Bayesian Value-at-Risk and Expected Shortfall for a Large Portfolio (Multi- and Univariate Approaches)

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Bayesian assessments of value-at-risk and expected shortfall for a given portfolio of dimension \( n \) can be based either on the \( n \)-variate predictive distribution of future returns of individual assets, or on the univariate model for portfolio volatility. In both cases, the Bayesian VaR and ES fully take into account parameter uncertainty and non-linear relationship between ordinary and logarithmic returns. We use the \( n \)-variate type I MSF-SBEKK(1,1) volatility model proposed specially to cope with large \( n \). We compare empirical results obtained using this (more demanding) multivariate approach and the much simpler univariate approach based on modelling volatility of the whole portfolio (of a given structure).

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1. Introduction

For investors and banks accurate evaluation of risk is very important. An underestimation of risk could throw them into bankruptcy. On the other hand, an overestimation of risk may have a negative effect on their profits.

Two risk measures are very popular: value-at-risk (VaR) and expected shortfall (ES).

The VaR measures the least portfolio loss that may happen with a given probability \( \alpha \) (usually 0.05 or smaller) over a certain time horizon (most often from 1 to 10 trading days). From mathematical point of view the VaR is a quantile of the Profit-and-Loss distribution of a given portfolio over a certain time horizon. A disadvantage of this measure is that it does not inform us about the potential size of loss that exceeds the VaR. Moreover, the VaR is not a coherent measure – it lacks the property of sub-additivity, i.e. the VaR of a portfolio may be greater than the sum of the VaR levels of the constituents of the portfolio [1]. Despite theoretical discussions, the VaR has become the standard measure of market risk used both by financial institutions and by their regulators; see [2]. According to the Capital Adequacy Directive by the Bank of International Settlement in Basel, banks are required to hold a certain amount of capital as a protection against (among others) adverse movements in the market values of assets. A bank should possess sufficient capital “to cover losses on the bank’s trading portfolio over a 10-day holding period in 99% of occasions”; see [3].

Expected Shortfall is defined as the conditional expectation of the loss given that it exceeds the VaR level; see [4]. Expected Shortfall is a sub-additive, coherent measure. The property of sub-additivity means that the ES for a portfolio is not greater than the sum of its sub-portfolios’ ES.

The VaR and ES are characteristics of the distribution of the future portfolio value (conditional on historical data on asset prices) and are closely related to its tails. More popular and traditional approaches to the VaR and ES assessment are based on parametric statistical models (with explicit assumptions on the conditional distribution of future returns) or methods related to the extreme value theory. Other approaches are based on non-parametric models, such as Conditional Autoregressive Expectiles (CARE, see [5]) and Conditional Autoregressive Value at Risk (CAVaR, see [2]). These approaches directly focus on the Expected Shortfall and on the \( \alpha \)-quantile modelled non-parametrically.

In this paper we discuss and compare VaR and ES assessments based on multi- and univariate parametric models. The multivariate approach is much more difficult, as it explicitly takes into account the full conditional covariance structure of asset prices: individual volatilities and correlations. Most of multivariate models in financial econometrics either belong to the Multivariate GARCH (MGARCH) or multivariate stochastic volatility (MSV) classes or are based on copulas; see [6, 7]. These models are difficult to estimate; only a few of them could be practical tools for large portfolios. A solution to the problem of simple, but parsimonious, multivariate volatility modelling is a hybrid model proposed by Osiewalski [8], see also [9]. This hybrid model is based on scalar BEKK (SBEKK) correlation structure and the simplest MSV specification, the multiplicative stochastic factor (MSF) model.

The VaR and ES require only the distribution of the future value of the portfolio; it can be derived using a univariate model for the historical values of the portfolio.

In our comparison we use the Bayesian MSF-SBEKK type I model for portfolios of dimension \( n = 50 \) and the univariate specification obtained from this model by taking \( n = 1 \). In the next section we discuss basic notions and introduce notation. Section 3 is devoted to our models proposed for the assessment of VaR and ES. Section 4 contains empirical results. Section 5 concludes.

