating each coefficient to zero, yield a set of over-determined differential equations for $a_0(X)$, $a_i(X)$, $b_i(X)$ and $\xi(X)$.

Step 3. Solve the system of over-determined differential equations obtained in Step 2 for $a_i(X)$, $b_i(X)$, $(i = 0, 1, \ldots, M)$ and $\xi(X)$ by the use of *Maple* or *Mathematica*.

Step 4. Use the results obtained in above steps to derive a series of fundamental solutions of Eq. (3), since the solutions of Eq. (6) have been well known for us. Thus, we can obtain the exact solutions of Eq. (3).

Remark 1: From the general solution of Eq. (6) we have the ratio

$$\frac{G'(\xi)}{G(\xi)} = \mathrm{i}\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\,\frac{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right) - B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right) + B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}, \tag{7}$$

where A and B are arbitrary constants.

Remark 2: It is necessary to point out that by adding the term $\left(\frac{G'}{G}\right)^j\sqrt{\sigma[1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2]}$ into (5), the ansatz proposed here is more general than the ansatz in the original $\left(\frac{G'}{G}\right)$-expansion method [27]. Therefore, the generalized extended $\left(\frac{G'}{G}\right)$-expansion method is more powerful than the original $\left(\frac{G'}{G}\right)$-expansion method and some new types of traveling wave solutions and solitary wave solutions would be expected for some NPDEs. If we choose the parameters in (5) and (6) to take special values, the original $\left(\frac{G'}{G}\right)$-expansion method can be recovered by our proposed method.

3. Applications

In this section, we will apply the generalized extended $\left(\frac{G'}{G}\right)$-expansion method to construct the exact solutions of two nonlinear evolution equations with variable coefficients via the nonlinear Schrödinger equation with the gain and variable coefficients (1) and the higher-order nonlinear Schrödinger equation with variable coefficients (2).

3.1. Example 1. The nonlinear Schrödinger equation with the gain and variable coefficients

In order to obtain the exact solutions of Eq. (1), we assume that the solution of this equation can be written in the form

$$u(x,t)=\upsilon(x,t)\exp(\mathrm{i}\theta(x,t)), \tag{8}$$

where $\upsilon(x,t)$ and $\theta(x,t)$ are amplitude and phase functions, respectively. Substituting (8) into (1) and separating the real and imaginary parts, we obtain

$$-\upsilon\theta_x+\frac{1}{2}\beta(x)\left[\upsilon_{tt}-\upsilon(\theta_t)^2\right]+\alpha(x)\upsilon^3=0\,, \tag{9}$$

and

$$\upsilon_x+\frac{1}{2}\beta(x)[2\upsilon_t\theta_t+\upsilon\theta_{tt}]-\gamma(x)\upsilon=0\,. \tag{10}$$

Balancing υ_{tt} and υ^3 in Eq. (8), we have $M=1$. We assume that Eqs. (9) and (10) have the following formal solutions:

$$\upsilon(\xi)=a_0(x)+a_1(x)\left(\frac{G'(\xi)}{G(\xi)}\right)+b_1(x)\sqrt{\sigma\left[1+\frac{1}{\mu}\left(\frac{G'(\xi)}{G(\xi)}\right)^2\right]}\,, \tag{11}$$

and

$$\theta(x,t)=f(x)t^2+g(x)t+h(x)\,, \tag{12}$$

where $G=G(\xi)$ satisfies Eq. (6) and $\xi=p(x)t+q(x)$. The functions $f(x)$, $g(x)$, $h(x)$, $p(x)$ and $q(x)$ are differentiable functions of the normalized propagation distance x to be determined. Substituting (11) and (12) into (9) and (10), collecting all terms with the same powers of $t^i\left(\frac{G'}{G}\right)^j\left(\sqrt{\sigma\left[1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right]}\right)^k$, $i=0,1,2$, $j=0,1,2,3$, $k=0,1$, together and setting them to zero, we have a system of over-determined differential equations which can be solved by *Maple* or *Mathematica* to get the following cases:

Case 1.

