Longitudinal Magnetoresistance of Metals Calculated on the Basis of a Single-Band Electron Theory

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The longitudinal magnetoresistance of metals at different temperatures and strengths of the external magnetic field is examined with the aid of a simple single-band theory. The calculated data are compared with the experimental ones. The theoretical ratio between the longitudinal and transversal magnetoresistance approaches unity, whereas the observed ratio is rather below that value. For some metals (Al, Pt) the theoretical results are within the range of the measured data for the transversal and longitudinal resistance, or remain close to the limits of this empirical interval. In average, a relatively good agreement between experiment and theory is obtained for Al, Pt, Cu, Ag and In metals.

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1. Basic properties of the electron model

In general, the longitudinal magnotoresistance of metals seems to be less often investigated than the transversal effect. But a large experimental basis of a study on the magnetoresistance in general, makes a joint theoretical treatment of the transversal and longitudinal phenomena too large to be considered in a satisfactory manner within a framework of a single paper. This motivated me to present the longitudinal problem separately from the transversal one which is examined in [1].

It can be noted that the physical parameters of a metal investigated experimentally are left essentially the same, or almost the same, in both transversal and longitudinal case. Since only directions of the examined current become different, our attention is focused, in the first step, on a comparison of the data for the longitudinal magnetoresistance with those obtained for their transversal magnetoresistance counterparts.

This kind of comparison is still more justified in the framework of the applied theory, because the longitudinal and transversal kind of magnetoresistance are, in fact, approached there on an equal footing. The formulae are based on the same one-electron approximation and, for reasons of the crystal symmetry chosen with respect to the magnetic field, a treatment of the longitudinal problem becomes much similar to that of the transversal one.

The model applied for the electron transport is descending from a single-band approach to the electric resistance which is next modified by the presence of the magnetic field. At the beginning, if the magnetic field is absent and only the electric field is active, the resistance of a metal which is isotropic with respect to a change of the Cartesian coordinate system x, y and z, is represented by a well-known formula

$$\overline{\overline{\varrho}}(0) = \frac{m}{n_{\mathrm{s}} e^2 \tau(0)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

The argument of zero entering both ϱ and the relaxation time τ in (1) refers to the magnetic induction $B = 0$, the electron mass is m and n_{s} is the concentration of the electron carriers.

If the magnetic field is added to the electric field in direction parallel to the axis z, there is usually assumed another resistance tensor, namely [2]:

$$\overline{\overline{\varrho}}(B) = \frac{m}{n_{\mathrm{s}} e^2 \tau(B)} \begin{pmatrix} 1 & -\xi & 0 \\ \xi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where

$$\xi = \tau(B)\Omega_0\,. \tag{3}$$

Here $\tau(B)$ is the relaxation time in the presence of the magnetic field induction B and

$$\Omega_0 = \frac{eB}{mc} \tag{4}$$

is the angular frequency of the electron gyration in the magnetic field. In fact B in (4) is equal to B_z.

Usually it is assumed that in the presence of B different than zero the tensor (2) replaces tensor (1). This implies only a slight, if any, change of $\tau(0)$, so

$$\tau(B) \approx \tau(0)\,. \tag{5}$$

Our aim is to point out that (5) is by no means true. A basic property of $\tau(0)$ is that it does not change appreciably with the change of the intensity of an external electric field. This property does not apply, however, for the magnetic field and $\tau(B)$. A rather simple quantum-mechanical calculation shows (see [1]) that, in fact, only the product ξ presented in (3) is independent of B, but not $\tau(B)$ alone. Therefore, we have that

$$\xi = \mathrm{const}\,. \tag{6}$$

But the same result of (6) can be attained also in an elementary way. The Peierls idea was [3, 4] that the main reason for magnetoresistance is the change of a free-electron progressive motion into a rotational motion due

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to the magnetic field. In reality, the rotational motion is similar to a scattered motion in the sense that electrons change their direction of momentum roughly into an opposite direction. For the circulating electrons this momentum change is coming after the time period

$$\tau = \frac{T_{\rm per}}{2}\,, \tag{7}$$

where $T_{\rm per}$ is a full period of the circulation. Formula (7) holds because after the time equal to $T_{\rm per}/2$ a quasi-progressive motion along an electron orbit performed in one direction is changed into a similar motion in an opposite direction.

The time interval in (7) can be identified with the relaxation time in view of a similar property of $\tau(0)$ in the presence of the electric field alone: this $\tau(0)$ is a time interval in which an electron changes drastically its momentum due to a scattering process on the metal defects. In case of the presence of the magnetic field, the direction change of momentum, instead of defects, is dictated by the electron gyration due to the action of the magnetic field induction of the strength B.