2. Portfolio value-at-risk and expected shortfall - concepts and notation

Consider a portfolio kept at present time (\( T \)) and consisting of \( n \) assets; \( a_i \) denotes the number of units of
asset $i$ possessed now and $S_{t,i}$ is the price of asset $i$ at
time $t$ ($S_{t,i} > 0$, $a_i > 0$ for $i = 1, \ldots, n$, $t = 1, \ldots, T$),
thus $W_t = \sum_{i=1}^{n} a_i S_{t,i}$ is the time $t$ value of this
portfolio. The $s$-period return rate on the portfolio
is $R_{t,T+s}^* = (W_{t+s} - W_t)/W_t = \sum_{i=1}^{n} \omega_{t,i} R_{t,T+s,i}$, where
$R_{t,T+s,i} = (S_{t+s,i} - S_{t,i})/S_{t,i}$ is the $s$-period return rate
on asset $i$ and $\omega_{t,i} = a_i S_{t,i}/W_t$ is the share of asset $i$ in
the time $t$ portfolio value. For most results the condition
$a_i > 0$ is not required (short sale is allowed), only $W_t > 0$
has to be assumed. Note that the sum of $\omega_{t,i}$ over the
assets ($i = 1, \ldots, n$) is always 1 by construction.

Assume that we observe the $n$-variate time series of
individual return rates for $t = 1, \ldots, T$ and we are inter-
ested in forecasting $R_{t,T+s}^*$, the $s$-period ahead return
on the portfolio kept at time $T$. Forecasting $R_{t,T+s}^*$
is closely related to the definition of $\text{VaR}_{T,T+s}^L$ and
$\text{VaR}_{T,T+s}^S$, the $s$-period ahead Value-at-Risk (VaR) for,
respectively, long and short trading positions of the port-
folio. If $\Psi$ denotes the observations on asset prices up
to time $T$, then $\text{VaR}_{T,T+s}^L(\alpha)$ for a given probability level
$\alpha$ is defined by the following equality:

$$Pr \left[ R_{T,T+s}^* \leq \text{VaR}_{T,T+s}^L(\alpha) \right] = \alpha,$$

which can be written as

$$Pr \left[ -1 + \sum_{i=1}^{n} \omega_{t,i} \exp \left( \sum_{j=1}^{s} r_{T+j,i} \right) \leq \frac{- \text{VaR}_{T,T+s}^L(\alpha)}{W_T} \right] = \alpha,$$

for details see [11].

The $s$-period ahead Expected Shortfall (ES) for a long trading
position at a given probability level $\alpha$, denoted by
$\text{ES}_{T,T+s}^L(\alpha)$, is defined by

$$\text{ES}_{T,T+s}^L(\alpha) = -E \left[ W_{T+s} - W_T | W_{T+s} \leq W_T - \text{VaR}_{T,T+s}^L(\alpha), \Psi_T \right],$$

and for a short trading position:

$$\text{ES}_{T,T+s}^S(\alpha) = E \left[ W_{T+s} - W_T | W_{T+s} \geq W_T + \text{VaR}_{T,T+s}^S(\alpha), \Psi_T \right].$$

Using the logarithmic return rates we can rewrite (7) and (8) as:

$$\text{ES}_{T,T+s}^L(\alpha) = -E \left[ W_T \sum_{i=1}^{n} \omega_{t,i} \exp \left( \sum_{j=1}^{s} r_{T+j,i} \right) - W_T | W_{T+s} \leq W_T - \text{VaR}_{T,T+s}^L(\alpha), \Psi_T \right],$$

$$\text{ES}_{T,T+s}^S(\alpha) = E \left[ W_T \sum_{i=1}^{n} \omega_{t,i} \exp \left( \sum_{j=1}^{s} r_{T+j,i} \right) - W_T | W_{T+s} \geq W_T + \text{VaR}_{T,T+s}^S(\alpha), \Psi_T \right],$$

The VaR and ES are related to quantiles of some non-
linear function of future logarithmic returns*. Condition-

* The usual linear approximation $\exp \left( \sum_{j=1}^{s} r_{t+j,i} \right) \approx 1 + \sum_{j=1}^{s} r_{t+j,i}$
can lead to serious errors, especially when $s$ is so large that the
$s$-period ahead return distribution is diffuse, see [11].

