Optical and Acoustical Methods in Science and Technology

Determination of Electromagnetic Emitter Position from Passive Radar Bearings as Linear Least Squares Problem

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Mathematical formulation of an inverse problem of radiolocation is discussed. Determination of electromagnetic emitter position from passive radar bearings is transformed into a linear least squares problem and equivalent system of normal equations. Eventually spatial localisation of emitter is reduced to solving system of linear algebraic equations with entries of coefficient matrix and free term depending on bearings in a nonlinear way. Regularisation algorithm is proposed.

PACS numbers: 41.20.Jb, 03.50.De

1. Introduction

Objects in air space investigated by passive radars (i.e. receiving signals only) can be mostly approximated well as point electromagnetic emitters. Hence localisation of such objects in three-dimensional space equipped with Cartesian coordinates $Oxyz$ can be reduced to finding a position $\boldsymbol{p} = [x, y, z]$ of an object-representing point emitter from bearings produced by a system of passive radar stations. Let Oxy be plane approximating Earth surface in operation area of a system of n ($n > 1$) radar stations with fixed positions $\boldsymbol{p}_i = [x_i, y_i, z_i]$, $i = 1, 2, \ldots, n$. In local spherical coordinates (of origin in a station position) a geometrical bearing from each station is registered as azimuth ϑ_i (in Oxy plane) and elevation β_i (perpendicularly to Oxy plane) of radius connecting a station and an investigated object, $[x - x_i, y - y_i, z - z_i]$, that relates the unknown object position by two equations

$$\frac{x - x_i}{y - y_i} = \tan\vartheta_i$$

and

$$\frac{z - z_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} = \tan\beta_i\,, \tag{1}$$

where $-\pi < \vartheta_i \le +\pi$ and $-\frac{1}{2}\pi \le \beta_i \le +\frac{1}{2}\pi$ are assumed [1 ÷ 5]. The distance between a station and an emitter is not measured in the radiolocation model considered here.

2. Formulation of problem

For each bearing (ϑ_i, β_i) the system of Eqs. (1) can be transformed into a system of equations linear with respect to unknown coordinates of emitter position, in dependence of signs of values of trigonometric functions involved

$$x - \tan\vartheta_i y = x_i - \tan\vartheta_i y_i$$

and

$$-\tan\beta_i y + \cos\vartheta_i z = -\tan\beta_i y_i + \cos\vartheta_i z_i\,,$$

for ($|\vartheta_i| \le \frac{1}{4}\pi$ or $|\vartheta_i| \ge \frac{3}{4}\pi$) and $|\beta_i| \le \frac{1}{4}\pi$,

$$x - \tan\vartheta_i y = x_i - \tan\vartheta_i y_i$$

and

$$y - \cos\vartheta_i \cot\beta_i z = y_i - \cos\vartheta_i \cot\beta_i z_i\,,$$

for ($|\vartheta_i| \le \frac{1}{4}\pi$ or $|\vartheta_i| \ge \frac{3}{4}\pi$) and $\frac{1}{4}\pi < |\beta_i| \le \frac{1}{2}\pi$;

$$-\cot\vartheta_i x + y = -\cot\vartheta_i x_i + y_i$$

and

$$-\tan\beta_i x + \sin\vartheta_i z = -\tan\beta_i x_i + \sin\vartheta_i z_i\,,$$

for $\frac{1}{4}\pi < |\vartheta_i| < \frac{3}{4}\pi$ and $|\beta_i| \le \frac{1}{4}\pi$;

$$-\cot\vartheta_i x + y = -\cot\vartheta_i x_i + y_i$$

and

$$x - \sin\vartheta_i \cot\beta_i z = x_i - \sin\vartheta_i \cot\beta_i z_i\,,$$

for $\frac{1}{4}\pi < |\vartheta_i| < \frac{3}{4}\pi$ and $\frac{1}{4}\pi < |\beta_i| \le \frac{1}{2}\pi$. These equations can be rewritten in the following equivalent matrix forms:

$$\boldsymbol{A}_i\boldsymbol{p} = \boldsymbol{a}_i\,, \quad \boldsymbol{A}_i \equiv \begin{bmatrix} 1 & -\tan\vartheta_i & 0 \\ 0 & -\tan\beta_i & \cos\vartheta_i \end{bmatrix},$$

$$\boldsymbol{a}_i \equiv \begin{bmatrix} x_i - \tan\vartheta_i y_i \\ -\tan\beta_i y_i + \cos\vartheta_i z_i \end{bmatrix}, \tag{2}$$

for ($|\vartheta_i| \le \frac{1}{4}\pi$ or $|\vartheta_i| \ge \frac{3}{4}\pi$) and $|\beta_i| \le \frac{1}{4}\pi$;