Because of a fundamental relation

$$T_{\rm per} = \frac{2\pi}{\Omega_0}\,, \tag{8}$$

we obtain from (4), (7) and (8) the formula

$$\tau = \frac{\pi}{\Omega_0} = \frac{\pi mc}{eB} = \tau(B)\,, \tag{9}$$

which makes the relaxation time evidently dependent on B. In the next step, the product entering (3), or (6), becomes

$$\xi = \tau(B)\Omega_0 = \pi\,. \tag{10}$$

This result is applied in calculations in the remainder part of the present paper.

2. The change of the electric resistance due to the magnetic field

We assume that an effective resistance tensor $\overline{\overline{\varrho}}^{\,\rm tot}$ which reproduces the electric field components E_λ from the current components j_ν, namely (see e.g. [2])

$$E_\lambda = \sum_\nu \varrho^{\rm tot}_{\lambda\nu} j_\nu \tag{11}$$

is a sum $\overline{\overline{\varrho}}(0)$ in (1) and $\overline{\overline{\varrho}}(B)$ in (2):

$$\overline{\overline{\varrho}}^{\,\rm tot}(B) = \overline{\overline{\varrho}}(0) + \overline{\overline{\varrho}}(B)\,. \tag{12}$$

Evidently, the component E_z in (11) which is directed along the magnetic field is provided by a sum

$$\varrho^{\rm tot}_{zz}(B) = \varrho_{zz}(0) + \varrho_{zz}(B)\,, \tag{13}$$

which on the basis of (1) and (2) becomes

$$\varrho^{\rm tot}_{zz}(B) = \frac{m}{n_{\rm s}e^2}\left[\frac{1}{\tau(0)} + \frac{1}{\tau(B)}\right]. \tag{14}$$

Therefore, in the presence of both the electric and magnetic field, the effective relaxation time is

$$\frac{1}{\tau_{\rm eff}} = \frac{1}{\tau(0)} + \frac{1}{\tau(B)}\,. \tag{15}$$

The formula (15) represents an application of the Matthiessen rule for the component relaxation times entering (1) and (2).

A constant ξ taken from (3) or (10) implies that

$$\frac{1}{\tau(B)} = \frac{\Omega_0}{\xi} = \frac{\Omega_0}{\pi}\,. \tag{16}$$

In the computational practice applied to the longitudinal magnetoresistance we are interested in the ratio

$$\frac{\Delta\varrho_{zz}}{\varrho_{zz}(0)} = \frac{\varrho^{\rm tot}_{zz}(B) - \varrho_{zz}(0)}{\varrho_{zz}(0)} = \frac{\frac{1}{\tau(0)} + \frac{1}{\tau(B)} - \frac{1}{\tau(0)}}{\frac{1}{\tau(0)}}$$

$$= \frac{\tau(0)}{\tau(B)} = \frac{\Omega_0\tau(0)}{\pi}\,. \tag{17}$$

The last step in (17) is obtained from the formula (16).

TABLE I

Empirical ratio β of the longitudinal and transversal magnetoresistance of metals examined in the present paper; see also Tables II–VI and [7]. The theoretical β [see (18)] approaches unity.

Metal	Temperature [K]	Magnetic field [kG]	β
Na	20.4	*ca.* 16.5	0.37
Be	*ca.* 79	*ca.* 4.6	0.07
	ca. 80	*ca.* 6.75	0.07
Al	77.2	5.2	0.38
		20.1	0.38
		23.5	0.35
	20.4	4.1	0.39
		10.3	0.44
		14.6	0.45
		23.5	0.47
	14.2	10.3	0.43
		14.6	0.45
		23.5	0.47
Pt	20.4	*ca.* 32	0.51
	14.0	*ca.* 32	0.47
	4.22	*ca.* 32	0.30
Cu	78	100	0.33
		200	0.41
		300	0.49
	4.2	30	0.73
		60	0.64
		90	0.51
		120	0.41
		150	0.35
Cd	78	100	0.67
		200	0.75
		300	0.76
Ga	*ca.* 79	100	1.07
		200	1.10
		300	1.12
In	20.4	*ca.* 25	0.54
	14.0	*ca.* 25	0.34

TABLE II

Theoretical longitudinal magnetoresistance compared with similar transversal magnetoresistance [see (17) and (18)] and the experimental data [7]. Metals of Ia and IIa group of the periodic table are taken into account. The constant ξ is taken from (10), the empirical correcting factor r fits the relaxation time $\tau(0)$ from Table VII to the measurement temperature; see (19). Symbol tr indicates the transversal data.