In equalities (2) and (4) we have the ordinary return
rate $R_{T,T+s}^*$. In practice, the logarithmic return rates
$r_{t+1,i} = \ln (S_{t+1,i}/S_{t,i}) = \ln (R_{t+1,i} + 1)$ are the quanti-
ties being modelled. They can take any real value, easily
aggregate over time, and modelling them can be more
justified in view of the data; see [10]. Using the logarith-
mic return rates we can rewrite (2) and (4) as:

$$Pr \left[ R_{T,T+s}^* \leq \text{VaR}_{T,T+s}^L(\alpha) \right] = \alpha.$$
approach is advocated for determining the s-period ahead VaR and ES. Foundations of Bayesian VaR assessment are presented in [11]. The same approach is used here for the ES assessment. Similarly as in [11], we consider two modelling strategies for the portfolio VaR and ES. The first one amounts to assuming some n-variate model for individual logarithmic returns \( r_{t,i} \) and obtaining the \( \alpha \)- and \( (1-\alpha) \)-quantile of the predictive distribution of

\[
R_{T:T+s} = -1 + \sum_{i=1}^{\omega} r_{T,i} \exp \left( \sum_{j=1}^{\omega} r_{T+j,i} \right),
\]

a nonlinear function of future returns. The second approach amounts to directly modelling univariate series of portfolio logarithmic returns to directly modelling univariate series of portfolio logarithmic returns and examining the predictive distribution of \( r^W_{T+s} = \ln (W_{T+s}/W_T) \). The two approaches (n- and univariate) are not formally coherent and their comparison is an empirical task, undertaken in this paper. The question is whether a univariate specification from a flexible parametric family can explain and predict portfolio returns better than n-variate models that require huge simplifications in order to cope with large n. In the case of VaR assessments for a long trading position, the results presented by Osiewalski and Pajor [11] are not conclusive, although reasonable performance of our n-variate hybrid model is somewhat surprising. In this paper we also consider VaR assessments for a short trading position and the ES.

### 3. The hybrid VAR(1)-MSF-SBEKK type I Bayesian model

First we consider a multivariate specification for individual assets. Let \( r_t = [r_{t,1}, \ldots, r_{t,n}] \) denote n-variate observations on logarithmic return rates, which we model using the basic VAR(1) framework:

\[
r_t = d_0 + r_{t-1} \Phi + \eta_t, \quad t = 1, \ldots, T + s.
\]  

(11)

The \( n(n+1) \) elements of the row vector \( d_0 = [d_0 \ (\text{vec} \Phi)'] \) are common parameters, which can be treated as a priori independent of all other (model-specific) parameters; we can assume for them some multivariate prior, e.g. standard Normal \( N (0, I_{n(n+1)}) \), truncated by the restriction that all eigenvalues of \( \Phi \) lie inside the unit circle.

Following [9], we specify the conditional distribution of the residual process \( \eta_t \) by conditioning on its past \( (\Phi_{t-1}) \), some univariate latent process \( (g_t) \) and the parameters. We assume the so-called type I hybrid specification:

\[
\eta_t = \phi \ln g_{t-1} + \sigma^2 \eta_t, \quad [\phi, \sigma^2] \overset{iid}{\sim} N (0_{(1,1)}), I_{n+1}) .
\]

(12)

\[
\text{H}_t = (1 - \beta - \gamma) A + \beta \text{vec} (\eta_{t-1}) + \gamma \text{H}_{t-1}.
\]

(13)

That is, \( \eta_t \) and \( \text{H}_t \) are conditionally Normal with mean vector \( 0 \) and covariance matrix \( g_t \text{H}_t \), where \( g_t \) is a latent process and \( \text{H}_t \) is a square matrix of order \( n \) that has the scalar BEKK(1,1) structure. Thus, the conditional distribution of \( r_t \) (given its past and latent variables) is Normal with mean \( m_t = d_0 + r_{t-1} \Phi \) and covariance matrix \( g_t \text{H}_t \).