$$a_0(x)=0\,,\quad a_1(x)=0\,,\quad b_1(x)=c_1\mathrm{e}^{\int\gamma(x)\,\mathrm{d}x}\,,\quad p(x)=c_2\,,\quad q(x)=-c_2c_3\int\beta(x)\,\mathrm{d}x+c_4\,,\quad f(x)=0\,,$$
$$g(x)=c_3\,,\quad h(x)=-\frac{1}{2}(\mu c_2^2+c_3^2)\int\beta(x)\,\mathrm{d}x+c_5\,,\quad \alpha(x)=\left(\frac{-\mu c_2^2\beta(x)}{c_1^2\sigma}\right)\mathrm{e}^{-2\int\gamma(x)\,\mathrm{d}x}\,,\quad \beta(x)=\beta(x)\,,$$
$$\gamma(x)=\gamma(x)\,, \tag{13}$$

In this case, we have the exact solution

$$u(\xi)=c_1\sqrt{\sigma\left[1-\frac{|\mu|\,\mathrm{sgn}(\mu)}{\mu}\left(\frac{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}(\mu)}\right)-B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}(\mu)}\right)}{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}(\mu)}\right)+B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}(\mu)}\right)}\right)^2\right]}$$
$$\times\exp\left(\mathrm{i}\left(c_3t-\frac{1}{2}(\mu c_2^2+c_3^2)\int\beta(x)\,\mathrm{d}x+c_5\right)+\int\gamma(x)\,\mathrm{d}x\right), \tag{14}$$

where

$$\xi=c_2\left(t-c_3\int\beta(x)\,\mathrm{d}x\right)+c_4\,. \tag{15}$$

Case 2.

$$a_0(x)=0\,,\quad a_1(x)=c_1\,\mathrm{e}^{\int\gamma(x)\,\mathrm{d}x}\,,\quad b_1(x)=0\,,\quad p(x)=c_2\,,\quad q(x)=-c_2c_3\int\beta(x)\,\mathrm{d}x+c_4\,,\quad f(x)=0\,,$$
$$g(x)=c_3\,,\quad h(x)=\frac{1}{2}\big(2\mu c_2^2-c_3^2\big)\int\beta(x)\,\mathrm{d}x+c_5\,,\quad \alpha(x)=\frac{-c_2^2\beta(x)}{c_1^2}\,\mathrm{e}^{-2\int\gamma(x)\,\mathrm{d}x}\,,\quad \beta(x)=\beta(x)\,,$$
$$\gamma(x)=\gamma(x)\,. \tag{16}$$

In this case, we have the exact solution

$$u(\xi)=\mathrm{i}c_1\sqrt{|\mu|\operatorname{sgn}(\mu)}\left(\frac{A\exp\Big(\mathrm{i}\xi\sqrt{|\mu|\operatorname{sgn}(\mu)}\Big)-B\exp\Big(-\mathrm{i}\xi\sqrt{|\mu|\operatorname{sgn}(\mu)}\Big)}{A\exp\Big(\mathrm{i}\xi\sqrt{|\mu|\operatorname{sgn}(\mu)}\Big)+B\exp\Big(-\mathrm{i}\xi\sqrt{|\mu|\operatorname{sgn}(\mu)}\Big)}\right)$$
$$\times\exp\bigg(\mathrm{i}\bigg(c_3t+\frac{1}{2}\big(2\mu c_2^2-c_3^2\big)\int\beta(x)\,\mathrm{d}x+c_5\bigg)+\int\gamma(x)\,\mathrm{d}x\bigg), \tag{17}$$

where ξ has the same form (15).

Case 3.

$$a_0(x)=0\,,\quad a_1(x)=0\,,\quad b_1(x)=c_1\sqrt{f(x)}\,\mathrm{e}^{\int\gamma(x)\,\mathrm{d}x},\quad p(x)=c_2f(x)\,,\quad q(x)=\frac{1}{2}c_2c_3f(x)+c_4\,,$$
$$f(x)=\frac{1}{2\int\beta(x)\,\mathrm{d}x+c_6}\,,\quad g(x)=c_3f(x)\,,\quad h(x)=\frac{1}{4}(\mu c_2^2+c_3^2)f(x)+c_5\,,$$
$$\gamma(x)=\gamma(x)\,,\alpha(x)=\frac{\mathrm{d}f(x)}{\mathrm{d}x}\frac{\mu c_2^2}{2\sigma c_1^2f(x)}\,\mathrm{e}^{-2\int\gamma(x)\,\mathrm{d}x}\,,\quad \beta(x)=\beta(x)\,. \tag{18}$$

