Optical and Acoustical Methods in Science and Technology

Open End Correction for Arbitrary Mode Propagating in a Cylindrical Acoustic Waveguide

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The paper presents a method of theoretical derivation and numerical calculation of the open-end correction coefficient for an arbitrary cut-on mode propagating in acoustic waveguide. Actually, the so-called open-end correction coefficient of acoustic tube, frequently discussed in literature, refers to specific conditions, when the wave heading the outlet is the plane wave. It follows from the fact that the plane wave is a commonly applied approximation when considering phenomena in duct-like devices or systems (tubes, musical instruments, heating or ventilation systems). The aim of the paper is to extend the concept of the open-end correction on the so-called higher Bessel modes, that under some conditions can also propagate in a duct. Theoretical results, forming the basis for numerical calculations, were obtained by considering diffraction at the duct end and applying the Wiener–Hopf factorization method. As a result, the formula for the acoustic field inside the duct was derived. For each Bessel mode present in the incident wave the reflected wave is composed of all cut-on modes of the same circumferential order. Each mode present in the reflected wave is characterized by the complex reflection/coupling coefficient, argument of which describes phase change at the duct end and therefore the open-end correction coefficient can be attributed to each coupled pair of modes.

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1. Introduction

The aim of this paper consists in an attempt to generalise the concept of the so-called waveguide open end correction onto higher allowable modes that may propagate freely in a cylindrical duct provided the so-called reduced excitation frequency $\omega_{\mathrm{red}} = \omega a/c = ka$ (where ω — radial velocity, a — waveguide radius, c — speed of sound, k — wave number), known also as the diffraction parameter or the Helmholtz number, is higher than the cut-off frequency for the mode in question [1]. For a perfectly rigid waveguide that, in view of its properties, constitutes a good model for many components of machines, devices and/or systems used in practice and for that reason is the most common object of investigation, the principal mode (that with the lowest cut-off frequency equalling 0) is the plane wave [2]. The first non-symmetric mode appears for the reduced excitation frequency $\omega_{\mathrm{red}} > 1.84$, and the next, symmetric, for $\omega_{\mathrm{red}} > 3.83$, i.e. above frequencies corresponding to about 200 Hz and 420 Hz, respectively, in a duct with diameter of 1 m. As the plane wave approximation is the most commonly used approach to phenomena occurring in a cylindrical waveguide, the open end correction values quoted in the literature refer, strictly speaking, to the correction calculated specifically for the plane wave. The plane wave approximation, according to numerous authors [2–5], proves insufficient in many cases and may even lead to false conclusions in such important matters as e.g. directivity of wave energy radiation outside the duct (for the plane wave, the maximum radiation occurs along the duct axis, while no energy at all is emitted in this direction from higher modes [4]). For that reason, especially in the case of waveguides with large diameters or high excitation frequencies, description of phenomena occurring in such system obtained by means of the plane wave approximation may prove too rough. In order to obtain a more adequate description of these effects, closer to these occurring in actual ducts, it may be beneficial to calculate theoretically and/or numerically the duct outlet corrections for modes higher than the plane wave. This in fact is the purpose of this paper.

As a result of wave (plane or higher mode) reflection, a phase change at the duct end occurs that can be associated with argument of the complex reflection coefficient [2, 6–8]. Therefore, acoustic velocity loop and acoustic pressure node are no longer located at the open waveguide outlet, as is usually assumed in basic models of the phenomena. That point occurs a little further outside the

