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On the Applying of the Van Der Pauw Method to Anisotropic Media

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The paper deals with the ways of finding an electrical conductivity tensor of a plane and anisotropically conductive sample. On the basis of the analysis of currents flowing through the electrodes at vertices of a rectangular sample, we derived specific electrical conductivity tensor calculation formula. The novelty of the present article is the use of the rectangular shape sample with arbitrarily directed sides in respect of the principal axes of the tensor. The influence of values of electrode lengths and the tensor components on the accuracy of these formulae is investigated. An iterative algorithm of increase of precision of calculating the tensor's components is presented.

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1. Introduction

The paper examines the methods of calculating of the specific electric conductivity tensor

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}, \quad \det\sigma = \sigma_{11}\sigma_{22} - (\sigma_{12})^2 > 0 \quad (1)$$

of the sample, which is of a thin ($d \ll a, b$) rectangular shape $a \times b$ and which has anisotropic electric conductivity. Let us assume that current electrodes of identical form are installed on the vertices of the sample and the following electric measurements are done (Fig. 1), where $I_\mathrm{h}, I_\mathrm{v}$ is strength of current that flows in horizontal and vertical directions; $\Delta\varphi_\mathrm{h} = |\varphi_4 - \varphi_3|$ and $\Delta\varphi_\mathrm{v} = |\varphi_3 - \varphi_2|$ are, respectively, the absolute values of potential differences.

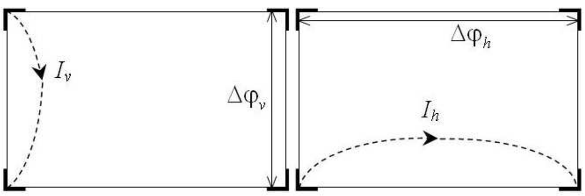

Fig. 1. Measurements of current and potential differences for finding the parameters k and $\det\sigma$.

In Refs. [1] and [2] Price has examined the case where lengths l of the electrodes of the sample are extremely small and sides of the sample are parallel to the principal axes of the tensor (1), i.e. when the tensor may be expressed in the following form:

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad (2)$$

and proved that the components σ_1, σ_2 may be calculated by solving the equation: $\frac{\Delta\varphi_\mathrm{h}}{I_\mathrm{h}} = \frac{-8}{\pi d\sqrt{\det\sigma}} \ln\left(\tanh\left(\sqrt{\frac{\sigma_1}{\sigma_2}}\frac{b}{a}\frac{\pi}{2}\right)\right)$, and the number $\sqrt{\det\sigma}$ satisfies the Van der Pauw Eq. [3]

$$\exp\left(-\pi d\sqrt{\det\sigma}\frac{|\Delta\varphi_\mathrm{h}|}{I_\mathrm{h}}\right) + \exp\left(-\pi d\sqrt{\det\sigma}\frac{|\Delta\varphi_\mathrm{v}|}{I_\mathrm{v}}\right) = 1\,. \quad (3)$$

The main disadvantage of this method is that directions of the principal axes of the tensor a priori are not known, therefore it is especially difficult to prepare a sample. In order to avoid this problem and calculate all three components of the tensor (1) (irrespective of a manner of cutting the sample), article of the authors [4] provide a system of transcendent equations that are designed for calculating the latter. The parameters of the system are found by using the sample, which already has five electrodes, and performing more measurements.

We would like to stress novelty of the present article. This article concentrates on finding the tensor (1) and solutions that are applied therein are based on the mathematical methods (especially — conformal mapping) which are used in the aforementioned articles. However we use a sample with arbitrarily directed (in respect of the principal axes of the tensor) sides and present simple formulae of calculation of all three components of the tensor of general form (1), as well as an iteration algorithm of increase of precision of calculating the tensor's components.