The presence of the latent AR(1) process in the conditional covariance matrix helps in explaining outlying observations, and the dependence on the past data (through the SBEKK structure of \( \text{H}_t \)) prevents the entries of the conditional covariance matrix \( g_t \text{H}_t \) from sharing the same dynamic pattern. Thus the model has time-varying conditional correlations without introducing more latent processes. In fact, the hybrid model defined by (12)-(14) nests two simple basic structures. In the limiting case when \( \sigma^2 \to 0 \) and \( \phi = 0 \) we are in the SBEKK model, while \( \beta = 0 \) and \( \gamma = 0 \) lead to the MSF case.

In (14) \( A \) is a free symmetric positive definite matrix of order \( n \), with an inverted Wishart prior distribution (in fact, for \( A^{-1} \) we assume the Wishart prior with \( n \) degrees of freedom and mean \( I_n \)), and \( \beta \) and \( \gamma \) are free scalar parameters, jointly uniformly distributed over the unit simplex. As regards initial conditions for \( \text{H}_t \), we take \( H_0 = h_0 I_n \) and treat \( h_0 \geq 0 \) as an additional parameter, a priori Exponentially distributed with mean 1. For the parameters of the latent process we use the same priors as in [13]: for \( \phi \): Normal with mean 0 and variance 100, truncated to \((-1,1) \), for \( \sigma^2 \): Exponential with mean 200: \( g_0 \) is fixed (equals 1).

In order to obtain the required quantiles of the predictive distribution of future logarithmic returns, we follow the approximation explained in [9]. That is, we use OLS for the VAR(1) parameters and replace \( A \) by the empirical covariance matrix of the OLS residuals from the VAR(1) part. The Bayesian analysis for the remaining parameters and future return rates is then based on the conditional posterior and predictive distributions given the particular values of the highly dimensional parameters \( (d \) and \( A \). These conditional distributions are sampled using the Gibbs scheme with Metropolis-Hastings steps, as shown in detail in [9].

In order to make the univariate model of portfolio value comparable to the n-variate volatility model of individual assets, we consider for the portfolio logarithmic returns the univariate AR(1) specification with the error term described by the hybrid SV-GARCH(1,1) process, which is the \( n = 1 \) special case of the MSF-SBEKK(1,1) structure. So we assume

\[
r_t = \delta_0 + \delta r_{t-1} + \varepsilon_t,
\]

(15)

\[
\varepsilon_t = \zeta_t \sqrt{g_t} h_t,
\]

(16)

\[
\ln g_t = \phi \ln g_{t-1} + \sigma^2 y_t, \quad [\phi, \sigma^2] \sim iidN (0_{(2,1)}), I_2)
\]

(17)

\[
H_t = (1 - \beta - \gamma) A + \beta (\varepsilon_{t-1} \varepsilon_{t-1}) + \gamma H_{t-1}, \quad t = 1, \ldots, T, \ldots, T + s.
\]

(18)

We take the prior distribution corresponding to the previous (n-variate) case (now with \( n = 1 \)). Now we do not face the dimensionality problem, but for comparison with the n-variate model, the posterior and predictive distribution is sampled (using the Gibbs scheme with Metropolis-Hastings steps) conditionally on preliminary non-Bayesian (OLS) estimates of \( \delta_0, \delta, \hat{a} \) as in the n-variate case.

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4. Empirical results

As the dataset we use the same stock data as in [11]; they represent 50 companies and cover the period May 13, 2005 – February 23, 2010. In February or March 2010 these companies were included in the two important indices of the Warsaw Stock Exchange, namely WIG20 and mWIG40. We start with the “tick” loss if $\gamma = \frac{p - 1}{t}$; it is applied directly to the series $D_{t-2t-1}$ of daily value changes $D_{t-2t-1} = W_{t-s} - W_{t}$ (not to the logarithmic returns); thus, $q_t(\alpha)$ denotes the conditional $\alpha$-quantile of $D_{t-1:t}$, $I_{(-\infty,0)}(\cdot)$ is the characteristic function of the interval $(-\infty,0)$. We also use the CARE model with asymmetric slope:

$$
\hat{\gamma} = \arg\min_{\gamma} \sum_{t=1}^{T} \left| \tau - I_{(-\infty,0)}(D_{t-1:t} - \mu(\tau)) \right| (D_{t-1:t} - \mu(\tau))^2 |\Psi_t| .
$$