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outlet and therefore, in order to calculate the system's eigenfrequencies, it is necessary to add certain value Δl to the actual waveguide length. It is therefore convenient to define the so-called effective waveguide length:

$$l_{\mathrm{ef}} = l + \Delta l\,, \tag{1}$$

where l is the actual duct length. It follows from the above definition that for a duct with one end open, $l_{\mathrm{ef}} = (2n+1)\lambda/4$. Therefore, Δl represents an increase of the actual duct length compensating diffraction phenomena occurring at its open end and related to change of the reflected wave's phase, that for the problem discussed here can be expressed as the product of the open end correction α and duct radius a, $\Delta l = \alpha a$ [2, 9]. The correction value depends not only on the type of wave propagating towards the outlet but also on geometrical features characterising the outlet, among which one can number existence of a baffle with given radius (from zero, corresponding to the unbaffled duct, to the classical case of infinite plane baffle); duct profile (constant or variable radius, e.g. duct terminated with an acoustic horn); edge type (sharp, rounded etc.) [2, 9–12]. The issue discussed herein constitutes therefore a special case of acoustic wave radiation in the cylindrical waveguide outlet area with given boundary conditions reflecting geometrical and physical characteristics of a system. The problem, for a wide variety of boundary conditions, is extensively represented in the literature, although it must be noted that analytical solutions have been obtained only for a narrow class of configurations. As an example of attempts made to calculate the open end correction value for plane wave, one can quote results obtained in the low frequency approximation ($\omega_{\mathrm{red}} = ka \to 0$) by the following authors:

— Rayleigh (1945): $\alpha = 0.785$ — a limiting case corresponding to baffle radius tending to zero; $\alpha = 0.824$ — for infinite baffle [9];

— Levine and Schwinger (1948): $\alpha = 0.6133$ — for cylindrical waveguide without baffle [10];

— Nomure (1960): $\alpha = 0.8217$ — a result obtained by means of numerical analysis carried out for cylindrical waveguide with infinite baffle [11].

The above analytically obtained open end correction values have been confirmed later by means of numerical and experimental methods carried out for different baffle radii by Ando [12]. Further research revealed that the value is strongly affected by the duct outlet edge geometry [13].

2. Theoretical fundamentals

Solution of the wave equation in semi-infinite perfectly rigid cylindrical waveguide stretching from $z = -\infty$ to $z = 0$, with the outlet at $z = 0$, for harmonic excitation $\exp(\mathrm{i}\omega t)$ has in the cylindrical coordinates (ρ, φ, z) the form

$$\begin{aligned}\Phi_{ml}(\rho,\varphi,z) = A_{ml}\,\mathrm{e}^{\mathrm{i}m\varphi}\Bigg[&\frac{J_m\big(\mu_{ml}\frac{\rho}{a}\big)}{J_m(\mu_{ml})}\,\mathrm{e}^{\mathrm{i}\gamma_{ml}z}\\ &+\sum_{n=1/0}^{N_m} R_{mln}\frac{J_m\big(\mu_{mn}\frac{\rho}{a}\big)}{J_m(\mu_{mn})}\,\mathrm{e}^{-\mathrm{i}\gamma_{mn}z}\Bigg],\end{aligned} \tag{2}$$

where A_{ml} — complex amplitude of the incident wave's mode (m,l); J_m — the Bessel function of order m; R_{mln} — coefficient of reflection/transformation of mode (m,l) into mode (m,n); μ_{ml}, μ_{mn} — roots of derivative of the Bessel function J_m; γ_{ml} — the longitudinal wave number of mode (m,l); N_m — index of the highest mode propagating in a duct at given value of the diffraction parameter ka. The reflection/transformation coefficient is defined as:

$$R_{mln} = \frac{B_{mn}}{A_{ml}}\,, \tag{3}$$

where B_{mn} is the complex amplitude of wave reflected/transformed at the duct open end. The coefficient can be represented by its modulus and phase θ_{mln} according to the formula

$$R_{mln} = -|R_{mln}|\,\mathrm{e}^{\mathrm{i}\theta_{mln}}. \tag{4}$$

Given phase θ_{mln} of the plane wave reflection coefficient, one can calculate the so-called open end correction representing the plane wave phase change occurring at duct outlet normalised by the diffraction parameter ka:

$$\alpha_{mll} = \frac{\theta_{mll}}{2\gamma_{ml}a}\,, \tag{5}$$

where γ_{ml} is the longitudinal wave number

$$\gamma_{ml} = \sqrt{k^2 - \left(\frac{\mu_{ml}}{a}\right)^2}\,. \tag{6}$$

Detailed calculations lead to the following formula for the reflection coefficient phase [2]:

