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Hamiltonian Approach to Multiple Coupled Nonlinear Oscillators

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In this paper, Hamiltonian approach is extended to investigate coupled nonlinear multi-degree-of-freedom oscillatory systems. At the beginning of the study, basic principles of Hamiltonian approach are provided for multiple coupled oscillators. In the next section, the amplitude–frequency relation for the two-degree-of-freedom nonlinear mechanical systems is obtained via Hamiltonian approach. The natural frequency expression is compared to show the agreement between the present and published results. Additionally, the approximate and numerical periodic solutions are plotted.

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1. Introduction

A vibrating system can be generally modeled as a 1-degree-of-freedom (1-DOF) system, which can give a good explanation of its essential dynamic characteristics; however, sometimes multi-degree-of-freedom (m-DOF) model has to be used to outline the complex nonlinear phenomenon [1].

A great deal of work has been devoted to the 1-DOF nonlinear oscillatory systems. In general an exact analytical solution to a given nonlinear problem is difficult, sometimes impossible. Plenty of approximate techniques have appeared in open literature, among which one can include the homotopy perturbation method [2], max-min approach [3], energy balance method [4].

Recently 2-DOF oscillation systems have been investigated [5–7]. In all studies the system reduces to one decoupled equation and one coupled equation by using a transformation. But for a general case of a mass and a spring system, such treatment becomes invalid even for a simple 2-DOF system. In order to obtain the closed form solutions of coupled nonlinear m-DOF oscillatory systems, Hamiltonian approach is implemented in this paper.

This study is an extension of the authors' previous work [8] which has focused on the application of variational approach to the nonlinear systems. At the beginning of the study, the basic principles of Hamiltonian approach are supplied for multiple coupled oscillators. In the following section, a nonlinear mechanical system (2-DOF) is considered as an application. The nonlinear natural frequencies are derived via Hamiltonian approach. The expressions are compared to show the agreement between the present and published results. Additionally the approximate and numerical periodic solutions are plotted.

2. Hamiltonian approach for nonlinear systems

Let us consider the motion of a multiple coupled oscillator

$$m_i\delta_{ij}\ddot{x}_j + f_i(x_1, x_2, \ldots, x_n) = 0\,, \quad i, j = 1, 2, \ldots, n\,, \tag{1}$$

where δ_{ij} is the Kronecker delta.

Hamiltonian approach, proposed by He [9] for nonlinear oscillators is extended for the multiple coupled nonlinear oscillators. Thereby Hamiltonian of Eq. (1) can be written as follows:

$$H(x_1, x_2, \ldots, x_n) = \frac{1}{2}\sum_{i=1}^{n} m_i\dot{x}_j^2 + F(x_1, x_2, \ldots, x_n)\,, \tag{2}$$

where T $(= 2\pi/\omega)$ is the period of the nonlinear system and $\partial F/\partial x_i = f_i$. In Eq. (2) the kinetic energy (E) and potential energy (T) can be respectively expressed as $E = \frac{1}{2}\sum_{i=1}^{n} m_i\dot{x}_j^2$ and $T = F(x_1, x_2, \ldots, x_n)$.

Throughout the oscillation since the system is conservative, the total energy remains unchanged during the motion; Hamiltonian of the oscillator becomes a constant value, $H = E + T = H_0$. A new function $\hat{H}(u)$ is defined by integrating Eq. (2) over the quarter period

$$\hat{H}(x_1, x_2, \ldots, x_n) = \int_0^{T/4} \frac{1}{2}\left[\sum_{i=1}^{n} m_i\dot{x}_j^2 + F(x_1, x_2, \ldots, x_n)\right]\mathrm{d}t = \frac{T}{4}H_0\,. \tag{3}$$

We assume that the approximate solution for the displacements can be expressed as

$$x_i(t) = A_i\cos\omega t, \tag{4}$$

where A_i and ω are the amplitudes and the frequency of the oscillation. Inserting Eq. (4) into (3) results in

$$\hat{H}(A_1, A_2, \ldots, A_N, \omega) = \int_0^{T}\left[\frac{1}{2}\omega^2\sin^2(\omega t)\sum_{i=1}^{N} m_iA_i + F(A_1\cos\omega t, A_2\cos\omega t, \ldots, A_N\cos\omega t)\right]\mathrm{d}t\,. \tag{5}$$

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Frequency–amplitude relation is obtained by setting

$$\frac{\partial}{\partial A_i}\left[\frac{\partial \hat{H}}{\partial T}\right] = 0 \quad \text{or} \quad \frac{\partial}{\partial A_i}\left[\frac{\partial \hat{H}}{\partial (1/\omega)}\right] = 0\,. \tag{6}$$

3. Applications

In this section an example for the coupled nonlinear oscillators will be given to show the efficiency of the extended Hamiltonian approach previously discussed. The sketch of a 2-DOF mass–spring system is given in Fig. 1. In this example the system consists of equal masses of m moving in a frictionless ground and connected to three linear and nonlinear springs. The coefficients of linear springs are k_1, k_2, while the nonlinear spring coefficient is denoted by k_1^N and k_2^N. The absolute displacements of equal masses m are represented by the functions $x_1(t)$ and $x_2(t)$, respectively.

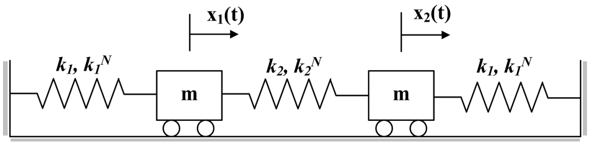

Fig. 1. 2-DOF mass–spring system.