In this case, we have the exact solution

$$u(\xi)=c_1\sqrt{\sigma\left[1-\frac{|\mu|\operatorname{sgn}(\mu)}{\mu}\left(\frac{A\exp\Big(\mathrm{i}\xi\sqrt{|\mu|\operatorname{sgn}(\mu)}\Big)-B\exp\Big(-\mathrm{i}\xi\sqrt{|\mu|\operatorname{sgn}(\mu)}\Big)}{A\exp\Big(\mathrm{i}\xi\sqrt{|\mu|\operatorname{sgn}(\mu)}\Big)+B\exp\Big(-\mathrm{i}\xi\sqrt{|\mu|\operatorname{sgn}(\mu)}\Big)}\right)^2\right]}$$
$$\times\left(\sqrt{\frac{1}{2\int\beta(x)\,\mathrm{d}x+c_6}}\right)\exp\left(\mathrm{i}\left(\frac{t^2+c_3t+\frac{1}{4}\big(\mu c_2^2+c_3^2\big)}{2\int\beta(x)\,\mathrm{d}x+c_6}+c_5\right)+\int\gamma(x)\,\mathrm{d}x\right), \tag{19}$$

where

$$\xi=\frac{c_2(2t+c_3)+c_4}{2\big(2\int\beta(x)\,\mathrm{d}x+c_6\big)}\,. \tag{20}$$

Case 4.

$$a_0(x)=0\,,\quad a_1(x)=c_1\sqrt{f(x)}\,\mathrm{e}^{\int\gamma(x)\,\mathrm{d}x}\,,\quad b_1(x)=0\,,\quad p(x)=c_2f(x)\,,\quad q(x)=\frac{1}{2}c_2c_3f(x)+c_4\,,$$
$$f(x)=\frac{1}{2\int\beta(x)\,\mathrm{d}x+c_6}\,,\quad g(x)=c_3f(x)\,,\quad h(x)=\frac{-1}{4}\big(2\mu c_2^2-c_3^2\big)f(x)+c_5\,,\quad \beta(x)=\beta(x)\,,$$
$$\alpha(x)=\frac{\mathrm{d}f(x)}{\mathrm{d}x}\frac{c_2^2}{2c_1^2f(x)}\,\mathrm{e}^{-2\int\gamma(x)\,\mathrm{d}x}\,,\quad \gamma(x)=\gamma(x)\,, \tag{21}$$

In this case, we have the exact solution

$$u(\xi)=\mathrm{i}c_1\sqrt{\frac{|\mu|\operatorname{sgn}(\mu)}{2\int\beta(x)\,\mathrm{d}x+c_6}}\,\frac{A\exp\Big(\mathrm{i}\xi\sqrt{|\mu|\operatorname{sgn}(\mu)}\Big)-B\exp\Big(-\mathrm{i}\xi\sqrt{|\mu|\operatorname{sgn}(\mu)}\Big)}{A\exp\Big(\mathrm{i}\xi\sqrt{|\mu|\operatorname{sgn}(\mu)}\Big)+B\exp\Big(-\mathrm{i}\xi\sqrt{|\mu|\operatorname{sgn}(\mu)}\Big)}$$
$$\times\exp\left(\mathrm{i}\left(\frac{t^2+c_3t-\frac{1}{4}\big(2\mu c_2^2-c_3^2\big)}{2\int\beta(x)\,\mathrm{d}x+c_6}+c_5\right)+\int\gamma(x)\,\mathrm{d}x\right), \tag{22}$$

where ξ has the same form (20).

Case 5.

$$a_0(x)=0\,,\quad a_1(x)=c_1\sigma\sqrt{\frac{\sigma}{\mu}}\,\mathrm{e}^{\int\gamma(x)\,\mathrm{d}x}\,,\quad b_1(x)=c_1\,\mathrm{e}^{\int\gamma(x)\,\mathrm{d}x}\,,\quad p(x)=c_2\,,\quad q(x)=-c_2c_3\int\beta(x)\,\mathrm{d}x+c_4\,,$$
$$f(x)=0\,,\quad g(x)=c_3\,,\quad h(x)=\frac{1}{4}\left(\mu c_2^2-2c_3^2\right)\int\beta(x)\,\mathrm{d}x+c_5\,,\quad \alpha(x)=\frac{-\mu c_2^2\beta(x)}{4\sigma c_1^2}\,\mathrm{e}^{-2\int\gamma(x)\,\mathrm{d}x}\,,$$
$$\beta(x)=\beta(x)\,,\quad \gamma(x)=\gamma(x)\,. \tag{23}$$

