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Calculation of Anderson Localization Length by Using Generalized Quantum Kicked Rotator Analytically

A. Gharaati, F. Bahadori and H. Boroumandi

Department of Physics, Payame Noor University (PNU), Tehran, I.R. of Iran

We use the generalized quantum kicked rotator model and its relation with the Anderson model. We calculate localization length analytically. For this reason, we consider a one-dimensional rotator model for a special potential whose an impulse is applied at equal time intervals, T. We obtain time evolution of the wave function between two successive impulses by an evolution matrix. We change this model to the tight-binding model for a particle on a one-dimensional lattice. At special case the wave function is localized, and then we derive the Anderson localization length for the system analytically.

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1. Introduction

We study quantum systems for Hamiltonians whose classical counterparts are chaotic. The classical model of kicked rotator was chaotic at special conditions and so one of the most extensively investigated systems in the field of classical chaos theory [1, 2]. This model is applicable in very different branches of physics, for example, related model with the Anderson localization in solid-state physics [3, 4]. The Anderson localization is due to disorder, the localization problem for a particle on a one-dimensional lattice with a pseudorandom potential whose particle states change from being extended to localized. This sudden change is referred to as a metal–insulator transition because the change in the nature of particle wave function corresponds to change in conductivity from conducting to insulating [5–7].

Gremple et al. have considered the potential of the form

$$V(x) = 2\arctan(k\cos(x)/2 - \varepsilon)\,,$$

where $k, \varepsilon \in R$, and numerically showed that the localization length (LL) (γ^{-1}) satisfies the equation

$$2k\cosh\gamma = \sqrt{1+(\varepsilon-k)^2} + \sqrt{1+(\varepsilon+k)^2},$$

which is exactly similar to their analytically calculated by Thouless [4, 8–10].

In this paper, we study a one-dimensional model of potential as,

$$V(x,t) = 2\arctan\left(\frac{k\cos(x)}{2(1-s\cos(x))} - E\right) \times \sum_n \delta(t-nT)\,, \quad \text{where} \quad |s|<1\,, \quad \text{and} \quad k, E \in R\,. \tag{1}$$

This potential has the remarkable property as: (a) it approaches to a delta-function when $s \to 1$ and (b) it takes the standard kicked rotator form when $s \to 0$. See Fig. 1.

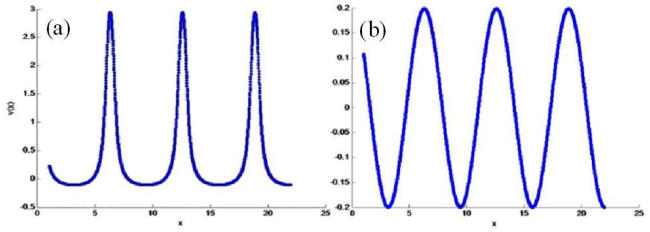

Fig. 1. Variation of potential $V(x)$ for E = const and (a) $k = 0.1$, $s = 0.99$ and (b) $k = 0.1$, $s = 0.0$.

2. Derivation of the wave function of the GQKR

We consider the Schrödinger equation for generalized quantum kicked-rotator (GQKR) model of the form

$$H\psi(x,t) = \left(\frac{I^2}{2} + V(x)\right)\psi(x,t)\,, \tag{2}$$

where x and I are the angle and angular momentum variables and potential $V(x)$ is a periodic function. The wave function at time t_0 and t are related through the following integral form of [11]:

$$\psi(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} H(t')\,\mathrm{d}t'\right)\psi(t_0)\,, \tag{3}$$

where $\hbar$ is Plank's constant. For $t = nT^+$ and $t = (n+1)T^-$, where "$-$" and "$+$" signs indicate the times immediately before and after the impulses, respectively. Equations (2) and (3) give

$$\psi(nT^+) = \exp\left(-\frac{\mathrm{i}}{\hbar}V(x)\right)\psi(nT^-)\,, \quad nT^- < t < nT^+, \tag{4a}$$

$$\psi[(n+1)T^-] = \exp\left(-\frac{\mathrm{i}}{\hbar}H_0T\right)\psi(nT^+)\,, \quad nT^+ < t < (n+1)T^-. \tag{4b}$$

Combining Eqs. (4), the wave function after the $(n+1)$-th impulse, in terms of that after the n-th, one obtains

$$\psi[(n+1)T^+] = U(k,x,T)\psi(nT^+)\,, \tag{5}$$

where

$$U(k,x,T) = \mathrm{e}^{\frac{-\mathrm{i}}{\hbar}V(x)}\mathrm{e}^{\frac{-\mathrm{i}}{\hbar}H_0T}. \tag{6}$$