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2. Method of determination

First of all we shall review results achieved in [2] and [4]. After a linear transformation (not conformal) of the variables, in case of the rectangular shape sample, the conduction of which is expressed by tensor σ, we may examine a parallelogram shape sample with electric conductivity, which is of isotropic type and which equals to $\sqrt{\det \sigma}$. Measurements of such parallelogram are indicated in Fig. 2b. Having made a conformal mapping

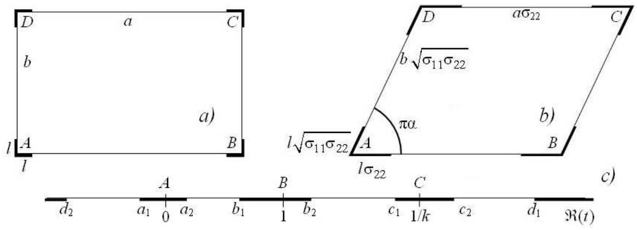

Fig. 2. Mapping of a rectangular form sample into parallelogram and image of the sample contour at the real axis.

of the top complex half-plane t to the parallelogram with the integral $w = a\sigma_{22}\int_0^t f_{\alpha,k}(q)\,\mathrm{d}q/A_{\alpha,k}$, where $f_{\alpha,k}(q) = q^{\alpha-1}(1-q)^{-\alpha}(1-kq)^{\alpha-1}$, $A_{\alpha,k} = \int_0^1 |f_{\alpha,k}(q)|\,\mathrm{d}q$, we obtain images of electrodes on the real axis $\Re(t)$, which are indicated in Fig. 2c. Let us assume that the values $\alpha, k, \det\sigma$ are known. In such case the components of the unknown σ may be calculated according to the following formulae:

$$\sigma_{11} = \lambda \frac{a}{b}\frac{A_{\alpha,k}}{A_{\alpha,1-k}}\,, \qquad \sigma_{22} = \lambda \frac{b}{a}\frac{A_{\alpha,1-k}}{A_{\alpha,k}}\,,$$

$$\sigma_{12} = \lambda\cos(\alpha\pi)\,, \tag{4}$$

where $\lambda = \sqrt{\det\sigma}/\sin(\alpha\pi)$. Thus the problem is to find values of $\alpha, k, \det\sigma$. We shall note that integrals $A_{\alpha,k}$ are hypergeometric functions which are used in majority mathematical software as standard functions (for example Maple environment), therefore calculation of their values causes no problem. When electrodes are short ($l \approx 0$), the articles [2] and [4] state that $\det\sigma$ is found by solving Van der Pauw Eq. (3), and

$$k = \exp\left(-\pi d\sqrt{\det\sigma}\,\frac{|\Delta\varphi_{\mathrm{h}}|}{I_{\mathrm{h}}}\right). \tag{5}$$

Now we shall present a new method of calculation of the outstanding unknown parameter α. We shall perform measurements of the currents I_1 and I_2, the directions of which coincide with diagonals (Fig. 3). If in calculation of I_1 we do not take into account the impact of electrodes B and D and consider that a difference of potentials between electrodes equals to V, we get

$$I_1 = Vd\sqrt{\det\sigma}$$
$$\times\,\frac{\int_{a_1}^{a_2} |(q-a_1)(q-a_2)(q-c_1)(q-c_2)|^{-1/2}\,\mathrm{d}q}{\int_{a_2}^{c_1} |(q-a_1)(q-a_2)(q-c_1)(q-c_2)|^{-1/2}\,\mathrm{d}q}\,. \tag{6}$$

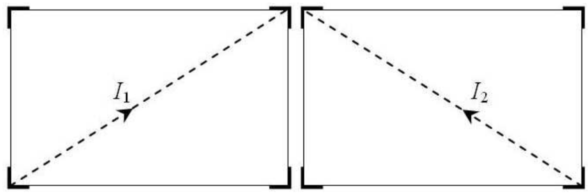

Fig. 3. Measurements of current for finding the parameter α.