In this case, we have the exact solution

$$u(\xi)=c_1\left\{\mathrm{i}\sigma\sqrt{\frac{\sigma|\mu|\,\mathrm{sgn}\,(\mu)}{\mu}}\,\frac{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)-B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)+B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}\right.$$
$$\left.+\sqrt{\sigma\left[1-\frac{|\mu|\,\mathrm{sgn}\,(\mu)}{\mu}\left(\frac{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)-B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)+B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}\right)^2\right]}\right\}$$
$$\times\exp\left(\mathrm{i}\left(c_3t+\frac{1}{4}\left(\mu c_2^2-2c_3^2\right)\int\beta(x)\,\mathrm{d}x+c_5\right)+\int\gamma(x)\,\mathrm{d}x\right), \tag{24}$$

where ξ has the same form (15).

Case 6.

$$a_0(x)=0\,,\quad a_1(x)=c_1\sigma\sqrt{\frac{\sigma f(x)}{\mu}}\,\mathrm{e}^{\int\gamma(x)\,\mathrm{d}x}\,,\quad b_1(x)=c_1\sqrt{f(x)}\,\mathrm{e}^{\int\gamma(x)\,\mathrm{d}x}\,,\quad p(x)=c_2f(x)\,,$$
$$q(x)=\frac{1}{2}c_2c_3f(x)+c_4\,,\quad f(x)=\frac{1}{2\int\beta(x)\,\mathrm{d}x+c_6}\,,\quad g(x)=c_3f(x)\,,\quad h(x)=\frac{-1}{8}\left(\mu c_2^2-2c_3^2\right)f(x)+c_5\,,$$
$$\beta(x)=\beta(x)\,,\quad \alpha(x)=\frac{\mathrm{d}f(x)}{\mathrm{d}x}\,\frac{\mu c_2^2}{8\sigma c_1^2f(x)}\,\mathrm{e}^{-2\int\gamma(x)\,\mathrm{d}x}\,,\quad \gamma(x)=\gamma(x)\,. \tag{25}$$

In this case, we have the exact solution

$$u(\xi)=c_1\left\{\mathrm{i}\sigma\sqrt{\frac{\sigma|\mu|\,\mathrm{sgn}\,(\mu)}{\mu}}\left(\frac{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)-B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)+B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}\right)\right.$$
$$\left.+\sqrt{\sigma\left[1-\frac{|\mu|\,\mathrm{sgn}\,(\mu)}{\mu}\left(\frac{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)-B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)+B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}\right)^2\right]}\right\}$$
$$\times\left(\sqrt{\frac{1}{2\int\beta(x)\,\mathrm{d}x+c_6}}\right)\exp\left(\mathrm{i}\left(\frac{t^2+c_3t-\frac{1}{8}\left(\mu c_2^2-2c_3^2\right)}{2\int\beta(x)\,\mathrm{d}x+c_6}+c_5\right)+\int\gamma(x)\,\mathrm{d}x\right), \tag{26}$$

where ξ has the same form (20). In particular, we deduce from (14) that the solitary wave solutions of Eq. (1) are derived as follows:

If $B=0$, $A\neq 0$ and $\mu<0$, then we obtain

$$u(\xi)=c_1\sqrt{-\sigma}\,\mathrm{csch}\left(\sqrt{-\mu}\xi\right)\exp\left(\mathrm{i}\left(c_3t-\frac{1}{2}\left(\mu c_2^2+c_3^2\right)\int\beta(x)\,\mathrm{d}x+c_5\right)+\int\gamma(x)\,\mathrm{d}x\right), \tag{27}$$

while, if $B\neq 0$, $B^2>A^2$ and $\mu<0$, then we obtain

$$u(\xi)=c_1\sqrt{\sigma}\,\mathrm{sech}\left(\sqrt{-\mu}\xi+\xi_0\right)\exp\left(\mathrm{i}\left(c_3t-\frac{1}{2}\left(\mu c_2^2+c_3^2\right)\int\beta(x)\,\mathrm{d}x+c_5\right)+\int\gamma(x)\,\mathrm{d}x\right), \tag{28}$$

where $\xi_0=\tanh^{-1}\left(\frac{A}{B}\right)$. Similarly, we can find more solitary wave solutions of Eq. (1) using (17), (19), (22), (24) and (26) but we omitted them for simplicity.