We expand this wave function in terms of the eigenfunction of free particle, $\varphi_n = (1/\sqrt{2\pi})\mathrm{e}^{\mathrm{i}nx}$. Thus,

$$\psi(nT^+) = \sum_{j=-\infty}^{\infty} C_j(nT^+)\varphi_j(x)\,. \tag{7}$$

Inserting (7) into (5), multiplying both sides by $\varphi_m^*(x)$ and integrating over the interval $(0, 2\pi)$ gives

$$\begin{aligned} C_j[(n+1)T^+] &= \sum_j U_{mj}C_j(nT^+) \\ &= A_{ml}\Omega_{lj}C_j(nT^+)\,, \end{aligned} \tag{8}$$

where U_{mj}, A_{ml} and Ω_{lj} are the matrix elements of the form

$$U_{mj} = \langle\varphi_m\Big|\mathrm{e}^{\frac{-\mathrm{i}}{\hbar}H_0T}\,\mathrm{e}^{\frac{-\mathrm{i}}{\hbar}V(x)}\Big|\varphi_j\rangle\,, \tag{9}$$

$$A_{ml} = \int_0^{2\pi}\varphi_m^*(x)\mathrm{e}^{\frac{-\mathrm{i}}{\hbar}V(x)}\varphi_l(x)\,\mathrm{d}\theta\,, \tag{10}$$

$$\begin{aligned} \Omega_{lj} &= \int_0^{2\pi}\varphi_l^*(x)\mathrm{e}^{\frac{-\mathrm{i}}{\hbar}H_0T}\varphi_j(x)\,\mathrm{d}\theta = \mathrm{e}^{-\mathrm{i}\hbar\omega_jT}\delta_{lj}\,, \\ \omega_j &= \frac{j^2}{2}\,. \end{aligned} \tag{11}$$

Now we have derived the matrix elements of U by considering special potential, Eq. (1), and obtained analytically the wave function of the kicked rotator. Combining Eq. (1) and Eq. (9) gives the matrix element between sites in the form

$$\begin{aligned} U_{mj} = A_{ml}\Omega_{lj} &= \langle m\Big|\mathrm{e}^{\frac{-\mathrm{i}}{\hbar}V(\theta)}\Big|j\rangle\,\mathrm{e}^{-\mathrm{i}\hbar\left(\frac{j^2}{2}\right)T} \\ &= \langle m\left|\frac{1-\mathrm{i}[k\cos x/2(1-s\cos x)-E]}{1+\mathrm{i}[k\cos x/2(1-s\cos x)-E]}\right|j\rangle \\ &\quad\times\mathrm{e}^{-\mathrm{i}\hbar\left(\frac{j^2}{2}\right)T}. \end{aligned} \tag{12}$$

By calculating the integral of Eq. (10), the elements A_{mj} are given by

$$A_{mj} = \begin{cases} \left(\frac{1+\mathrm{i}E'}{1-\mathrm{i}E'}\right)\left(-\frac{\mathrm{i}r^*}{2\sqrt{1+r^2}}\right)\left[z(r)^2 + \frac{2\mathrm{i}}{r^*}z(r) + 1\right]z(r)^{|m-j|-1}, & m \neq j, \\ \left(\frac{1+\mathrm{i}E'}{1-\mathrm{i}E'}\right)\left(\frac{1}{\sqrt{1+r^2}}\right)\left(1+\frac{r^*}{r}\right) - \frac{r^*}{r}, & m = j, \end{cases} \tag{13}$$

where

$$\begin{aligned} z(r) &= \frac{\mathrm{i}}{r}(1-\sqrt{1+r^2})\,, \quad r = \frac{K+E's+\mathrm{i}s}{1-\mathrm{i}E'}\,, \\ K &= k/2\hbar\,, \quad E' = E/\hbar\,. \end{aligned}$$

In order to obtain a compact formula, we define

$$\frac{1}{r} = \sinh(\gamma+\mathrm{i}\mu)\,. \tag{14}$$

Then

$$z(r) = -\mathrm{i}\,\mathrm{e}^{-(\gamma+\mathrm{i}\mu)} \quad\text{and}\quad z(r)z^*(r) = \mathrm{e}^{-2\gamma}, \tag{15}$$

where γ and μ are real numbers. Inserting Eq. (14) and Eq. (15) in Eq. (13), gives

$$\begin{aligned} A_{mj} &= \frac{1+\mathrm{i}E'}{1-\mathrm{i}E'}\frac{\sinh(\gamma+\mathrm{i}\mu)}{\sinh(\gamma-\mathrm{i}\mu)} \\ &\quad\times\left(\frac{2\cos\mu\sinh\gamma}{\cosh(\gamma+\mathrm{i}\mu)} - \delta_{mj}\right)z_-^{|m-j|}. \end{aligned} \tag{16}$$

Then,

$$\begin{aligned} U_{mj} &= \frac{1+\mathrm{i}E'}{1-\mathrm{i}E'}\frac{\sinh(\gamma+\mathrm{i}\mu)}{\sinh(\gamma-\mathrm{i}\mu)} \\ &\quad\times\left(\frac{2\cos\mu\sinh\gamma}{\cosh(\gamma+\mathrm{i}\mu)} - \delta_{mj}\right)z_-^{|m-j|}\,\mathrm{e}^{\frac{-\mathrm{i}\hbar j^2}{2}T}. \end{aligned} \tag{17}$$

By considering Eq. (13) and Eq. (16) and substituting $n = m - j$ we could obtain the wave function of the GQKR analytically. The matrix elements A_{mj} show interaction energy between impulse m-th and j-th for GQKR. By increasing $|m-j|$ the matrix elements A_{mj} fall off exponentially.