$$\theta_{mln} = \frac{1}{2}\begin{cases} Y_m(\gamma_{ml}) + Y_m(\gamma_{mn}), \\ \quad l+n \text{ — even number}, \\ Y_m(\gamma_{ml}) + Y_m(\gamma_{mn}) + \pi, \\ \quad l+n \text{ — odd number}, \end{cases} \tag{7}$$

where

$$\begin{aligned}Y_m(w) = \frac{2wa}{\pi} - \Omega_m(va) - \mathrm{i}\lim_{N\to\infty}\Bigg[&\sum_{n=N_m+1}^{N}\frac{\gamma_{mn}+w}{\gamma_{mn}-w}\\ &-\frac{1}{\pi}\int_{-\gamma N}^{\gamma N}\frac{\Omega_m(v'a)}{w'-w}\,\mathrm{d}w'\Bigg],\end{aligned} \tag{8}$$

and

$$v = \sqrt{k^2 - w^2}\,. \tag{9}$$

Function $\Omega_m(va)$ is equal, to an additive constant, to argument of derivative of the Hankel function of first order $H_m^{(1)'}$,

$$\Omega_m(va) = \arg H_m^{(1)'}(va) \pm \frac{\pi}{2} = \arctan\frac{N_m'(va)}{J_m'(va)} \pm \frac{\pi}{2}\,, \tag{10}$$

where N_m is the Neumann function of order m, with sign "+" applicable to $m = 0$ and "−" to $m \neq 0$. The argument was selected so that

$$\Omega_m(0) = 0\,, \quad \Omega_m(\mu_{mn}) = \begin{cases} n\pi, & \text{for } m = 0, \\ (n-1)\pi, & \text{for } m \neq 0, \end{cases} \tag{11}$$

where a different value for $m = 0$ results from the fact that roots μ_{mn} of the Bessel function for $m = 0$ are numbered starting from $n = 0$.

It can be seen from the above that for the diffraction parameter values $0 < ka < \mu_{01}$, where $\mu_{01} = 3.83$, only the plane wave may propagate freely in a duct with reflection coefficient

$$\theta_{000} = Y_0(k)\,, \tag{12}$$

while for the diffraction parameter values $\mu_{01} < ka < \mu_{02}$, where $\mu_{02} = 7.02$, both the plane wave and the first symmetric mode are allowed. Phase of the coefficient representing transformation of plane wave into the first allowable mode occurring in the course of reflection at the open end is then

$$\theta_{001} = \frac{1}{2}\big[Y_0(k) + Y_0(\gamma_{01}) + \pi\big]\,. \tag{13}$$

For mode $(0,1)$, phase of the $(0,1) \to (0,1)$ reflection coefficient is simply

$$\theta_{011} = Y_0(\gamma_{01})\,, \tag{14}$$

while phase of the $(0,1) \to (0,0)$ coefficient representing transformation into plane wave can be expressed as

$$\theta_{010} = \frac{1}{2}\big[Y_0(k) + Y_0(\gamma_{01}) + \pi\big]\,. \tag{15}$$

3. Numerical calculation results and conclusions

Numerical calculation of $Y(w)$ values carried out according to (8) in MATLAB computing environment allowed to obtain plots of the open end correction.

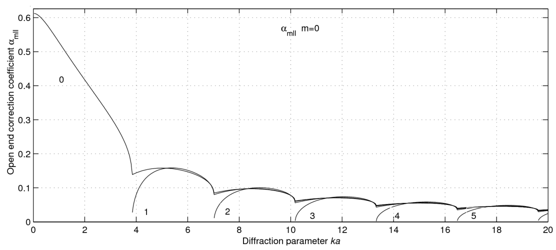

Fig. 1. The open end correction for plane wave $m = 0$, $l = 0$ compared to its values for consecutive axisymmetric modes $l = 1, 2, 3, 4, 5$.