3.2. Example 2. The higher-order nonlinear Schrödinger equation with variable coefficients

In order to obtain the exact solutions of Eq. (2), we assume that the solution of Eq. (2) has the same form (8). Substituting (8) into (2) and separating the real and imaginary parts, we obtain

$$\upsilon_x = (-2\upsilon_t\theta_t - \upsilon\theta_{tt})\alpha_1(x) + \Big(\upsilon_{ttt} - 3\upsilon_t(\theta_t)^2 - 3\upsilon\theta_t\theta_{tt}\Big)\alpha_3(x) + \upsilon^2\upsilon_t(3\alpha_4(x) + 2\alpha_5(x)) + \upsilon\Gamma(x)\,, \tag{29}$$

and

$$\upsilon\theta_x = \Big(\upsilon_{tt} - \upsilon(\theta_t)^2\Big)\alpha_1(x) + \Big(3\upsilon_{tt}\theta_t + 3\upsilon_t\theta_{tt} - \upsilon(\theta_t)^3 + \upsilon\theta_{ttt}\Big)\alpha_3(x) + \upsilon^3\alpha_2(x) + \upsilon^3\theta_t\alpha_4(x)\,. \tag{30}$$

Balancing υ_{ttt} and $\upsilon^2\upsilon_t$ in Eq. (29), we have $M = 1$. In order to search for explicit solutions, we deduce that Eqs. (29) and (30) have the same formal solutions (11) and (12). Consequently, we have the following cases:

Case 1.

$$\begin{aligned} & a_0(x) = 0\,, \quad a_1(x) = 0\,, \quad b_1(x) = c_1\mathrm{e}^{\int\Gamma(x)\,\mathrm{d}x}\,, \quad p(x) = c_2\,, \\ & q(x) = -c_2\left[2c_3\int\alpha_1(x)\,\mathrm{d}x + (\mu c_2^2 + 3c_3^2)\int\alpha_3(x)\,\mathrm{d}x\right] + c_4\,, \\ & h(x) = -(\mu c_2^2 + c_3^2)\int\alpha_1(x)\,\mathrm{d}x - c_3(3\mu c_2^2 + c_3^2)\int\alpha_3(x)\,\mathrm{d}x + c_5\,, \quad f(x) = 0\,, \quad g(x) = c_3\,, \quad \alpha_1(x) = \alpha_1(x)\,, \\ & \alpha_3(x) = \alpha_3(x)\,, \quad \alpha_4(x) = \alpha_4(x)\,, \quad \alpha_2(x) = -c_3\alpha_4(x) - \frac{2\mu c_2^2}{\sigma c_1^2}[\alpha_1(x) + 3c_3\alpha_3(x)]\mathrm{e}^{-2\int\Gamma(x)\,\mathrm{d}x}\,, \\ & \alpha_5(x) = \frac{-3}{2}\alpha_4(x) - \frac{3\mu c_2^2}{\sigma c_1^2}\alpha_3(x)\mathrm{e}^{-2\int\Gamma(x)\,\mathrm{d}x}\,, \quad \Gamma(x) = \Gamma(x)\,. \end{aligned} \tag{31}$$

In this case, we have the exact solution

$$\begin{aligned} u(\xi) = {} & c_1\sqrt{\sigma\left[1 - \frac{|\mu|\,\mathrm{sgn}\,(\mu)}{\mu}\left(\frac{A\exp\Big(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\Big) - B\exp\Big(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\Big)}{A\exp\Big(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\Big) + B\exp\Big(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\Big)}\right)^2\right]} \\ & \times\exp\left(\mathrm{i}\left(c_3t - (\mu c_2^2 + c_3^2)\int\alpha_1(x)\,\mathrm{d}x - c_3(3\mu c_2^2 + c_3^2)\int\alpha_3(x)\,\mathrm{d}x + c_5\right) + \int\Gamma(x)\,\mathrm{d}x\right), \end{aligned} \tag{32}$$

where

$$\xi = c_2\left\{t - \left[2c_3\int\alpha_1(x)\,\mathrm{d}x + (\mu c_2^2 + 3c_3^2)\int\alpha_3(x)\,\mathrm{d}x\right]\right\} + c_4. \tag{33}$$