Using conditional expectiles we estimate the Expected Shortfall by:

$$
ES_{t:t+1}(\alpha) = - \left[ 1 + \frac{\tau}{(1 - 2\tau)\alpha} \right] \mu_t(\alpha) + \left[ \frac{\tau}{(1 - 2\tau)\alpha} \right] E(D_{t-1:t}|\Psi_t),
$$

see [5]. Note that in the case of $E(D_{t-1:t}|\Psi_t) = 0$, inserting $\mu_t(\alpha)$ from (20) into (23) gives the following equality:

$$
ES_{t:t+1}(\alpha) = \lambda_0 + \lambda_1 \alpha ES_{t-1:t}(\alpha) + \lambda_2 |D_{t-2:t-1}| + \lambda_3 |D_{t-2:t-1}| I_{(-\infty,0)}(D_{t-2:t-1}),
$$

where $\lambda_1 = \gamma_1$, $\lambda_2 = - \left[ 1 + \frac{\tau}{(1 - 2\tau)\alpha} \right] \gamma_1$, $i = 0, 2, 3$.

To obtain ES for a short trading position we apply (23) to the series of daily value changes multiplied by minus one (i.e. $-D_{t:t+1}$), for details see [5]).

The losses are generally calculated as $L_s = \frac{1}{p} \sum_{t=T}^{T+p-1} l_t^{i,t+s}$, where for $l_t^{i,t+s}$ and $j \in \{L, S\}$ we have

- the “tick” loss if

$$
l_t^{i,t+s} = \begin{cases} 
(\alpha - 1) [D_{t:t+s} + VaR_{t:t+s}(\alpha)], & \text{if } D_{t:t+s} < -VaR_{t:t+s}(\alpha), \\
\alpha [D_{t:t+s} + VaR_{t:t+s}(\alpha)], & \text{if } D_{t:t+s} \geq -VaR_{t:t+s}(\alpha);
\end{cases}
$$

- the Lopez loss if

$$
l_t^{i,t+s} = \begin{cases} 
1 + [D_{t:t+s} + VaR_{t:t+s}(\alpha)]^2, & \text{if } D_{t:t+s} < -VaR_{t:t+s}(\alpha), \\
0, & \text{if } D_{t:t+s} \geq -VaR_{t:t+s}(\alpha);
\end{cases}
$$

We consider two portfolios: one consisting of one unit of each asset, i.e. $a = [a_1, a_2, \ldots, a_n] = [1, \ldots, 1]$ and the other of $a_{\tau,i} = \frac{1}{\sum_{j=1}^{n} a_{\tau,j}}$ units of asset $i$, that is $\omega_{\tau,i} = 1/50$, where $i = 1, \ldots, 50$, and $\tau$ represents May 12, 2009.

In order to compare one period ahead VaR and ES obtained in two different ways, i.e. using $n$-variate MSF-SBEKK model for individual assets or its univariate counterpart for the portfolio value, we use popular non-Bayesian criteria. They include the failure rate and $p$-value for the Kupec test as well as the “tick” and the Lopez loss functions (defined below); see Tables I-IV. For the sake of comparison we also use some non-parametric models, specially designed for direct VaR or ES assessment. In particular, we apply the CAViaR model with asymmetric slope:
The mean error. Another measure of predictive out-of-sample performance for ES is the mean absolute error: \[ \text{MAE}_t^S = \begin{cases} 1 + [D_{t:t+s} - \text{VaR}_{t:t+s}^S(\alpha)]^2, & \text{if } D_{t:t+s} > \text{VaR}_{t:t+s}^S(\alpha), \\ 0, & \text{if } D_{t:t+s} \leq \text{VaR}_{t:t+s}^S(\alpha); \end{cases} \]

see e.g. [14–16].