Metal symbol	Temperature [K]	Factor r	Magnetic field [kG]	Theory	Experiment
Na	78	0.189	16.5	0.0157	0.0006
	20.4	0.00435	16.5	0.68	0.029
	20.4	0.0043	16.0	0.67tr	0.078tr
K	14.0	0.0155	35.0	0.52	0.0249
Be	79.7	0.0114	4.85	0.012	0.19
	79.1	0.0127	4.49	0.010tr	2.56tr
	79.7	0.0114	6.7	0.017	0.23
	78.9	0.0171	6.8	0.011tr	3.08tr

TABLE III

Theoretical longitudinal magnetoresistance compared with similar transversal magnetoresistance [see (17) and (18)] and the experimental data [7]. Here only one IIIa group metal (Al) is taken into account. The meaning of parameters applied in calculations is the same as in Table II. A printing error of r in [7] for the temperature 20.4 K is removed by taking r from the experimental data for the transversal magnetoresistance given in the same reference handbook.

Metal symbol	Temperature [K]	Factor r	Magnetic field [kG]	Theory	Experiment
Al	78	0.121	100	0.037	0.078
			200	0.074	0.216
			300	0.152	0.358
	77.2	0.158	5.2	0.0014	0.0003
	77.2	0.161	5.2	0.0014tr	0.0008tr
			12.6	0.0036	0.0012
			20.1	0.0057	0.0033
			20.1	0.0056tr	0.0087tr
			23.5	0.0067	0.0038
			23.5	0.0065tr	0.0110tr
	20.4	0.0676	4.1	0.0027	0.0013
		0.0675	4.1	0.0027tr	0.0033tr
			10.3	0.0068	0.0075
			10.3	0.0068tr	0.0172tr
			14.6	0.0097	0.0138
			14.6	0.0097tr	0.0308tr
			23.5	0.0155	0.0299
			23.5	0.0156tr	0.0640tr
	14.2	0.0668	4.1	0.0028	0.0016
		0.0665			
			10.3	0.0069	0.0080
			10.3	0.0069tr	0.0185tr
			14.6	0.0098	0.0146
			14.6	0.0098tr	0.0326tr
			23.5	0.0158	0.0316
			23.5	0.0158tr	0.0678tr

A characteristic point obtained because of symmetry assumed for tensors (1) and (2) is that the transversal magnetoresistance calculated in x and y directions of the coordinate axes is expressed with the aid of the same formula as the longitudinal magnetoresistance in (17) examined along the direction of z. We have

$$\frac{\Delta\varrho_{xx}}{\varrho_{xx}(0)} = \frac{\Delta\varrho_{yy}}{\varrho_{yy}(0)} = \frac{\Delta\varrho_{zz}}{\varrho_{zz}(0)} = \frac{\Omega_0}{\xi}\tau(0)\,. \tag{18}$$

In the remainder of the paper the results of (17) and (18) are applied in calculating the magnetoresistance of several metal cases. The calculations are next compared (Tables I–VI) with the experimental data. The measurements depend both on B and the metal temperature. The same property holds for the theoretical data: the dependence on temperature is due to the presence of $\tau(0)$ in (17) which is also a temperature dependent parameter.

TABLE IV

Theoretical longitudinal magnetoresistance compared with transversal magnetoresistance [see (17) and (18)] and the experimental data [7]. Only one VIIIa group metal (Pt) is taken into account. The meaning of parameters applied in calculations is the same as in Table II.

Metal	Temperature [K]	Factor r	Magnetic field [kG]	Theory	Experiment
Pt	20.4	0.0066	33.47	0.26	0.146
		0.0067	30.7	0.23tr	0.285tr
	14.0	0.0034	33.4	0.50	0.333
		0.0034	30.7	0.46tr	0.716tr
	4.22	0.0024	33.4	0.70	0.405
		0.0022	30.7	0.70tr	1.338tr

TABLE V

Theoretical magnetoresistance of two Ib group metals (Cu, Ag) compared with the experimental data [7]. For Cu both longitudinal and transversal resistance is examined [see respectively Eqs. (17) and (18)], for Ag only the longitudinal resistance calculated according to (17) is considered. The meaning of parameters applied in the calculations is the same as in Table II.