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$$\boldsymbol{B}_i\boldsymbol{p} = \boldsymbol{b}_i\,, \quad \boldsymbol{B}_i \equiv \begin{bmatrix} 1 & -\tan\vartheta_i & 0 \\ 0 & 1 & -\cos\vartheta_i\cot\beta_i \end{bmatrix},$$

$$\boldsymbol{b}_i \equiv \begin{bmatrix} x_i - \tan\vartheta_i y_i \\ y_i - \cos\vartheta_i\cot\beta_i z_i \end{bmatrix}, \tag{3}$$

for ($|\vartheta_i| \le \frac{1}{4}\pi$ or $|\vartheta_i| \ge \frac{3}{4}\pi$) and $\frac{1}{4}\pi < |\beta_i| \le \frac{1}{2}\pi$;

$$\boldsymbol{C}_i\boldsymbol{p} = \boldsymbol{c}_i\,, \quad \boldsymbol{C}_i \equiv \begin{bmatrix} -\cot\vartheta_i & 1 & 0 \\ -\tan\beta_i & 0 & \sin\vartheta_i \end{bmatrix},$$

$$\boldsymbol{c}_i \equiv \begin{bmatrix} -\cot\vartheta_i x_i + y_i \\ -\tan\beta_i x_i + \sin\vartheta_i z_i \end{bmatrix}, \tag{4}$$

for $\frac{1}{4}\pi < |\vartheta_i| < \frac{3}{4}\pi$ and $|\beta_i| \le \frac{1}{4}\pi$;

$$\boldsymbol{D}_i\boldsymbol{p} = \boldsymbol{d}_i\,, \quad \boldsymbol{D}_i \equiv \begin{bmatrix} -\cot\vartheta_i & 1 & 0 \\ 1 & 0 & -\sin\vartheta_i\cot\beta_i \end{bmatrix},$$

$$\boldsymbol{d}_i \equiv \begin{bmatrix} -\cot\vartheta_i x_i + y_i \\ x_i - \sin\vartheta_i\cot\beta_i z_i \end{bmatrix}, \tag{5}$$

for $\frac{1}{4}\pi < |\vartheta_i| < \frac{3}{4}\pi$ and $\frac{1}{4}\pi < |\beta_i| \le \frac{1}{2}\pi$.

All elements of coefficient matrices of these systems of linear algebraic equations are of modulus smaller than 1 which results in avoiding apparent singularity related with tangent and cotangent functions of azimuth and elevation.

3. Method of solution and regularisation

If the number of radar stations is greater than one ($n > 1$), then the relation between coordinates $\boldsymbol{p}$ of an investigated object (unknown) and bearings (known) with respect to $\boldsymbol{p}$ takes the form of an overestimated system of $2n$ linear algebraic equations with 3 unknowns

$$\boldsymbol{E}\boldsymbol{p} = \boldsymbol{e}\,, \quad \boldsymbol{E} \equiv \begin{bmatrix} \boldsymbol{E}_1 \\ \vdots \\ \boldsymbol{E}_n \end{bmatrix}, \quad \boldsymbol{e} \equiv \begin{bmatrix} \boldsymbol{e}_1 \\ \vdots \\ \boldsymbol{e}_n \end{bmatrix}, \tag{6}$$

where each pair $(\boldsymbol{E}_i, \boldsymbol{e}_i)$ is of one of forms (2)–(5) and $\boldsymbol{E}$, $\boldsymbol{e}$ are block matrices. The number of equations is smaller or larger than the number of unknowns for one bearing or more than one bearings, respectively. Obviously, this system is compatible and has unique solution for at least two different bearings. For real bearings perturbed data $(\tilde{\boldsymbol{E}}_i, \tilde{\boldsymbol{e}}_i)$, $i = 1, 2, \ldots, n$, are given instead of the exact ones $(\boldsymbol{E}_i, \boldsymbol{e}_i)$ and the system of inexact equations

$$\tilde{\boldsymbol{E}}\tilde{\boldsymbol{p}} = \tilde{\boldsymbol{e}}\,, \quad \tilde{\boldsymbol{E}} \equiv \begin{bmatrix} \tilde{\boldsymbol{E}}_1 \\ \vdots \\ \tilde{\boldsymbol{E}}_n \end{bmatrix}, \quad \tilde{\boldsymbol{e}} \equiv \begin{bmatrix} \tilde{\boldsymbol{e}}_1 \\ \vdots \\ \tilde{\boldsymbol{e}}_n \end{bmatrix} \tag{7}$$

is inconsistent in general for at least two different bearings and undetermined for one bearing. Since only the case when there exists a unique solution is interesting in applications, the problem must be qualified as being ill-posed. A natural way to overcome this drawback is re-formulating it as a linear least squares problem

$$\boldsymbol{E}\boldsymbol{p} \approx \boldsymbol{e}\,, \tag{8}$$

and solving a corresponding inexact problem

$$\tilde{\boldsymbol{E}}\tilde{\boldsymbol{p}} \approx \tilde{\boldsymbol{e}}\,. \tag{9}$$

