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The Competing Interactions on a Cayley Tree-Like Lattice: Pentagonal Chandelier

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Different types of lattice spin systems with competing interactions have rich and interesting phase diagrams. In this study we present some new results for such systems involving the Ising spin system (i.e. $\sigma = \pm 1$) using a generalization of the Cayley tree-like lattice approximation. We study the phase diagrams for the Ising model on a Cayley tree-like lattice, a new lattice type called pentagonal chandelier, with competing nearest--neighbor interactions J_1, prolonged next-nearest-neighbor interactions J_p and one-level next-nearest-neighbor senary interactions $J_{l_1}^{(6)}$. The colored phase diagrams contain some multicritical Lifshitz points that are at nonzero temperature and many modulated new phases. We also investigate the variation of the wave vector with temperature in the modulated phase and the Lyapunov exponent associated with the trajectory of the system.

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1. Introduction

The Ising model was originally devised to study the ferromagnetic ordering of the magnetic moments in some solids in statistical mechanics. The model consists of discrete variables called spins that can be in one of two states. Recently, the Ising model on a Cayley tree with different competing interactions has been studied extensively because of the appearance of nontrivial magnetic orderings and some important applications such as physical, chemical and biological systems, and even in sociology [1–4] (see references in [1]). More complicated models are studied on tree-like lattices, with the hope to discover new phases or unusual types of behaviors. The important point is that statistical mechanics on trees involve nonlinear recursion equations and are naturally connected to the rich world of dynamical systems, a world presently under intense investigation (see references in [1]). Moreover, the Ising model has found some applications physical, chemical and biological systems, and even in sociology. The Cayley tree is not a realistic lattice; however, its amazing topology makes the exact calculation of various quantities possible. For many problems the solution on a tree is much simpler than on a regular lattice and is equivalent to the standard Bethe–Peierls theory [5]. In the literature, there have been working many different lattice types similar to the Cayley trees [6, 7]. In this paper, we produce a Cayley tree-like lattice [5, 6] which we called as a *pentagonal chandelier* from the configuration model given in Figs. 1, 2. In recent years, investigation of phase diagrams of the Ising model has attracted increased attention.

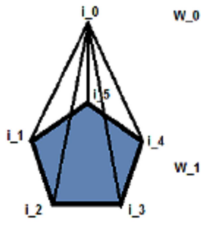

Fig. 1. The first-generation branch of the semi-infinite Cayley tree-like lattice: a pentagonal chandelier lattice with $k = 5$. The spin in the root x_0 that called the 0th level is i_0. W_0 and W_1 of a pentagonal chandelier lattice consist of $\{i_0\}$ and $\{i_1, i_2, i_3, i_4, i_5\}$, respectively.

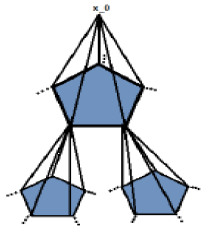

Fig. 2. The first and second-generation branch of a pentagonal chandelier lattice.

The aim of this paper is to clarify the role of order $k = 5$ of the Cayley tree-like lattice which we called pentagonal chandelier as studied before the order $k = 3$ and $k = 4$ [8–10]. In this paper, we study the phase diagrams for the Ising model on a Cayley tree-like lattice [8–10], a new lattice type called pentagonal chan-

delier, with competing nearest-neighbor interactions J_1, prolonged next-nearest-neighbor interactions J_p and one--level next-nearest-neighbor senary interactions $J_{l_1}^{(6)}$. The diagrams contain some multicritical Lifshitz points that are at nonzero temperature and many modulated new phases. We also plot the variation of the wave vector with temperature in the modulated phase and the Lyapunov exponent associated with the trajectory of the system.

2. Technical preliminaries and the model

In this section we present basic concepts of the Cayley tree and our notations for the Hamiltonian model.