Case 2.

$$\begin{aligned} & a_0(x) = 0\,, \quad a_1(x) = c_1\mathrm{e}^{\int\Gamma(x)\,\mathrm{d}x}\,, \quad b_1(x) = 0\,, \quad p(x) = c_2\,, \\ & q(x) = -c_2\left[2c_3\int\alpha_1(x)\,\mathrm{d}x + (-2\mu c_2^2 + 3c_3^2)\int\alpha_3(x)\,\mathrm{d}x\right] + c_4\,, \\ & h(x) = (2\mu c_2^2 - c_3^2)\int\alpha_1(x)\,\mathrm{d}x + c_3(6\mu c_2^2 - c_3^2)\int\alpha_3(x)\,\mathrm{d}x + c_5\,, \quad f(x) = 0\,, \quad g(x) = c_3\,, \quad \alpha_1(x) = \alpha_1(x)\,, \\ & \alpha_3(x) = \alpha_3(x)\,, \quad \alpha_4(x) = \alpha_4(x)\,, \quad \alpha_2(x) = -c_3\alpha_4(x) - \frac{2c_2^2}{c_1^2}[\alpha_1(x) + 3c_3\alpha_3(x)]\mathrm{e}^{-2\int\Gamma(x)\,\mathrm{d}x}\,, \\ & \alpha_5(x) = \frac{-3}{2}\alpha_4(x) - \frac{3c_2^2}{c_1^2}\alpha_3(x)\mathrm{e}^{-2\int\Gamma(x)\,\mathrm{d}x}\,, \quad \Gamma(x) = \Gamma(x)\,. \end{aligned} \tag{34}$$

In this case, we have the exact solution

$$\begin{aligned} u(\xi) = {} & \mathrm{i}c_1\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\frac{A\exp\Big(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\Big) - B\exp\Big(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\Big)}{A\exp\Big(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\Big) + B\exp\Big(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\Big)} \\ & \times\exp\left(\mathrm{i}\left(c_3t + (2\mu c_2^2 - c_3^2)\int\alpha_1(x)\,\mathrm{d}x + c_3(6\mu c_2^2 - c_3^2)\int\alpha_3(x)\,\mathrm{d}x + c_5\right) + \int\Gamma(x)\,\mathrm{d}x\right), \end{aligned} \tag{35}$$

where

$$\xi = c_2\left\{t - \left[2c_3\int\alpha_1(x)\,\mathrm{d}x + (-2\mu c_2^2 + 3c_3^2)\int\alpha_3(x)\,\mathrm{d}x\right]\right\} + c_4\,. \tag{36}$$

Case 3.

$$\begin{aligned}
&a_0(x)=0\,,\quad a_1(x)=c_1\,\mathrm{e}^{\int \Gamma(x)\,\mathrm{d}x}\,,\quad b_1(x)=c_1\sigma\sqrt{\frac{\mu}{\sigma}}\,\mathrm{e}^{\int \Gamma(x)\,\mathrm{d}x}\,,\quad p(x)=c_2\,,\\
&q(x)=\frac{1}{2}c_2\left[-4c_3\int\alpha_1(x)\,\mathrm{d}x+\left(\mu c_2^2-6c_3^2\right)\int\alpha_3(x)\,\mathrm{d}x\right]+c_4\,,\\
&h(x)=\frac{1}{2}\left(\mu c_2^2-2c_3^2\right)\int\alpha_1(x)\,\mathrm{d}x+\frac{1}{2}c_3\left(3\mu c_2^2-2c_3^2\right)\int\alpha_3(x)\,\mathrm{d}x+c_5\,,\quad f(x)=0\,,\quad g(x)=c_3\,,\quad \alpha_1(x)=\alpha_1(x)\,,\\
&\alpha_3(x)=\alpha_3(x)\,,\quad \alpha_4(x)=\alpha_4(x)\,,\quad \alpha_2(x)=-c_3\alpha_4(x)-\frac{c_2^2}{2c_1^2}\left[\alpha_1(x)+3c_3\alpha_3(x)\right]\mathrm{e}^{-2\int \Gamma(x)\,\mathrm{d}x}\,,\\
&\alpha_5(x)=\frac{-3}{2}\alpha_4(x)-\frac{3c_2^2}{4c_1^2}\alpha_3(x)\,\mathrm{e}^{-2\int \Gamma(x)\,\mathrm{d}x}\,,\quad \Gamma(x)=\Gamma(x)\,.
\end{aligned}\tag{37}$$