We shall calculate approximate values of integral limits a_1, a_2, c_1, c_2. While paying account to the lengths of electrodes as indicated in Fig. 2b,c and presuming that $l \approx 0$, we obtain $a_1, a_2 \approx 0, c_1, c_2 \approx 1/k, f_{\alpha,k}(q) \approx q^{\alpha-1}$. Thus

$$l\sigma_{22} = \frac{a\sigma_{22}}{A_{\alpha,k}}\int_0^{a_2} f_{\alpha,k}(q)\,\mathrm{d}q \approx \frac{a\sigma_{22}}{A_{\alpha,k}}\int_0^{a_2} q^{\alpha-1}\,\mathrm{d}q$$
$$= \frac{a\sigma_{22}}{A_{\alpha,k}}\frac{a_2^{-\alpha}}{\alpha}\,,$$

therefore $a_2 = (\frac{l\alpha A_{\alpha,k}}{a})^{1/\alpha}$. Having followed by the same way and having marked $s_\alpha = (\sqrt{\sigma_{11}/\sigma_{22}})^{1/\alpha}$ we obtained the approximate expressions of the outstanding limits: $a_1 = -a_2 s_\alpha, c_1 = [1-(1-k)a_2 s_\alpha]/k$, $c_2 = [1+(1-k)a_2]/k$. Now, when expanding integrals of the current I_1 into series within the neighborhood of the parameter $a_2 = 0$ (or $l = 0$, because $a_2 \to 0$ only when $l \to 0$), we obtain: $I_1 = Vd\sqrt{\det\sigma}\frac{\pi/2+\mathrm{O}(l^2)}{\ln\left(\frac{4}{(1+s_\alpha)\sqrt{k(1-k)}}\right)-\ln(a_2)+\mathrm{O}(l)}$. Similar expression of I_2 will be obtained after replacing α with $1-\alpha$, therefore relation of the currents I_1 and I_2 at the accuracy of the vanishing parameter l equals to

$$\frac{I_1}{I_2} = \Big\{\ln\left(4/(1+s_{1-\alpha})\sqrt{k(1-k)}\right)$$
$$-\frac{1}{1-\alpha}\left[\ln(l/a)+\ln((1-\alpha)A_{\alpha,k})\right]\Big\}$$
$$\Big/\Big\{\ln\left(4/(1+s_\alpha)\sqrt{k(1-k)}\right)$$
$$-\frac{1}{\alpha}\left[\ln(l/a)+\ln(\alpha A_{\alpha,k})\right]\Big\}\,. \tag{7}$$

Having proceeded in Eq. (7) to the limit, where $l \to 0$, we obtain the equation $\frac{I_1}{I_2} = \frac{\alpha}{1-\alpha}$ and its solution

$$\alpha = \frac{1}{1+I_2/I_1}\,. \tag{8}$$

3. Discussion and conclusions

1. Peculiarity of the proposed method lies in the fact that complexity of the formulae (3), (4), (5), (8) of calculation of tensor's components and necessary measurements does not exceed complexity of the formulae and measurements of the classic Van der Pauw method. Moreover, they do not depend on the size of the rectangular shape sample and length of electrodes.

2. All formulae of the presented method become accurate only within the limit, where lengths of all electrodes

l approach zero (similar to the first equation (3) in Van der Pauw article [3]). However, lengths of electrodes of the sample are not equal to zero, therefore an important task of the assessment of the errors in the presented method originates. If we agree that the direct goal is finding the three parameters $|\Delta\varphi_{\mathrm{h}}|/I_{\mathrm{h}}, |\Delta\varphi_{\mathrm{v}}|/I_{\mathrm{v}}, I_1/I_2$, when components $\sigma_{11}, \sigma_{22}, \sigma_{12}$ of the tensor are known, the method which is presented in this article is the solution of the inverse problem. When assessing the accuracy of solution of the inverse problem we avail of the fact that the direct problem with electrodes of any length l may be sufficiently accurately solved by applying methods of complex analysis [5].

3. Everybody knows high accuracy of Van der Pauw (3) solution $\sqrt{\det\sigma}$ even in case of large enough l. We shall examine errors of solution of Eq. (8) of parameter α. As already mentioned, the solution becomes accurate when $l = 0$ or in case of isotropic conduction ($\alpha = 1/2$), because in the latter case $I_1 = I_2$. Figure 4 illustrates the errors of the already calculated α values for the entire interval of variation of the parameter α. This interval depends upon the expression of the tensor on the principal axes (2), i.e. upon ratio of its components σ_1/σ_2 and may be defined in the inequality: $|\alpha - 1/2| \le \frac{1}{\pi}\arcsin\left(\left|\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}\right|\right)$. Figure 4 reveals that in the cases where $\sigma_1/\sigma_2 \le 4$ and $l/a \le 0.05l$, which occur most often, the absolute errors of the calculated α values do not exceed 0.02 and relative errors do not exceed 5%.