2.1. The lattice spin systems: Cayley tree and Cayley tree-like lattice

A Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles with exactly $k+1$ edges issuing from each vertex. Let denote the Cayley tree as $\Gamma^k = (V, \Lambda)$, where V is the set of vertices of Γ^k, Λ is the set of edges of Γ^k. Two vertices x and y, $x, y \in V$ are called *nearest-neighbors* if there exists an edge $l \in \Lambda$ connecting them, which is denoted by $l = \langle x, y\rangle$. The distance $d(x, y)$, $x, y \in V$, on the Cayley tree Γ^k, is the number of edges in the shortest path from x to y. For a fixed $x^0 \in V$ we set $W_n = \{x \in V | d(x, x^0) = n\}$, $V_n = \{x \in V | d(x, x^0) \leq n\}$ and L_n denotes the set of edges in V_n. The fixed vertex x^0 is called the 0-th level and the vertices in W_n are called the n-th level. For the sake of simplicity we put $|x| = d(x, x^0)$, $x \in V$. Two vertices $x, y \in V$ are called *the next-nearest--neighbors* if $d(x, y) = 2$. The next-nearest-neighbor vertices x and y are called *prolonged next-nearest-neighbors* if $|x| \neq |y|$ and is denoted by $\rangle \overset{\square}{x}, y\langle$. We will consider a semi-infinite Cayley tree Γ_+^5 of 5th order, i.e. an infinite graph without cycles with 6 edges issuing from each vertex except for x^0 which has only 5 edges. In this work we use the same concepts and definitions for the Cayley tree-like lattice as in the Cayley tree.

2.2. The Hamiltonian model

For the Ising model with spin values in $\Phi = \{-1, 1\}$, the relevant Hamiltonian with the coupling constants $J_1, J_p, J_{l_1}^{(6)} \in R$ are

$$H(\sigma) = -J_1 \sum_{\langle x_1, x_2\rangle} \sigma(x_1)\sigma(x_2) - J_p \sum_{\rangle \overset{\square}{x_1}, x_2\langle} \sigma(x_1)\sigma(x_2) - J_{l_1}^{(6)} - \sum_{\rangle x_1, x_2, \ldots, x_6\langle} \sigma(x_1)\sigma(x_2)\ldots\sigma(x_6). \quad (1)$$

The case $J_p = J_{l_1}^{(6)} = 0$ was considered in Refs. [11–13]. In the case of $J_p = J_{l_1}^{(6)} = 0$, the Ising model (1) is exactly solvable [14] and its phase diagram consists of ferromagnetic and antiferromagnetic phases only. In the presence of J_p with $J_{l_1}^{(6)} = 0$ for $k = 2$ this model was considered by Vannimenus [1]. He proved that phase diagram contains new modulated phase with the expected paramagnetic and ferromagnetic ones. The case $J_{l_1}^{(6)} = 0$ on the Cayley tree of order 2 was considered in [2]. Lately the model (1) on a Cayley tree of arbitrary order k with the case $J_{l_1}^{(6)} = 0$ was studied in [15] as a Vannimenus extension result. Also the quadrable interaction case $J_{l_1}^{(4)} \neq 0$ for $k = 3$ was newly investigated on triangular chandelier in [8].

3. The recursion relations

In order to produce the recurrent equations, we consider the relation of the partition function on V_n to the partition function on subsets of V_{n-1}. Given the initial conditions on V_1, the recurrence equations indicate how their influence propagates down the tree.

Let $Z^{(n)}\begin{pmatrix} i_1, i_2, i_3, i_4, i_5 \\ i_0 \end{pmatrix}$ be the partition function on V_n where the spin in the root x^0 is i_0 and the 5 spins in the proceeding ones are $i_1, i_2, \ldots, i_5$. There are a priori 2^6 different $Z^{(n)}$ to consider. One can show that there are only four independent variables, namely

$$\begin{aligned} z_1 &= Z^{(n)}\begin{pmatrix} +,+,+,+,+ \\ + \end{pmatrix}, \\ z_2 &= Z^{(n)}\begin{pmatrix} -,-,-,-,- \\ + \end{pmatrix}, \\ z_3 &= Z^{(n)}\begin{pmatrix} +,+,+,+,+ \\ - \end{pmatrix}, \\ z_4 &= Z^{(n)}\begin{pmatrix} -,-,-,-,- \\ - \end{pmatrix}. \end{aligned} \quad (2)$$