Metal	Temperature [K]	Factor r	Magnetic field [kG]	Theory	Experiment
Cu	78	0.155	100	0.098	0.03
		0.141		0.108tr	0.09tr
			200	0.196	0.11
				0.217tr	0.27tr
			300	0.293	0.23
				0.325tr	0.47tr
	4.2	0.0089	30	0.51	0.337
		0.0089		0.51tr	0.461tr
			60	1.02	0.719
				1.02tr	1.121tr
			90	1.53	1.018
				1.53tr	1.99tr
			120	2.04	1.318
				2.04tr	3.18tr
			150	2.55	1.61
				2.55tr	4.64tr
Ag	20	0.0096	20	0.467	0.190
	10	0.0067	20	0.669	0.288
	5	0.0064	20	0.700	0.302

An experimental factor of r depending on the absolute temperature T couples $\tau(0)$ at the temperature of the resistance measurement with $\tau(B=0, T=273\ \mathrm{K})$ which is the relaxation time $\tau(0)$ at $T=273$ K. These $\tau(0)$ are listed in numerous tables (see [5, 6]), in the present paper they are reported in Table VII.

We have the relation

$$\frac{1}{r}\tau(B=0,\ T=273\ \mathrm{K}) = \tau(B=0,\ T)\,. \tag{19}$$

On the right-hand side of (19) T is the temperature of the resistance measurement, the factor r on the left-hand side can be obtained from the experimental data for the ratio of the electric resistance at given T to the electric resistance at $T=273$ K; see [7]. Evidently, the relaxation time $\tau(0)$ increases with a decrease of T.

The measurements of the magnetoresistance are done regularly at T smaller than 273 K. In consequence, r in (19) is usually smaller than 1 and $\tau(0)$ entering (17) and (18) becomes much larger than $\tau(0)$ listed in Table VII.

The field B in kilogauss is substituted into the formulae (17) and (18) which give the data of Tables I–VI. In the metal practice the number of gauss is equal to that of oersteds quoted in numerous experiments (see e.g. [8]). A useful formula

$$\Omega_0^{(1\mathrm{G})} = 1.76\times 10^7\ \mathrm{radians/s} \tag{20}$$

transforms the gauss units into the frequency Ω_0 applied in the calculations.

An evident consequence of (17) and (18) is that both transversal and longitudinal magnetoresistance should be proportional to B; see (4). This property is roughly confirmed for all measured data reported in Tables I–VI.

TABLE VI

Longitudinal and transversal magnetoresistance of IIb and IIIb group of metals (Cd, Ga, In). Theoretical data of (17) and (18) are compared with the experimental data in [7]. The meaning of parameters applied in calculations is the same as in Table II.

Metal	Temperature [K]	Factor r	Magnetic field [kG]	Theory	Experiment
Cd	78	0.22	100	0.0143	0.161
		0.22		0.0143tr	0.24tr
			200	0.0285	0.435
				0.0285tr	0.58tr
			300	0.0428	0.705
				0.0428tr	0.93tr
Ga	78	0.2	100	0.0048	0.30
	80	0.21		0.0045tr	0.28tr
			200	0.0095	0.625
				0.0091tr	0.57tr
			300	0.0143	0.96
				0.0136tr	0.86tr
In	20.4	0.023	25.74	0.024	0.045
		0.023	25.0	0.023tr	0.083tr
	14.0	0.0105	25.74	0.052	0.101
		0.0085	25.0	0.063tr	0.30tr

3. Discussion

As a rule the transversal magnetoresistance data attained experimentally at given B and T are larger than the longitudinal ones; see Table I where the empirical ratios β of the longitudinal measurements to the transversal ones are listed for several metals. An exception is the gallium metal for which approximately $\beta \approx 1$. In general, our theory, which implies an equal size of resistance in both longitudinal and transversal field directions [see (18)], cannot explain $\beta < 1$. Nevertheless, the discrepancies of the experimental β from the theoretical result of $\beta = 1$ are not large. Moreover, for many measurements, the theoretical data are between the experimental results obtained separately for the transversal and longitudinal case, or close to this transversal-longitudinal interval; see especially Al in Table III and Pt in Table IV.

The theoretical ratio (17) can be referred to the corresponding experimental ratio obtained from [7]. This gives

$$s = \frac{[\Delta\varrho_{zz}/\varrho_{zz}(0)]^{\text{theor}}}{[\Delta\varrho_{zz}/\varrho_{zz}(0)]^{\text{exp}}} \,. \tag{21}$$

The average of s calculated for different metals are given in Table VIII. A perfect agreement between the theory and experiment implies obviously

$$s = 1 \,. \tag{22}$$

The metals whose s are, in average, not far from the result in (19) are represented by Al, Co, Pt, Cu, In, also Ag. The remainder of the examined metals (Na, K, Be, Cd, Ga) give much poorer results.