For backtesting the forecasted Expected Shortfall we use different measures proposed by Zhu and Galbraith [17], Kaufmann and Patie [18], and Embrechts, Kaufmann and Patie [19]. We compute the mean error:

\[ ME_j^S(\alpha) = \text{ES}_S^{A:j}(\alpha) - \text{AL}_j^L(\alpha) \]  

where \( j \in \{L,S\} \), \( \text{ES}_S^{A:j}(\alpha) \) is the average predictive ES:

\[ \text{ES}_S^{A:L}(\alpha) = \frac{1}{J_L} \sum_{t=T}^{T+p-1} I_{(-\infty,0)} [D_{t:t+s} + \text{VaR}_{t:t+s}^L(\alpha)] \text{ES}_{t:t+s}^L(\alpha), \]

\[ \text{ES}_S^{A:S}(\alpha) = \frac{1}{J_S} \sum_{t=T}^{T+p-1} I_{(-\infty,0)} [D_{t:t+s} - \text{VaR}_{t:t+s}^S(\alpha)] \text{ES}_{t:t+s}^S(\alpha), \]

\( \text{AL}_j^L(\alpha) \) is the average loss on the portfolio when the loss is larger than \( \text{VaR}_{t:t+s}^L(\alpha) \):

\[ \text{AL}_L^L(\alpha) = \frac{1}{J_L} \sum_{t=T}^{T+p-1} I_{(-\infty,0)} [D_{t:t+s} + \text{VaR}_{t:t+s}^L(\alpha)] |D_{t:t+s}|, \]

\[ \text{AL}_L^S(\alpha) = \frac{1}{J_S} \sum_{t=T}^{T+p-1} I_{(-\infty,0)} [D_{t:t+s} - \text{VaR}_{t:t+s}^S(\alpha)] |D_{t:t+s}|, \]

\[ J_L^L = \sum_{t=T}^{T+p-1} I_{(-\infty,0)} [D_{t:t+s} + \text{VaR}_{t:t+s}^L(\alpha)], \]

\[ J_S^L = \sum_{t=T}^{T+p-1} I_{(-\infty,0)} [D_{t:t+s} - \text{VaR}_{t:t+s}^S(\alpha)], \]

As Zhu and Galbraith [17] have noticed, if the average predictive ES is lower (higher) than \( \text{AL}_L^L \) (“observed ES”), then the model tends to underestimate (overestimate) risk. A good forecast of ES will lead to a low absolute value of the mean error. Another measure of predictive out-of-sample performance for ES is the mean absolute error:

\[ \text{MAE}_L^L(\alpha) = \frac{1}{J_L} \sum_{t=T}^{T+p-1} I_{(-\infty,0)} [D_{t:t+s} + \text{VaR}_{t:t+s}^L(\alpha)] |\text{AL}_L^L(\alpha) - \text{ES}_{t:t+s}^L(\alpha)|, \]

for a long trading position;

\[ \text{MAE}_S^L(\alpha) = \frac{1}{J_S} \sum_{t=T}^{T+p-1} I_{(-\infty,0)} [D_{t:t+s} - \text{VaR}_{t:t+s}^S(\alpha)] |\text{AL}_L^S(\alpha) - \text{ES}_{t:t+s}^S(\alpha)|, \]

for a short trading position.

Also, the mean absolute percentage error can be considered:

\[ \text{MAPE}_L^L(\alpha) = \frac{1}{J_L} \sum_{t=T}^{T+p-1} I_{(-\infty,0)} [D_{t:t+s} + \text{VaR}_{t:t+s}^L(\alpha)] \frac{|\text{AL}_L^L(\alpha) - \text{ES}_{t:t+s}^L(\alpha)|}{|\text{AL}_L^L(\alpha)|}, \]

for a long trading position;

\[ \text{MAPE}_S^L(\alpha) = \frac{1}{J_S} \sum_{t=T}^{T+p-1} I_{(-\infty,0)} [D_{t:t+s} - \text{VaR}_{t:t+s}^S(\alpha)] \frac{|\text{AL}_L^S(\alpha) - \text{ES}_{t:t+s}^S(\alpha)|}{|\text{AL}_L^S(\alpha)|}, \]

for a short trading position.