Then arbitrary

$$Z^{(n)}\begin{pmatrix} i_1, i_2, i_3, i_4, i_5 \\ i_0 \end{pmatrix}$$

is a combination of z_1, z_2, z_3, z_4. Through the introduction of the new variables $u_i = \sqrt[5]{z_i}$, we produce the following recurrence system:

$$\begin{aligned} u_1' &= a\left[\sum_{r=0}^{5}\binom{5}{r} b^{5-2r} c^{(-1)^r} u_1^{5-r} u_2^r\right], \\ u_2' &= a^{-1}\left[\sum_{r=0}^{5}\binom{5}{r} b^{5-2r} c^{(-1)^{r+1}} u_3^{5-r} u_4^r\right], \\ u_3' &= a^{-1}\left[\sum_{r=0}^{5}\binom{5}{r} b^{-5+2r} c^{(-1)^r} u_1^{5-r} u_2^r\right], \\ u_4' &= a\left[\sum_{r=0}^{5}\binom{5}{r} b^{-5+2r} c^{(-1)^{r+1}} u_3^{5-r} u_4^r\right]. \end{aligned} \quad (3)$$

We obtain that the total partition function $Z^{(n)}$ is given in terms of (u_i) by

$$Z^{(n)} = (u_1 + u_2)^5 + (u_3 + u_4)^5. \quad (4)$$

For discussing of the phase diagrams in the Hamiltonian

three-parameter spaces, the following choice of reduced variables is convenient:

$$x = \frac{u_2 + u_3}{u_1 + u_4}, \quad y_1 = \frac{u_1 - u_4}{u_1 + u_4}, \quad y_2 = \frac{u_2 - u_3}{u_1 + u_4}. \tag{5}$$

The variable x is just a measure of the frustration of the nearest-neighbor bonds and is not an order parameter like y_1, y_2. Then the relations now have the following form:

$$x' = \frac{1}{a^2 D}\Bigg[\sum_{r=0}^{5}\binom{5}{r}\Big(b^{5-2r}c^{(-1)^{r+1}}(x-y_2)^{5-r}(1-y_1)^r + b^{-5+2r}c^{(-1)^r}(1+y_1)^{5-r}(x+y_2)^r\Big)\Bigg],$$

$$y_1' = \frac{1}{D}\Bigg[\sum_{r=0}^{5}\binom{5}{r}\Big(b^{5-2r}c^{(-1)^r}(1+y_1)^{5-r}(x+y_2)^r - b^{-5+2r}c^{(-1)^{r+1}}(x-y_2)^{5-r}(1-y_1)^r\Big)\Bigg],$$

$$y_2' = \frac{1}{a^2 D}\Bigg[\sum_{r=0}^{5}\binom{5}{r}\Big(b^{5-2r}c^{(-1)^{r+1}}(x-y_2)^{5-r}(1-y_1)^r - b^{-5+2r}c^{(-1)^r}(1+y_1)^{5-r}(x+y_2)^r\Big)\Bigg] \tag{6}$$

and

$$D(x, y_1, y_2) = \sum_{r=0}^{5}\binom{5}{r}\Big(b^{5-2r}c^{(-1)^r}(1+y_1)^{5-r}(x+y_2)^r + b^{-5+2r}c^{(-1)^{r+1}}(x-y_2)^{5-r}(1-y_1)^r\Big) \tag{7}$$

where

$$a = \exp(J_1/T), \quad b = \exp(J_\mathrm{p}/T), \quad c = \exp(J_{l_1}^{(6)}/T). \tag{8}$$

Then the average magnetization m for the n-th generation is given by

$$m = \frac{Z_+^{(n)} - Z_-^{(n)}}{Z_+^{(n)} + Z_-^{(n)}} \tag{9}$$

and the magnetization of the root x^0 is defined by

$$\langle\sigma_0\rangle = \lim_{n\to\infty}\frac{Z_+^{(n)} - Z_-^{(n)}}{Z_+^{(n)} + Z_-^{(n)}}. \tag{10}$$

Hence the average magnetization m for the n-th generation can be obtained by

$$m = \frac{(1+x+y_1+y_2)^5 - (1+x-y_1-y_2)^5}{(1+x+y_1+y_2)^5 + (1+x-y_1-y_2)^5}. \tag{11}$$

Below we will apply numerical methods to study detailed behavior of Eqs. (6) and (10).

