Optical and Acoustical Methods in Science and Technology

Fibonacci and Lucas Numbers for Real Indices and Some Applications

R. Wituła*

Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland

In this paper the notion of the Fibonacci and Lucas numbers is extended onto real indices. Next, these new numbers are used for calculating real powers of certain matrices. The presented method to the extension of elements of linear recurrence sequence to real indices ought to find practical application in wide understanding metrology and medical diagnostics.

PACS: 02.10.Yn, 02.10.De, 02.50.Cw

1. Introduction

One of the main reasons behind the generalization of the Fibonacci and Lucas numbers onto real indices were the authors' attempts at generalizing the powers of matrices onto any real (complex) exponents [1].

Surely, standard grounds for such generalization of the powers of any matrix A could be given by its Jordan decomposition

$$A = BJ(A)B^{-1}. \tag{1}$$

If $J(A)$ have a diagonal form

$$J(A) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix},$$

it would be possible, almost without any consequences, to assume that

$$A^x := B \begin{bmatrix} \lambda_1^x & & 0 \\ & \ddots & \\ 0 & & \lambda_n^x \end{bmatrix} B^{-1}, \quad x \in \mathsf{R} \quad (x \in \mathsf{C}),$$

where a standard definition of the complex power is utilized $\lambda^x := \exp(x \ln \lambda)$, $\ln \lambda := \ln|\lambda| + \mathrm{i} \arg \lambda$.

This method of generalizing the powers of matrices has some potential drawbacks associated with the numerical nature of a decomposition (1) (we are looking for not only the eigenvalues of matrices A, but also of a similarity matrix B). It turns out that in many specific cases, when it is possible to describe, in a recurrent manner, the elements of matrix A^n, $n \in \mathsf{N}$:

$$A^n = \begin{bmatrix} a_{11}(n) & a_{12}(n) & \ldots & a_{1k}(n) \\ \vdots & & & \\ a_{k1}(n) & a_{k2}(n) & \ldots & a_{kk}(n) \end{bmatrix},$$

it would be sufficient to replice argument $n \in \mathsf{N}$ with a more general argument $x \in \mathsf{R}$ ($x \in \mathsf{C}$), provided that we could "clearly" define $a_{ij}(x)$. An example of such approach is the publication [1] and Sect. 5 of this one.

2. Basic definitions

The following three known identities satisfied by the Fibonacci and Lucas numbers (denoted by F_n and L_n, $n \in \mathsf{N}$, respectively) form the basis of our generalization of the definition of F_n and L_n for real indices n:

$$\sqrt{5}F_n = \alpha^n - \beta^n, \tag{2}$$

$$F_{n+1} - \beta F_n = \alpha^n, \tag{3}$$

and

$$L_n = 2\alpha^n - \sqrt{5}F_n\,, \tag{4}$$

where $n \in \mathsf{N}$, $\alpha := \frac{1+\sqrt{5}}{2}$ and $\beta := \frac{1-\sqrt{5}}{2}$ ((3) and (4) was discovered by Rabinowitz [2] and, independently, by Wituła [3]). Such choice of the identities is by no means accidental, but, in our opinion, results from the fact that the number of technical procedures connected with the process of generalizing definition F_n and L_n onto indices $n \in \mathsf{R}$ should be as small as possible. Accordingly, as indicated below, conditions (2)–(4) guarantee:

- optimal initial conditions,
- recursive extension procedure,
- simple relationships between the Fibonacci and the Lucas numbers.

What more do we need? Thus, let us get down to work.

The generalized Fibonacci numbers F_s are given first for $s \in [0, 1)$ by (see (2)):

$$\sqrt{5}F_s = \alpha^s - \mathrm{e}^{\mathrm{i}\pi s}(-\beta)^s, \tag{5}$$

and, in the next steps, for $s \in [1, 2)$, for $s \in [2, 3)$, ... and at last for $s \in [-1, 0)$, for $s \in [-2, -1)$, ..., all one's are defined by the relation (see (3)):

* e-mail: roman.witula@polsl.pl

$$F_{s+1} = \alpha^s + \beta F_s\,. \tag{6}$$

Remark 2.1. *We observe that $\alpha > 0$ and $-\beta > 0$. So in (5) only $\mathrm{e}^{\mathrm{i}\pi s}$ is a complex function.*

Moreover we set (see (4)):

$$L_s := 2\alpha^s + \sqrt{5}F_s\,, \quad s \in \mathsf{R}\,, \tag{7}$$

which defines the generalized Lucas numbers.