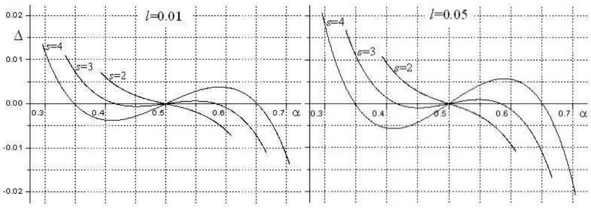

Fig. 4. Distribution of absolute errors Δ of parameter α (calculations performed by expression (8)) dependent on the ratio of the principle tensor components σ_1/σ_2 and the length of electrodes l. The sample dimensions are $a = b = 1$.

4. α is just a transitional parameter of the problem to be solved, therefore we shall examine errors of components of the main problem — the calculated tensor. Figure 5 shows that relative errors of the components σ_{11}, σ_{22} do not exceed 6% and hardly comprise tenth parts of a percent, while in case which was examined in [2] $\sigma_1/\sigma_2 \le 1.5$. For this reason we state that the presented method of solving the problem may be applied in practice when calculating σ_{11}, σ_{22}. However, we have to note that accuracy of the calculated σ_{12} is lower and its maximum relative error may reach 15%. Such difference in accuracy stems from formulae (4), because, for example, if the error of the calculated parameter α equals to 0.02, a variation of the function $\cos(\alpha\pi)/\sin(\alpha\pi)$ several times exceeds $1/\sin(\alpha\pi)$.

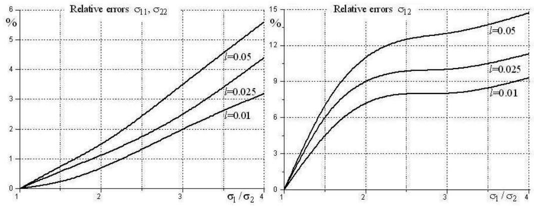

Fig. 5. Distribution of maximal relative errors (%) of tensor components σ_{11}, σ_{22} and σ_{12} (calculation performed by expressions (3)–(5), (8)) dependent on the ratio of the principle tensor components σ_1/σ_2 and the length of electrodes l. The sample dimensions are $a = b = 1$.

5. Accuracy of approximate solution $\sigma^{(0)} = (\sigma_{11}^{(0)}, \sigma_{22}^{(0)}, \sigma_{12}^{(0)})$ which was reached after applying the present method may be significantly increased if geometric sizes a, b, l of the sample are known (Fig. 2a). Let us mark the data obtained during the experiment by vector $\boldsymbol{E} = (|\Delta\varphi_{\mathrm{h}}|/I_{\mathrm{h}},\ |\Delta\varphi_{\mathrm{v}}|/I_{\mathrm{v}},\ I_1/I_2)$ and may the solution of the forward problem be realized in the following functions:

$$\begin{cases} f_1(\sigma_{11}, \sigma_{22}, \sigma_{12}) = |\Delta\varphi_{\mathrm{h}}|/I_{\mathrm{h}} = E_1, \\ f_2(\sigma_{11}, \sigma_{22}, \sigma_{12}) = |\Delta\varphi_{\mathrm{v}}|/I_{\mathrm{v}} = E_2, \\ f_3(\sigma_{11}, \sigma_{22}, \sigma_{12}) = I_1/I_2 = E_3. \end{cases} \tag{9}$$

System of Eqs. (9) (in respect of the unknown parameters $\sigma_{11}, \sigma_{22}, \sigma_{12}$) may be solved by applying the Newton method, while assigning the obtained approximate solution $\sigma^{(0)} = (\sigma_{11}^{(0)}, \sigma_{22}^{(0)}, \sigma_{12}^{(0)})$ to the primary approximation (upon the aforementioned method).

6. This method of solution proved to be especially effective and may be applied for a wide range of samples. For example, in the above examined samples we saw that the relative error may reach 15%. Meanwhile, after single iteration (10) errors of all components do not exceed 0.5%.

References

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