4. The phase analysis

It is convenient to know the broad features of the phase diagram before discussing the different transitions in more detail. This can be achieved numerically in a straightforward fashion. The recursion relations (2) provide us the numerically exact phase diagram in $(T/J_1, -J_\mathrm{p}/J_1, J_{l_1}^{(6)}/J_1)$ space. Let $T/J_1 = \alpha$, $-J_\mathrm{p}/J_1 = \beta$, $J_{l_1}^{(6)}/J_1 = \gamma$ and respectively $a = \exp(\alpha^{-1})$, $b = \exp(-\alpha^{-1}\beta)$ and $c = \exp(\alpha^{-1}\gamma)$. Starting from initial conditions which corresponds to boundary condition $\bar{\sigma}^{(n)}(V\backslash V_n) \equiv 1$, one iterates the recurrence relations (6) and observes their behavior after a large number of iterations. In the simplest situation a fixed point (x^*, y_1^*, y_2^*) is reached. If $y_1^* = 0$, $y_2^* = 0$, it corresponds to a paramagnetic phase or to a ferromagnetic phase if $y_1^*, y_2^* \neq 0$. From formula of average magnetization (11) follows that a situation where $y_1^*, y_2^* \neq 0$ but $m = 0$ cannot occur.

Secondary, the system may be periodic with period p, where case $p = 2$ corresponds to antiferromagnetic phase and case $p = 4$ corresponds to so-called antiphase, that is denoted $\langle 2\rangle$ for compactness. We consider periodic phases with period p where $p \leq 12$. All periodic phases with period $p > 12$ and aperiodic phase will be considered as modulated phase. The resultant phase diagrams for some values of γ, β are shown in Figs. 3–5.

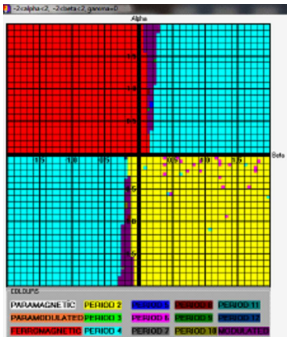

Fig. 3. Phase diagram of the model for $\gamma = 0$.

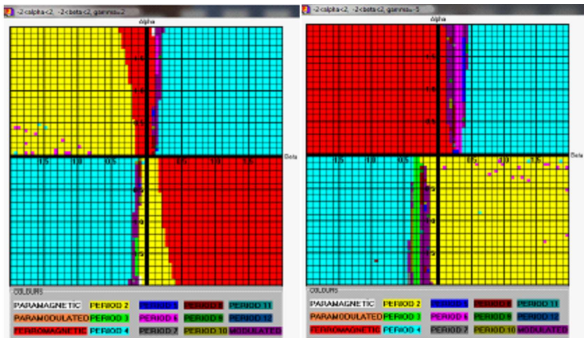

Fig. 4. Phase diagrams of the model for $\gamma = 2$ and $\gamma = -5$, respectively.

We consider the variation of the wavevector with temperature. A definition of the wave vector that is convenient for numerical purposes is

$$q = \lim_{N\to\infty}\left(\frac{1}{2}\frac{n}{N}\right), \tag{12}$$

where n is the number of times the magnetization (11)

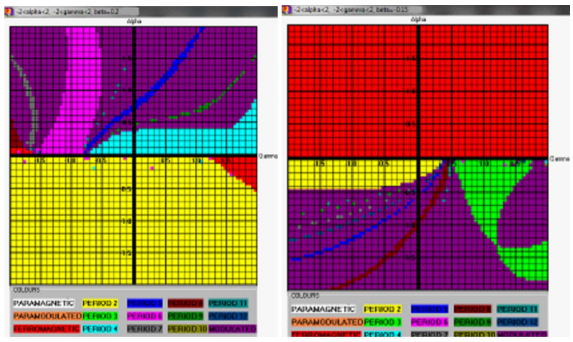

Fig. 5. Phase diagrams of the model for $\beta = 0.2$ and $\beta = -0.15$, respectively.