A linear least squares problem (8) is equivalent to a system of normal equations [4]:

$$\left(\tilde{\boldsymbol{E}}^{\mathrm{T}}\tilde{\boldsymbol{E}}\right)\tilde{\boldsymbol{p}} = \tilde{\boldsymbol{E}}^{\mathrm{T}}\tilde{\boldsymbol{e}}\,. \tag{10}$$

Problems (9) and (10) have the same unique solution if coefficient matrix $\tilde{\boldsymbol{E}}$ is of full rank, $\operatorname{rank}(\tilde{\boldsymbol{E}}) = 3$ (which implies that $\det(\tilde{\boldsymbol{E}}^{\mathrm{T}}\tilde{\boldsymbol{E}}) \neq 0$ and system (10) is consistent and determinate). For a system of at least two radar stations ($n > 1$) this condition is held if bearings from at least two different stations differ. This implies the approximate condition for effective localisation: the distance from a station to limits of an observed area g, the smallest distance between stations s, the smallest emitter altitude determined at the area limit w and the bearing error u (of azimuth or elevation, in radians) should satisfy the inequalities $\frac{s}{g} > u$ and $\frac{w}{g} > u$. System of linear algebraic Eqs. (8), when being determinate, can be solved using Gaussian elimination with partial or complete pivoting or better with using the Cholesky–Banachiewicz decomposition of coefficient matrix $\tilde{\boldsymbol{E}}^{\mathrm{T}}\tilde{\boldsymbol{E}}$ [5]. When the fractions in these inequalities are close to u, the accuracy of this algorithm can be improved via regularisation of the problem [6, 7]: instead of (10) a system of equations

$$\left(\tilde{\boldsymbol{E}}^{\mathrm{T}}\tilde{\boldsymbol{E}} + \alpha\boldsymbol{I}\right)\tilde{\boldsymbol{p}} = \tilde{\boldsymbol{E}}^{\mathrm{T}}\tilde{\boldsymbol{e}} \tag{11}$$

with the regularisation parameter $\alpha > 0$ (where $\boldsymbol{I}$ is the unit matrix) is solved using the same algorithms. For any data $(\tilde{\boldsymbol{E}}, \tilde{\boldsymbol{e}})$ a coefficient matrix in (11) is symmetric and positive-definite and there exists a unique solution of (11). Regularisation parameter can be chosen according to the discrepancy principle [5]:

$$\left\|\tilde{\boldsymbol{E}}\boldsymbol{p}_\alpha - \tilde{\boldsymbol{e}}\right\|_2 = \delta + \eta\left\|\boldsymbol{p}_\alpha\right\|_2\,, \tag{12}$$

where $\tilde{\boldsymbol{p}} = (\tilde{\boldsymbol{E}}^{\mathrm{T}}\tilde{\boldsymbol{E}} + \alpha\boldsymbol{I})^{-1}\tilde{\boldsymbol{E}}^{\mathrm{T}}\tilde{\boldsymbol{e}}$ and δ, η are data error estimates: $\|\tilde{\boldsymbol{E}} - \boldsymbol{E}\|_{\mathrm{F}} \le \eta$, $\|\tilde{\boldsymbol{e}} - \boldsymbol{e}\|_2 \le \delta$ in the Frobenius matrix norm and Euclidean vector norm. If for all bearings $|\tilde{\vartheta}_i - \vartheta_i| \le u$ and $|\tilde{\beta}_i - \beta_i| \le u$, then the errors are bounded in the following way:

$$\eta \le \|\boldsymbol{E}\|_{\mathrm{F}}\, u, \quad \delta \le \|\boldsymbol{e}\|_2\, u \tag{13}$$

and one can take $\eta \approx \|\tilde{\boldsymbol{E}}\|_{\mathrm{F}} u, \delta \approx \|\tilde{\boldsymbol{e}}\|_2 u$ as a good approximation. A system of Eqs. (11), (12) can be solved by applying method of secants or the Newton method to (12).

4. Conclusion

It was demonstrated that a problem of determining a point electromagnetic emitter position from passive radar bearing can be reduced to a system of linear algebraic equations and solved with using standard numerical methods. In some cases the regularisation may be useful. The appropriate algorithms were described.

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