Remark 2.2. *We note that (6) and (5) implies*

$$\begin{aligned} F_{s+1} &= \mathrm{e}^{\mathrm{i}\pi s}(-\beta)^s + (\sqrt{5}+\beta)F_s \\ &= \mathrm{e}^{\mathrm{i}\pi s}(-\beta)^s + \alpha F_s\,, \end{aligned} \tag{8}$$

which is the dual form of the formula (6).

Remark 2.3. *We note that similarly to our definition of numbers F_s and L_s, $s \in \mathsf{R}$, the $\Gamma(z)$ function for all $z \in \mathsf{C}\backslash\{0,-1,-2,\ldots\}$ can also be defined. First, by the Euler reflection formula*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{9}$$

we extend the definition of $\Gamma(z)$ from the strip $0 < \Re(z) \le \frac{1}{2}$ onto the strip $0 < \Re(z) < 1$. Next by the functional equation of $\Gamma(z)$:

$$\Gamma(z+1) = z\Gamma(z) \tag{10}$$

we extend the definition of $\Gamma(z)$ to all strips $k < \Re(z) < k+1$, $k \in \mathsf{Z}$. Along the line $\Re(z) = 1$, the definition of $\Gamma(z)$ by the continuity argument is given (from the integral definition of $\Gamma(z)$) which by (10) implies the definition of $\Gamma(z)$ for $\Re(z) = k$, $k = 2,3,\ldots$ and, simultaneously, blocks the possibility of defining $\Gamma(z)$ for $\Re(z) = k$, $k = 0,-1,-2,\ldots$

3. Fundamental properties

Now, some fundamental properties of F_s and L_s ($s \in \mathsf{R}$) are listed as follows:

$$\sqrt{5}F_s = \alpha^s - \mathrm{e}^{\mathrm{i}\pi s}(-\beta)^s, \tag{11}$$

$$L_s = \alpha^s + \mathrm{e}^{\mathrm{i}\pi s}(-\beta)^s, \tag{12}$$

$$F_{s+2} = F_{s+1} + F_s\,, \tag{13}$$

$$L_{s+2} = L_{s+1} + L_s\,, \tag{14}$$

$$F_{-s} = -\mathrm{e}^{-\mathrm{i}\pi s}F_s\,, \tag{15}$$

$$L_{-s} = \mathrm{e}^{-\mathrm{i}\pi s}L_s\,, \tag{16}$$

$$F_{2s} = F_s L_s\,, \tag{17}$$

$$L_s = F_{s+1} + F_{s-1}\,, \tag{18}$$

$$\mathrm{e}^{\mathrm{i}\pi t}L_{s-t} = F_{t+1}L_s - F_t L_{s+1}\,, \tag{19}$$

$$F_s F_t - F_{s-r}F_{t+r} = \mathrm{e}^{\mathrm{i}\pi(s-r)}F_r F_{t-s+r}\,, \tag{20}$$

$$F_{s+t+1} = F_{s+1}F_{t+1} + F_s F_t\,, \tag{21}$$

$$5F_s^2 = L_{2s} - 2\mathrm{e}^{\mathrm{i}\pi s}, \tag{22}$$

$$5F_s^3 = F_{3s} - 3\mathrm{e}^{\mathrm{i}\pi s}F_s\,, \tag{23}$$

$$25F_s^4 = L_{4s} - 4\mathrm{e}^{\mathrm{i}\pi s}L_{2s} + 6\mathrm{e}^{\mathrm{i}3\pi s}, \tag{24}$$

and, generally, we have

$$5^n F_s^{2n} = \sum_{k=0}^{n}\binom{2n}{k}(-1)^k\mathrm{e}^{\mathrm{i}k\pi s}L_{2(n-k)s}\,, \tag{25}$$

$$5^n F_s^{2n+1} = \sum_{k=0}^{n}\binom{2n+1}{k}(-1)^k\mathrm{e}^{\mathrm{i}k\pi s}F_{(2n+1-2k)s}\,, \tag{26}$$

$$L_s^{2n} = \sum_{k=0}^{n}\binom{2n}{k}\mathrm{e}^{\mathrm{i}k\pi s}L_{2(n-k)s}\,, \tag{27}$$

$$L_s^{2n+1} = \sum_{k=0}^{n}\binom{2n+1}{k}\mathrm{e}^{\mathrm{i}k\pi s}F_{(2n+1-2k)s}\,, \tag{28}$$