We also compute the backtesting measures proposed by Kaufmann and Patie [18]:

\[ V_1^{ES,L}(\alpha) = \frac{1}{J_L} \sum_{t=T}^{T+p-1} I_{(-\infty,0)} [D_{t:t+s} + \text{VaR}_{t:t+s}^L(\alpha)] [D_{t:t+1} + \text{ES}_{t:t+1}^L(\alpha)], \]

\[ V_1^{ES,S}(\alpha) = \frac{1}{J_S} \sum_{t=T}^{T+p-1} I_{(-\infty,0)} [D_{t:t+s} - \text{VaR}_{t:t+s}^S(\alpha)] [D_{t:t+1} - \text{ES}_{t:t+1}^S(\alpha)], \]
The particular criterion (see Table I-X). For example, case is in bold). Which model is better depends on VaR assessments, because it considers only values of \( D_{t,t+1} \) which exceed this threshold.

### 4.1. 1-day risk measures assessments

The outcomes of the Kupiec test for one period ahead VaR seem to indicate that our Bayesian assessment (based on the parametric MSF-SBEKK structure) competes with the one based on CAViaR (Table II; the best

- \( \alpha \): Value of the parameter
- \( \omega \): Long trading position
- \( \tau \): Short trading position

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>PF1</th>
<th>PF2</th>
</tr>
</thead>
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<td>AA</td>
<td>BB</td>
<td>CC</td>
</tr>
<tr>
<td>long trading position</td>
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<tr>
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<td></td>
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<tr>
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</tr>
<tr>
<td>0.1</td>
<td>1.820</td>
<td>1.891</td>
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MSF-SBEKK approach is helpful for portfolio risk assessment. The failure rates show that, for the portfolio with one unit of each asset, VaR tends to be underestimated in the univariate models (see Table I). In other words, the chance of a daily change in portfolio value being below \( -\text{VaR}_{t+1}(\alpha) \) (or above \( \text{VaR}_{t+1}(\alpha) \)), which should be \( \alpha \), is in fact greater.

Interestingly, the univariate SV-GARCH specification leads to VaR assessments that are highly correlated with the ones based on CAViaR (see Table V and Fig. 1).
Lopez loss function for VaR $L_i^{t+1}(\alpha)$ and VaR $S_i^{t+1}(\alpha)$:

<table>
<thead>
<tr>
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<td>0.01</td>
<td>63.316</td>
<td>68.848</td>
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<td>0.025</td>
<td>112.060</td>
<td>133.847</td>
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<tr>
<td>0.05</td>
<td>187.108</td>
<td>227.417</td>
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<tr>
<td>0.1</td>
<td>348.888</td>
<td>416.769</td>
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<table>
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<tr>
<th>$\alpha$</th>
<th>PF1</th>
<th>PF2</th>
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<tbody>
<tr>
<td>AA</td>
<td>BB</td>
<td>CC</td>
</tr>
<tr>
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<td>63.316</td>
<td>68.848</td>
</tr>
<tr>
<td>0.025</td>
<td>112.060</td>
<td>133.847</td>
</tr>
<tr>
<td>0.05</td>
<td>187.108</td>
<td>227.417</td>
</tr>
<tr>
<td>0.1</td>
<td>348.888</td>
<td>416.769</td>
</tr>
</tbody>
</table>

Correlation coefficients between $VaR_i^{t+1}(\alpha)$ for different methods of assessment, $\omega_{\tau,i} = 1/50$:

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<td>0.01</td>
<td>0.362</td>
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<tr>
<td>0.025</td>
<td>0.355</td>
<td>0.276</td>
<td>0.827</td>
<td>0.355</td>
<td>0.276</td>
<td>0.827</td>
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<tr>
<td>0.05</td>
<td>0.372</td>
<td>0.247</td>
<td>0.857</td>
<td>0.372</td>
<td>0.247</td>
<td>0.857</td>
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<tr>
<td>0.1</td>
<td>0.402</td>
<td>0.247</td>
<td>0.765</td>
<td>0.402</td>
<td>0.247</td>
<td>0.765</td>
</tr>
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</table>

Mean errors for $ES_i^{t+1}(\alpha)$ and $ES_i^{S,t+1}(\alpha)$:

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<td>0.01</td>
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<td>0.1</td>
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Mean absolute errors for $ES_i^{t+1}(\alpha)$ and $ES_i^{S,t+1}(\alpha)$:

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<td>BB</td>
<td>DD</td>
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<td>0.01</td>
<td>52.327</td>
<td>39.193</td>
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<tr>
<td>0.025</td>
<td>25.764</td>
<td>27.788</td>
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<tr>
<td>0.1</td>
<td>30.316</td>
<td>23.212</td>
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</table>