In this case, we have the exact solution

$$\begin{aligned}
u(\xi)=c_1&\left\{\mathrm{i}\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\frac{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)-B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)+B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}\right.\\
&\left.+\,\sigma\sqrt{\frac{\mu}{\sigma}}\sqrt{\sigma\left[1-\frac{|\mu|\,\mathrm{sgn}\,(\mu)}{\mu}\left(\frac{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)-B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}{A\exp\left(\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)+B\exp\left(-\mathrm{i}\xi\sqrt{|\mu|\,\mathrm{sgn}\,(\mu)}\right)}\right)^2\right]}\right\}\\
&\times\exp\left(\mathrm{i}\left(c_3t+\frac{1}{2}\left(\mu c_2^2-2c_3^2\right)\int\alpha_1(x)\,\mathrm{d}x+\frac{1}{2}c_3\left(3\mu c_2^2-2c_3^2\right)\int\alpha_3(x)\,\mathrm{d}x+c_5\right)+\int\Gamma(x)\,\mathrm{d}x\right),
\end{aligned}\tag{38}$$

where

$$\xi=c_2\left\{t-\frac{1}{2}\left[-4c_3\int\alpha_1(x)\,\mathrm{d}x+\left(\mu c_2^2-6c_3^2\right)\int\alpha_3(x)\,\mathrm{d}x\right]\right\}+c_4\,.\tag{39}$$

In particular, we deduce from (32) that the solitary wave solutions of Eq. (2) will be given as follows: If $B=0$, $A\neq 0$ and $\mu<0$, then we obtain

$$\begin{aligned}
u(\xi)&=c_1\sqrt{-\sigma}\,\mathrm{csch}\left(\sqrt{-\mu}\xi\right)\\
&\times\exp\left(\mathrm{i}\left(c_3t-\left(\mu c_2^2+c_3^2\right)\int\alpha_1(x)\,\mathrm{d}x-c_3\left(3\mu c_2^2+c_3^2\right)\int\alpha_3(x)\,\mathrm{d}x+c_5\right)+\int\Gamma(x)\,\mathrm{d}x\right),
\end{aligned}\tag{40}$$

while, if $B\neq 0$, $B^2>A^2$ and $\mu<0$, then we obtain

$$\begin{aligned}
u(\xi)&=c_1\sqrt{\sigma}\,\mathrm{sech}\left(\sqrt{-\mu}\xi+\xi_0\right)\\
&\times\exp\left(\mathrm{i}\left(c_3t-\left(\mu c_2^2+c_3^2\right)\int\alpha_1(x)\,\mathrm{d}x-c_3\left(3\mu c_2^2+c_3^2\right)\int\alpha_3(x)\,\mathrm{d}x+c_5\right)+\int\Gamma(x)\,\mathrm{d}x\right),
\end{aligned}\tag{41}$$

where $\xi_0=\tanh^{-1}\left(\frac{A}{B}\right)$. Similarly, we can find more solitary wave solutions of Eq. (2) using (35) and (38) but we omitted them for simplicity.

Remark 3: All solutions of this article have been checked with the *Maple* by putting them back into the original Eqs. (1) and (2).

4. Conclusions

In this article, the generalized extended $\left(\frac{G'}{G}\right)$-expansion method has been proposed to find out exact solutions of NLSEs. As applications of the proposed method, some new traveling wave solutions of the nonlinear Schrödinger equation with the gain and variable coefficients (1) and the higher-order nonlinear Schrödinger equation with variable coefficients (2) are successfully obtained. These solutions include hyperbolic function solutions and trigonometric function solutions. When the parameters are taken as special values, the solitary wave solutions are derived from the hyperbolic function solutions. This work shows that the generalized extended $\left(\frac{G'}{G}\right)$-expansion method is direct, effective and can be applied to many other NLSEs in mathematical physics.

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