Results obtained for the plane wave case $m = 0$, $l = 0$ (Fig. 1) and diffraction parameter ka from interval (0,3.83), i.e. for reduced frequencies lower than the cut-off frequency for the first symmetric mode ($m = 0$, $l = 1$), are consistent with results obtained by Levine and Schwinger [10].

For higher ka values, results presented in [10] are no longer accurate, as the employed model does not account for the possibility of plane wave transformation into other, higher-order axisymmetric modes (e.g. $m = 0$, $l = 1, 2, \ldots$), occurrence of which affects value of the reflection coefficient R_{000} and, in the process, magnitude of the open end correction α_{000}. Value of this correction for small frequencies ($ka \to 0$) corresponds to figures quoted in the literature, $\alpha_{000} = 0.613$, and applicable to the case of an unbaffled cylindrical duct. The presented graph allows to read out the correction values also for other excitation frequencies.

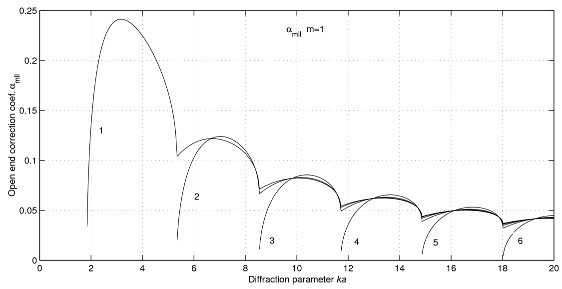

Fig. 2. The open end correction for the first circumferential mode $m = 1$, $l = 1, 2, 3, 4, 5, 6$.

As a rule, open end correction values are small for frequencies slightly above consecutive cut-off frequencies and then increase reaching values close to this obtained for the plane wave.

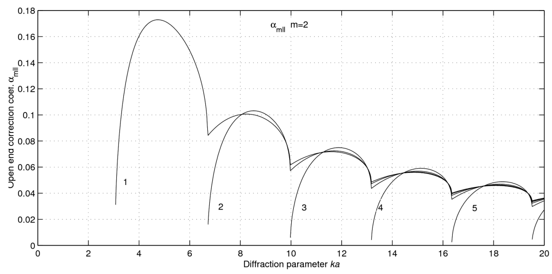

Fig. 3. The open end correction for circumferential mode $m = 2$, $l = 1, 2, 3, 4, 5$.

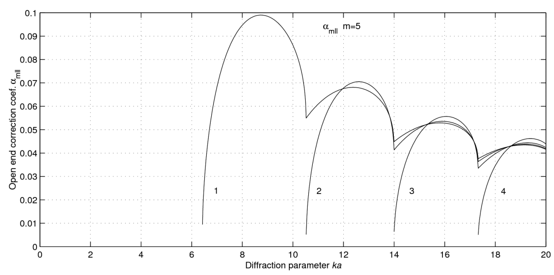

Fig. 4. The open end correction for circumferential mode $m = 5$, $l = 1, 2, 3, 4$.

Figure 2 presents analogously the outlet correction for consecutive modes of the same circumferential order $m = 1$. The first non-symmetric mode appears at $ka = 1.84$, with maximum open end correction α_{111} value amounting to about 0.24.

Figures 3 and 4 present open end correction plots for first few circumferential modes with indices $m = 2$ and $m = 5$ for which maximum values occur always for $l = 1$, on the understanding that for mode $(2, 1)$ the maximum value is about 0.19 compared to about 0.1 for mode $(5, 1)$.

All results presented above indicate strongly mode-dependent nature of the open end correction for cylindrical ducts. Except for the plane wave, values of the parameter for consecutive higher modes are small just above their cut-off frequencies and have local minima at frequencies corresponding to appearance of even higher modes. An interesting feature of the open end correction consists also in the fact that, except for frequencies close to the given mode's cut-off frequency, the values determined for consecutive modes of the same circumferential order m are close to each other which means that for all these modes one obtains similar values of the effective length l_{ef} and therefore loops of acoustic velocity fall approximately at the same distance Δl from the outlet. Analysis of the obtained results allows to define frequency ranges in which taking the open end correction may have significant effect on determination of the examined system parameters containing a duct-type component.

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