Mean absolute percentage errors for $ES_i^{t+1}(\alpha)$ and $ES_i^{S,t+1}(\alpha)$:

<table>
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<th>PF2</th>
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<tbody>
<tr>
<td>AA</td>
<td>BB</td>
<td>DD</td>
</tr>
<tr>
<td>0.01</td>
<td>0.245</td>
<td>0.196</td>
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<tr>
<td>0.025</td>
<td>0.145</td>
<td>0.152</td>
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<tr>
<td>0.05</td>
<td>0.270</td>
<td>0.212</td>
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<tr>
<td>0.1</td>
<td>0.276</td>
<td>0.210</td>
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</table>

For further DD denotes CAViaR CARE.

![Fig. 1. VaR](#)
The aim of the paper was to compare the \( n \)-variate and univariate approaches to risk assessment for a large portfolio. It depends on the particular criterion which approach is better. It seems that, for one-day VaR and ES assessments, univariate modelling (of portfolio value instead of portfolio components) is often sufficient. For \( s \)-day VAR estimates (where \( s = 2, 3, \ldots, 10 \)) the \( n \)-variate MSF-BEKK model turned out to be better (in terms of the Lopez loss function), but mean absolute errors for ES indicate that the univariate approaches are enough (especially for short trading positions).

Our empirical study shows that the simple hybrid SV-GARCH(1,1) model, which is the univariate version of the MSF-BEKK(1,1) model, behaves well and successfully competes with non-parametric specifications (CAViaR and CARE). Thus, our univariate hybrid model appears an interesting all-purpose alternative to different non-parametric models designed to focus on specific aspects of future returns (and not on their full predictive distribution).

Finally, the paper indicates that the Bayesian approach to VaR and ES analysis is fully relevant and practical. Remind that conditioning on observed data and inference on non-linear functions of unobserved quantities (future logarithmic returns) are necessary for any appropriate risk analysis. Both are natural and easy within Bayesian statistics, equipped with the Markov Chain Monte Carlo (MCMC) simulation tools.

### References

TABLE XI

$$\text{Var}_t^{1,t+s}(0.05)$$ and $$\text{ES}_t^{1,t+s}(0.05)$$ results based on the univariate SV-GARCH model, $$\omega_{r,t} = 1/50$$

<table>
<thead>
<tr>
<th>s</th>
<th>1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
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<td>FR</td>
<td>p-value for the Kupiec test</td>
<td>FR</td>
<td>p-value for the Kupiec test</td>
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<td>146.97</td>
<td>385.70</td>
<td>147.67</td>
<td>52.09</td>
<td>133.13</td>
<td>25.03</td>
<td>6.83</td>
<td>3.20</td>
<td>2.19</td>
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<tr>
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<td>19.452</td>
<td>45.275</td>
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<td>77.449</td>
<td>47.298</td>
<td>82.062</td>
<td>89.645</td>
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<td>21.522</td>
<td>37.278</td>
<td>29.045</td>
<td>34.402</td>
<td>37.639</td>
<td>39.669</td>
<td>59.760</td>
<td>88.012</td>
<td>73.866</td>
<td>87.931</td>
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<tr>
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<td>0.220</td>
<td>0.223</td>
<td>0.208</td>
<td>0.227</td>
<td>0.197</td>
<td>0.159</td>
<td>0.276</td>
<td>0.185</td>
<td>0.294</td>
<td>0.279</td>
</tr>
</tbody>
</table>

TABLE XII

$$\text{Var}_t^{1,t+s}(0.05)$$ and $$\text{ES}_t^{1,t+s}(0.05)$$ results based on the $$\eta$$-variate MSF-SBEKK model, $$\omega_{r,t} = 1/50$$

<table>
<thead>
<tr>
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<th>1</th>
<th>2</th>
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<td>58.107</td>
<td>88.012</td>
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<td>87.931</td>
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<td>0.182</td>
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<td>0.121</td>
<td>0.116</td>
<td>0.120</td>
<td>0.114</td>
<td>0.170</td>
<td>0.246</td>
<td>0.183</td>
<td>0.202</td>
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