On a Class of Multiattribute Utility Functions Invariant under Shift Transformations

J. Chudziak*
Department of Mathematics, University of Rzeszów, Rejtana 16 C
35-959 Rzeszów, Poland

We consider a class of multiattribute utility functions which are invariant with respect to the shifts having identical parameters for each attribute.

PACS numbers: 89.65.Gh

1. Introduction

Following [1, Definition 2, p. 19], we say that a utility function
\[ U \text{ is invariant to a continuous transformation } g \]
provided it satisfies the following functional equation
\[ U(g(x, z)) = k(z)U(x) + \ell(z), \tag{1} \]
with some functions \( k \) and \( \ell \). If \( g \) is a shift transformation, that is \( g(x, z) = x + z \), Eq. (1) reduces to
\[ U(x + z) = k(z)U(x) + \ell(z) \tag{2} \]
and a utility function \( U \) satisfying (2) is said to be invariant under shift transformation. Some results concerning utility functions invariant under shift transformation could be found e.g. in [1] and [3–5]. In a recent paper [2] this notion has been extended into the case of \( n \)-attribute utility functions. The problem in a natural way leads to the following generalization of (2)

\[ U(x_1 + z_1, \ldots, x_n + z_n) \]
\[ = k(z_1, \ldots, z_n)U(x_1, \ldots, x_n) + \ell(z_1, \ldots, z_n). \tag{3} \]

It turns out that in many cases (e.g. if the initial wealth of the decision maker is in the form of annuity payment which pays an amount \( z \) at every period for \( n \) successive periods) it is reasonable to assume that the utility function satisfies invariance just when the shift parameters are identical for each attribute, i.e. \( z_1 = \ldots = z_n = z \) with \( z \) in an interval of positive length. It is clear that in such a case Eq. (3) reduces to

\[ U(x_1 + z, \ldots, x_n + z) \]
\[ = k(z)U(x_1, \ldots, x_n) + \ell(z). \tag{4} \]

Equation (4) has been already solved in [2] under the assumptions that \( D \) is a non-empty open set, for every \( (x_1, \ldots, x_n) \in D \), the set \( V(x_1, \ldots, x_n) := \{ z \in \mathbb{R} | (x_1 + z, \ldots, x_n + z) \in D \} \) is an interval, \( U : D \rightarrow \mathbb{R}, k, \ell : V_D \rightarrow \mathbb{R} \) and a function given by (5) is non-constant for at least one \( (x_1, \ldots, x_n) \in D \). Then a triple \((U, k, \ell)\) satisfies Eq. (4) for all \((x_1, \ldots, x_n) \in D \) and \( z \in V(x_1, \ldots, x_n) \) if and only if one of the subsequent two conditions holds:

(i) there exist a nonconstant additive function \( a : \mathbb{R} \rightarrow \mathbb{R} \) and a function \( \psi : T \rightarrow \mathbb{R} \) such that

\[
\begin{cases}
  k(z) = 1 \\
  \ell(z) = a(z) \\
  U(x_1, \ldots, x_n) = \psi(x_2 - x_1, \ldots, x_n - x_1) + a(x_1)
\end{cases}
\]

for \( z \in V_D \), for \( z \in V_D \), for \( (x_1, \ldots, x_n) \in D \);

(ii) there exist a nonconstant exponential function \( e : \mathbb{R} \rightarrow \mathbb{R} \), a constant \( c \in \mathbb{R} \) and a not identically zero function \( \psi : T \rightarrow \mathbb{R} \) such that

\[ V(x_1, \ldots, x_n) \ni z \rightarrow U(x_1 + z, \ldots, x_n + z) \tag{5} \] is non-constant for at least one \( (x_1, \ldots, x_n) \in D \).