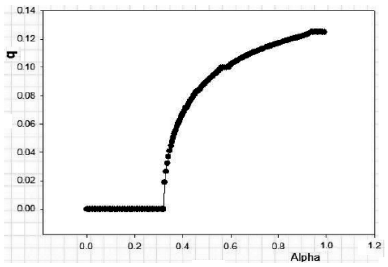

Fig. 6. Variation of the wave vector q: $\beta = 0.15$ for $\gamma = 0$.

changes sign during N successive iterations [1]. Typical graphs of q versus T are drawn in Figs. 6, 7. Lastly, we study the Lyapunov exponent of our model. For a more detailed investigation of $q(T)$, it is necessary to locate the main locking steps that must be present according to the general theory. These intervals may be very narrow, moreover the distinction between long-periodic cycles and truly aperiodic solutions is difficult to achieve numerically. It tells whether an infinitesimal perturbation of the initial conditions will have an infinitesimal effect (negative exponent) or will lead to a totally different trajectory (positive exponent). In practice the calculation of the Lyapunov exponent goes as follows. The recurrence equations are linearized around the successive points of the trajectory, yielding linear recurrence equations for the perturbations $(\delta x, \delta y, \delta z)$. In matrix form

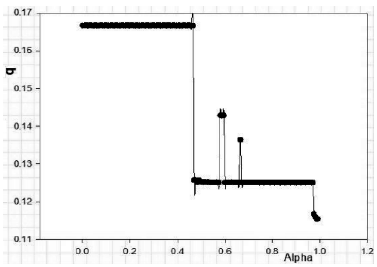

Fig. 7. Variation of the wave vector q: $\beta = 0.13$ for $\gamma = 2$.

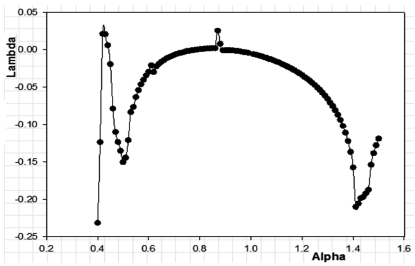

Fig. 8. Variation of the Lyapunov exponent λ: $\beta = 0.33$ with $\alpha \in [0.4, 1.5]$ for Fig. 3.

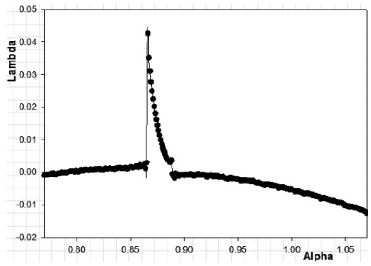

Fig. 9. Variation of the Lyapunov exponent λ: $\beta = 0.33$ with $\alpha \in [0.75, 1.1]$ for Fig. 3.

one has

$$V_{k+1} = \begin{pmatrix} \delta x' \\ \delta y' \\ \delta z' \end{pmatrix} = L_k \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}, \tag{13}$$

where the matrix L_k depends on the iteration step. The Lyapunov exponent λ is obtained by

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \log(\|V_N\|), \tag{14}$$

where $\|V_N\|$ denotes the norm of the vector V. Stable limit cycles may exist only for negative exponents. Numerically, it is found here that every region of negative λ coincides with the stability domain of a given cycle. Also the variation of the Lyapunov exponent for $\beta = 0.33$ with $\gamma = 0$ are presented in Figs. 8, 9. The vanishing of the Lyapunov exponent means that the set of trajectories is quasi-continuous, or in other terms that it has a zero frequency "phason" mode [1].

5. Conclusions

In this paper, we have clarified the role of order 5 of the Cayley tree-like lattice which we called pentagonal chandelier and the coupling constants. We also have studied the variation of the wave vector with temperature in the modulated phase and the Lyapunov exponent associated with the trajectory of the system. In this case, the phase diagrams and graphs of variation of wave vectors are changed completely.

Acknowledgments

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