The Hamiltonian of the system is easily established as

$$\hat{H}(x_1, x_2) = \frac{1}{2}\int_0^{T/4}\Big[m\dot{x}_1^2 + m\dot{x}_2^2 + k_1 x_1^2 + \frac{1}{2}k_1^N x_1^4 + k_2(x_1 - x_2)^2 + \frac{1}{2}k_2^N(x_1 - x_2)^4 + k_1 x_2^2 + \frac{1}{2}k_1^N x_2^4\Big]\mathrm{d}t\,. \tag{7}$$

Assuming the first approximate solutions in the form of

$$x_1(t) = A_1 \cos\omega t\,, \tag{8a}$$

$$x_2(t) = (A_1 - A)\cos\omega t\,. \tag{8b}$$

Substituting Eq. (8) into (7) leads to

$$\hat{H}(A, A_1, \omega) = \frac{1}{2}\int_0^{T/4}\Big\{m\omega^2\big[A_1^2 + (A_1 - A)^2\big] \times \sin^2\omega t + \frac{1}{2}A_1^2 k_1 \cos^2\omega t + \frac{1}{2}A_1^4 k_1^N \cos^4\omega t + k_2[A_1\cos\omega t - (A_1 - A)\cos\omega t]^2 + \frac{1}{2}k_2^N[A_1\cos\omega t - (A_1 - A)\cos\omega t]^4 + (A_1 - A)^2 k_1 \cos^2\omega t + \frac{1}{2}(A_1 - A)^4 k_1^N \cos^4\omega t\Big\}\mathrm{d}t\,. \tag{9}$$

Setting $\frac{\partial}{\partial A}\left[\frac{\partial \hat{H}}{\partial(1/\omega)}\right] = 0$ and $\frac{\partial}{\partial A_1}\left[\frac{\partial \hat{H}}{\partial(1/\omega)}\right] = 0$, we have two equations as follows:

$$4A_1k_1 + 9A^2A_1k_1^N + 3A_1^3k_1^N - 3A^3\big(k_1^N + k_2^N\big) - 4A_1m\omega^2 - A\big(4k_1 + 4k_2 + 9A_1^2k_1^N - 4m\omega^2\big) = 0\,, \tag{10a}$$

$$(A - 2A_1)\big(4k_1 + 3A^2k_1^N - 3AA_1k_1^N + 3A_1^2k_1^N - 4m\omega^2\big) = 0\,. \tag{10b}$$

The amplitude–frequency relation can be derived by the use of Eqs. (10a), (10b) as follows:

$$\omega(A) = \frac{1}{4}\sqrt{\frac{16k_1 + 32k_2 + 24A^2k_2^N + 3A^2k_1^N}{m}}\,. \tag{11}$$

The special cases given below are verified with the recent study of Ref. [6]:

(i) For $k_1 = 0$ and $k_1^N = 0 \Rightarrow \omega(A) = \sqrt{\frac{4k_2 + 3A^2k_2^N}{2m}}$;

(ii) For $k_1^N = 0 \Rightarrow \omega(A) = \sqrt{\frac{2k_1 + 4k_2 + 3A^2k_2^N}{2m}}$.

To obtain a more accurate result $x_1(t)$ and $x_2(t)$ are defined as follows:

$$x_1(t) = A_3\cos(\omega_2 t) + (A_2 - A_3)\cos(3\omega_2 t)\,, \tag{12a}$$

$$x_2(t) = A_5\cos(\omega_2 t) + (A_4 - A_5)\cos(3\omega_2 t)\,. \tag{12b}$$

For the second order approximation contrary to the first order approximation, periodic solutions could not be expressed in closed forms. Instead the results for both approximations will be demonstrated with the numerical results (standard 4th order Runge–Kutta). Thus using the numerical values $m = 1$, $k_1 = 7$, $k_2 = 2$, $k_1^N = 1$, $k_2^N = 5$ and $A = 10$, the periodic functions $x_1(t)$ and $x_2(t)$ are evaluated and plotted in Fig. 2.

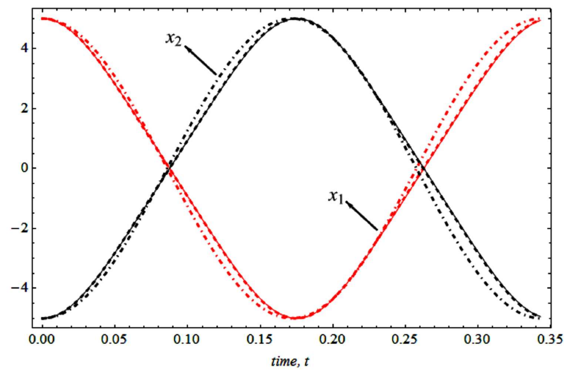

Fig. 2. Approximate and numerical periodic solutions for the system.

In this figure, three different lines are used. Those solid, dot-dashed and dashed represent the results of numerical, first and second order Hamiltonian approach methods, respectively. For both displacements, the results of the second order approximation overlap with the ones of numerical solution.

4. Conclusion

In this paper, Hamiltonian approach is extended for solving the motion of nonlinear m-DOF oscillation systems. One example has been presented and discussed. The closed form of the amplitude–frequency relations of 2-DOF system is presented and compared to the expressions from literature. The results of special cases of 2-DOF system perfectly match with the published results. Moreover, the results are refined by second order approximation.

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