In addition, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}F_s = -\mathrm{i}\pi\mathrm{e}^{\mathrm{i}\pi s}(-\beta)^s[1+2\ln(-\beta)] + F_s\ln\alpha\,. \tag{29}$$

4. Proofs of selected properties

(13). If (6) is applied three times, we get

$$\begin{aligned} F_{s+2} &= \alpha^{s+1} + \beta F_{s+1} = \alpha^{s+1} + \beta(\alpha^s + \beta F_s) \\ &= \alpha^s(\alpha+\beta) + \beta^2 F_s = \alpha^s + (\beta+1)F_s \\ &= F_{s+1} + F_s\,. \end{aligned}$$

(15). By (11) and by the equality $\alpha\beta = -1$ we obtain

$$\begin{aligned} \sqrt{5}F_{-s} &= \alpha^{-s} - \mathrm{e}^{-\mathrm{i}\pi s}(-\beta)^{-s} = (-\beta)^s - \mathrm{e}^{-\mathrm{i}\pi s}\alpha^s \\ &= -\mathrm{e}^{-\mathrm{i}\pi s}\sqrt{5}F_s\,. \end{aligned}$$

(17). From (5) and (12) follows.

(18). By (7) and (6) we have

$$\begin{aligned} L_s &= 2\alpha^s + (\beta-\alpha)F_s = (\alpha^s + \beta F_s) + (\alpha^s - \alpha F_s) \\ &= F_{s+1} + \alpha(\alpha^{s-1} - F_s) \\ &= F_{s+1} - \alpha\beta F_{s-1} = F_{s+1} + F_{s-1}\,. \end{aligned}$$

(19). At first, we note that

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\pi t}\alpha^s(-\beta)^t + \mathrm{e}^{\mathrm{i}\pi s}\alpha^t(-\beta)^s = \mathrm{e}^{\mathrm{i}\pi t}\alpha^t(-\beta)^t \\ &\quad\times(\alpha^{s-t} + \mathrm{e}^{\mathrm{i}\pi(s-t)}(-\beta)^{s-t}) \overset{(12)}{=} \mathrm{e}^{\mathrm{i}\pi t}L_{s-t}\,. \end{aligned}$$

On the other hand, using (6) and (8), we get

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\pi t}\alpha^s(-\beta)^t + \mathrm{e}^{\mathrm{i}\pi s}\alpha^t(-\beta)^s = (F_{s+1} - \beta F_s) \\ &\quad\times(F_{t+1} - \alpha F_t) + (F_{t+1} - \beta F_t)(F_{s+1} - \alpha F_s) \\ &\quad= 2(F_{s+1}F_{t+1} - F_s F_t) - (\alpha+\beta) \\ &\quad\times(F_s F_{t+1} + F_{s+1}F_t) = 2(F_{s+1}F_{t+1} - F_s F_t) \\ &\quad- F_s F_{t+1} - F_{s+1}F_t = F_{t+1}(2F_{s+1} - F_s) \\ &\quad- F_t(2F_s + F_{s+1}) \overset{(13),(18)}{=} F_{t+1}L_s - F_t L_{s+1}\,. \end{aligned}$$

(20). By (5) we obtain

$$5F_s F_t = (\alpha^s - \mathrm{e}^{\mathrm{i}\pi s}(-\beta)^s)(\alpha^t - \mathrm{e}^{\mathrm{i}\pi t}(-\beta)^t)$$

$$= \alpha^{s+t} + \mathrm{e}^{\mathrm{i}\,\pi(s+t)}(-\beta)^{s+t} - \mathrm{e}^{\mathrm{i}\,\pi s}\alpha^t(-\beta)^s - \mathrm{e}^{\mathrm{i}\,\pi t}\alpha^s(-\beta)^t. \quad (30)$$

Hence, the following formula is deduced:

$$5F_sF_t - 5F_{s-r}F_{t-r} = \mathrm{e}^{\mathrm{i}\,\pi t}(-\beta)^t\alpha^{s-r}(\mathrm{e}^{\mathrm{i}\,\pi r}(-\beta)^r - \alpha^r) + \mathrm{e}^{\mathrm{i}\,\pi(s-r)}(-\beta)^{s-r}\alpha^t(\alpha^r - \mathrm{e}^{\mathrm{i}\,\pi r}(-\beta)^r) \overset{(5)}{=} \mathrm{e}^{\mathrm{i}\,\pi(s-r)}(-\beta)^{s-r}\alpha^t\sqrt{5}F_r - \mathrm{e}^{\mathrm{i}\,\pi t}(-\beta)^t \times \alpha^{s-t}\sqrt{5}F_r = \mathrm{e}^{\mathrm{i}\,\pi(s-r)}(-\beta)^{s-r}\alpha^{s-r}\sqrt{5}F_r \times (\alpha^{t-s+r} - \mathrm{e}^{\mathrm{i}\,\pi(t-s+r)}(-\beta)^{t-s+r}) \overset{(5)}{=} 5\mathrm{e}^{\mathrm{i}\,\pi(s-r)}F_rF_{t-s+r}\,.$$