It should be noted that no correction for the effective mass of a metal electron is considered in the paper. Also a kind of preparation of the metal sample, neglected in the present approach, can much influence the ratio s.

TABLE VII

Relaxation times $\tau(0)$ (in 10^{-14} s) of metals examined in the present paper [5, 6]. These $\tau(0)$ refer to the temperature of 273 K.

Na	K	Be	Al	Pt
3.2	4.1	0.51	0.8	0.9
Cu	Ag	Cd	Ga	In
2.7	4.0	0.56	0.17	0.38

TABLE VIII

Average value of the ratio s of Eq. (21) calculated for different metals on the basis of the theoretical and experimental data for the longitudinal magnetoresistance collected in Tables I–V.

Na	K	Be	Al	Pt
24.5	20.7	0.065	0.73	1.67
Cu	Ag	Cd	Ga	In
1.7	2.4	0.07	0.018	0.52

Moreover, the polycrystalline solids examined with the aid of the present formalism should be expected to give, in general, better s than monocrystals. This is so because the magnetoresistance of monocrystals can be much sensitive to the crystal orientation with respect to direction of the magnetic field; see e.g. [2, 9].

A special interest is often attached to Bi which, in fact, is a semimetal. At the field of $B = 10$ kG the experiment shows [7], in average,

$$\frac{\Delta\varrho_{zz}}{\varrho_{zz}(0)} = 0.17 \tag{23}$$

obtained at the measurement temperature of 289 K which is not much different than 273 K. This implies the approximate value of $r \approx 1$.

A subsitution of $\tau(0) = 0.023 \times 10^{-14}$ s (see [5]), Ω_0 for $B = 10$ kG from (19) and ξ of (10) into the formula (17) gives

$$\frac{\Delta\varrho_{zz}}{\varrho_{zz}(0)} = \frac{10^4 \times 1.76 \times 10^7 \times 0.023 \times 10^{-14}}{\pi} \approx 0.13 \times 10^{-4}. \tag{24}$$

The ratio of the theoretical result (24) to the experimental one in (23) amounts to

$$s = 0.76 \times 10^{-4}, \tag{25}$$

which is much different from any s presented in Table VIII.

4. Summary

A change of the electric resistance of a metal, due to the presence of an external magnetic field and measured parallel to that field, is examined theoretically for several metal cases. The data are calculated with the aid of a simple single-band theory in which two relaxation times, that due to the action of the electric field and that dependent on the magnetic field, are separately considered and next combined according to the Matthiessen rule. The calculated results exhibit, in general, a proportionality of the magnetoresistance effect to the strength of the magnetic field induction B.

The magnetoresistance data obtained for individual metal cases are compared with experiment for the longitudinal magnetoresistance, but also, where possible, with the results for the transversal magnetoresistance. In average, a relatively good agreement between experiment and theory is obtained for Al, Pt, Cu, Ag and In metals. In many cases the longitudinal results are not much different from the transversal ones, in accordance with the theoretical expectations.

References

[1] S. Olszewski, M. Gluzinski, *Z. Naturforsch. A* **66**, 311 (2011).

[2] C. Kittel, *Quantum Theory of Solids*, 2nd ed., Wiley, New York 1976.

[3] R. Peierls, *The Behaviour of Metallic Conductors in Strong Magnetic Fields*, Leipziger Vortrage, S. Hirzel, Leipzig 1930, p. 75, (in German).

[4] R. Peierls, *Ann. Phys. (Leipzig)* **10**, 97 (1931).

[5] N.W. Ashcroft, N.D. Mermin, *Solid State Physics*, Holt, Rinehart & Winston, New York 1976.

[6] N.F. Mott, H. Jones, *Theory of the Properties of Metals and Alloys*, Clarendon Press, Oxford 1958.

[7] Landoldt-Bornstein 1959, *Numeric Values and Functions*, Vol. 2, Part 6, Springer, Berlin 1959.

[8] D.J. Griffith, *Introduction to Electrodynamics*, 3rd ed., Prentice Hall, New Jersey 1999.

[9] Y.P. Gaidukov, *Zh. Eksp. Teor. Fiz.* **37**, 1281 (1959).