3. Calculation of Anderson localization length

By considering Eq. (2) and a little algebra and changing of variable, we achieve the tight-binding model of the form [11]:

$$T_mu_m + \sum_{r\neq0} W_ru_{m+r} = Eu_m\,, \tag{18}$$

where u_m is the amplitude of the m-th site of a one-dimensional lattice and W_r is the hopping matrix element between sites at the r-th position and the diagonal potential T_m is chosen as

$$T_m = \tan((\omega - p_m^2/2)/2\hbar)\,, \tag{19}$$

and W_r is

$$W_r = \frac{-1}{2\pi}\int_0^{2\pi}\mathrm{d}x\mathrm{e}^{\mathrm{i}rt}\tan(V(x)/2\hbar)\,, \tag{20}$$

in which $W_r = W_{-r}$ and $W_0 = -E$. With this potential one gets a tight-binding model without any approximation as

$$\begin{aligned} &(T_m + W_0)u_m + W_1(u_{m-1} + u_{m+1}) \\ &\quad + W_2(u_{m-2} + u_{m+2}) + \ldots \\ &\quad + W_r(u_{m-r} + u_{m+r}) + \ldots = 0\,, \end{aligned} \tag{21}$$

where in the special case $s = 0$, Eq. (21) is restricted and $W_r = 0$ for $|r| > 1$. The diagonal potential T_m can be chosen in several ways. If T_m's are taken to be random and W_r is restricted to nearest-neighbor, Eq. (18) is well-known as the Anderson model [8].

According to Eq. (15), the matrix elements of U fall off away from the main diagonal exponentially. Then, the parameter $|z(t)|$ can be written as

$$|z(t)| = \mathrm{e}^{-\gamma}, \tag{22}$$

where γ is the inverse of LL. Now, we calculate LL analytically. For this reason by using Eq. (14) we have

$$\sinh(\gamma + \mathrm{i}\mu) = \frac{1 - \mathrm{i}E'}{K + E's + \mathrm{i}s} = \frac{1}{K'} - \mathrm{i}\frac{E''}{K'}, \tag{23}$$

where $E'' = \frac{s+E'K+E'^2 s}{K}$ and $K' = \frac{s^2+(K+E's)^2}{K}$ and by expanding the $\sinh(\gamma + \mathrm{i}\mu)$ we obtain

$$\sinh\gamma\cos\mu = \frac{1}{K'} \quad \text{and} \quad \sin\mu\cosh\gamma = -\frac{E''}{K'}. \tag{24}$$

Using the identity $\cosh^2 x = 1 + \sinh^2 x$ and define $t = \cosh^2\gamma$, we find

$$K'^2 t^2 - (1 + E''^2 + K'^2)t + E''^2 = 0. \tag{25}$$

Then, the solution of Eq. (25) gives the LL analytically as

$$2K'\cosh\gamma = \left[1 + (E'' - K')^2\right]^{\frac{1}{2}} + \left[1 + (E'' + K')^2\right]^{\frac{1}{2}}. \tag{26}$$

In this equation if $s = 0$ and $E \neq 0$ it reduces to

$$2K\cosh\gamma = \sqrt{1 + (E' - K)^2} + \sqrt{1 + (E' + K)^2}. \tag{27}$$

This equation is exactly the same as results in Ref. [8].

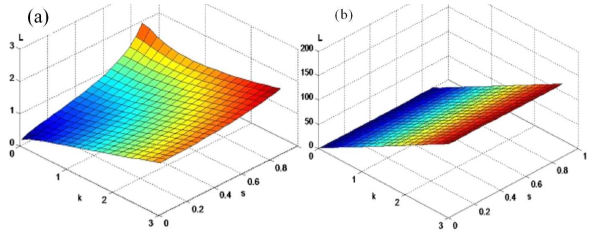

Fig. 2. Variation of localization length for $E = 0$ and (a) $\hbar = 1$ and (b) $\hbar = 0.01$.

We plot Eq. (26) for parameters k, s and $\hbar$. In Fig. 2, we can see that the LL grows with increase of the value of s. Figure 2a shows LL for $\hbar = 1$ and Fig. 2b shows LL for $\hbar = 0.01$. According to Fig. 2 it can be seen that by decreasing $\hbar$, the LL increases. This result has confirmed that in the quantum limit, the wave function is localized and the LL is small while in classical limit, the wave function expands and thus LL increases considerably.

4. Conclusions

We have considered a generalized potential for the kicked rotator model. We have derived analytically the wave function of the system after each impulse by using an evolution matrix U. The elements of this matrix exponentially fall off away from the main diagonal, which is exactly related to the Anderson model and localization of the wave function. We have shown the relationship between kicked-rotator models with tight-binding model for a particle in a one-dimensional lattice. So, the wave function is localized at special conditions. We have derived expression analytically for the Anderson localization length $1/\gamma$ for two parameters of b and k, and they have been compared with special case $b = 0$, numerically. In this case, they are completely compatible with the traditional quantum kicked rotator model.

Acknowledgments

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