(21). From (30) we deduce

$$5(F_{s+1}F_{t+1} + F_sF_t) = (\alpha^2+1)\alpha^{s+t} + (\beta^2+1)\mathrm{e}^{\mathrm{i}\,\pi(s+t)}(-\beta)^{s+t} = \sqrt{5}(\alpha^{s+t+1} + \mathrm{e}^{\mathrm{i}\,\pi(s+t)}(-\beta)^{s+t+1}) \overset{(5)}{=} 5F_{s+t+1}\,.$$

5. Some applications

Let us set

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^s := \begin{bmatrix} F_{s+1} & F_s \\ F_s & F_{s-1} \end{bmatrix}, \quad (31)$$

for every $s \in \mathsf{R}$.

Why is the relation (31) correct? First, we note that

$$\begin{bmatrix} F_{s+1} & F_s \\ F_s & F_{s-1} \end{bmatrix}\begin{bmatrix} F_{t+1} & F_t \\ F_t & F_{t-1} \end{bmatrix} = \begin{bmatrix} F_{s+1}F_{t+1} + F_sF_t & F_{s+1}F_t + F_sF_{t-1} \\ F_sF_{t+1} + F_{s-1}F_t & F_sF_t + F_{s-1}F_{t-1} \end{bmatrix} = \begin{bmatrix} F_{s+t+1} & F_{s+t} \\ F_{s+t} & F_{s+t-1} \end{bmatrix},$$

which means that (31) is correct for rational s. Next, it is sufficient to apply the continuous argument for all other $s \in \mathsf{R}$.

Similarly, we can define:

$$\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}^s := \begin{bmatrix} F_{s+1} & -F_s \\ -F_s & F_{s-1} \end{bmatrix}, \quad s \in \mathsf{R}\,, \quad (32)$$

and, more generally, it can be defined for every $\varepsilon \in C\backslash\{0\}$:

$$\begin{bmatrix} 1 & \varepsilon \\ \varepsilon^{-1} & 0 \end{bmatrix}^s := \begin{bmatrix} F_{s+1} & \varepsilon F_s \\ \varepsilon^{-1}F_s & F_{s-1} \end{bmatrix}, \quad s \in \mathsf{R}\,. \quad (33)$$

What else can be done? What other kind of matrices can be treated in a similar way?

There are quite a lot of publications concerning these issues (see [4–7]). For example in [4, 5] the foundations for the description of the following powers are found:

$$\begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix}^s, \quad x, y, s \in \mathsf{R}$$

in view of generalized bivariate Fibonacci polynomials. The most general procedure is contained in [6], making it possible to calculate

$$A^xB = F_x\,, \quad x \in \mathsf{R}\,,$$

where

$$A = \begin{pmatrix} c_1 & c_2 & \dots & c_{k-1} & c_k \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} -1 & 2 & 0 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{pmatrix},$$

$$F_x = \begin{pmatrix} f_x^1 & f_x^2 & \cdots & f_x^k \\ f_{x-1}^1 & f_{x-1}^2 & \cdots & f_{x-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ f_{x-k+1}^1 & f_{x-k+1}^2 & \cdots & f_{x-k+1}^k \end{pmatrix},$$

and for $x = n \in \mathsf{N}$ we have

$$f_n^i = \sum_{j=1}^{k} c_j f_{n-j}^i$$

with initial values $f_{1-k}^i, f_{2-k}^i, \dots, f_0^i$, where c_j $(1 \le j \le k)$ are constant coefficients.

6. Final remarks

Many authors have extended the Fibonacci and Lucas numbers to arbitrary real (complex) subscripts [8–12]. Contribution to this problem presented here is original. Also the adaptation of this method to the extension of elements of linear recurrence sequence to real indices is probably new.

The presented method oughts to find practical application, among others in photonics and in medical diagnostics [13–17].

In the author's opinion, the most important element of this paper is presented here the definition of real powers of matrices. The author was already concerning this problem in different works [1, 16].

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