However, it is easy to check that [2, Theorem 4.3, p. 9] remains true if, instead of the openness of \( D \) we assume that, for every \((x_1, \ldots, x_n) \in D \), the set \( V(x_1, \ldots, x_n) \) is an open interval. In order to formulate that result in such a modified version, we need to introduce the following notation. Let

\[ T := \{(x_2 - x_1, \ldots, x_n - x_1) | (x_1, \ldots, x_n) \in D\} \]
and, for every \((t_1, \ldots, t_{n-1}) \in T,\]

\[ V(t_1, \ldots, t_{n-1}) := \{ (x_1, \ldots, x_n) \in D | (x_2 - x_1, \ldots, x_n - x_1) = (t_1, \ldots, t_{n-1}) \}. \]

Furthermore, given a function \( \psi : T \rightarrow \mathbb{R} \), we set

\[ V_{\psi \neq 0} := \bigcup_{(x_1, \ldots, x_n) \in D, \psi(x_2 - x_1, \ldots, x_n - x_1) \neq 0} V(x_1, \ldots, x_n). \]

Theorem 1. Let \( D \) be a nonempty subset of \( \mathbb{R}^n \) such that, for every \((x_1, \ldots, x_n) \in D \), \( V(x_1, \ldots, x_n) \) is an open interval. Assume that \( U : D \rightarrow \mathbb{R}, k, \ell : V_D \rightarrow \mathbb{R} \) and a function given by (5) is non-constant for at least one \((x_1, \ldots, x_n) \in D \). Then a triple \((U, k, \ell)\) satisfies Eq. (4) for all \((x_1, \ldots, x_n) \in D \) and \( z \in V(x_1, \ldots, x_n) \) if and only if one of the subsequent two conditions holds:

(1) there exist a nonconstant additive function \( a : \mathbb{R} \rightarrow \mathbb{R} \) and a function \( \psi : T \rightarrow \mathbb{R} \) such that

\[
\begin{cases}
  k(z) = 1 \\
  \ell(z) = a(z) \\
  U(x_1, \ldots, x_n) = \psi(x_2 - x_1, \ldots, x_n - x_1) + a(x_1)
\end{cases}
\]

for \( z \in V_D \), for \( z \in V_D \), for \( (x_1, \ldots, x_n) \in D \);

(2) there exist a nonconstant exponential function \( e : \mathbb{R} \rightarrow \mathbb{R} \), a constant \( c \in \mathbb{R} \) and a not identically zero function \( \psi : T \rightarrow \mathbb{R} \) such that

\[ V(x_1, \ldots, x_n) \ni z \rightarrow U(x_1 + z, \ldots, x_n + z) \tag{5} \] is non-constant for at least one \( (x_1, \ldots, x_n) \in D \).
The aim of the present paper is to complete the results of [2] by determining all solutions of (4) such that a function given by (5) is constant for every \((x_1, \ldots, x_n) \in D\).

## 2. Results

We begin this section with the following remark

**Remark 1.** Since, for every \((x_1, \ldots, x_n) \in D\), \(V(x_1, \ldots, x_n)\) is an open interval containing 0, it is clear that, for every \((t_1, \ldots, t_{n-1}) \in T, V(t_1, \ldots, t_{n-1})\) is also an open interval containing 0. Moreover, for every \((t_1, \ldots, t_{n-1}) \in T, V(t_1, \ldots, t_{n-1})\) is symmetric with respect to 0. In fact, if \(z \in V(t_1, \ldots, t_{n-1})\) is an interval containing 0, this implies that for sufficiently large \(n \in \mathbb{N}\), we have \(z_0^+ \in V(x_1, \ldots, x_n)\), and so, in view of (9), for sufficiently large \(n \in \mathbb{N}\), we obtain

\[
\ell(z_0^+) = (1 - k(z_0^+))\psi(x_2 - x_1, \ldots, x_n - x_1) = \psi(x_2 - x_1, \ldots, x_n - x_1) = \psi(x_2 - x_1, \ldots, x_n - x_1) = d. \tag{10}
\]

Analogously, one can prove that, if \(V_D^- \neq \emptyset\), then there exists a \(d \in \mathbb{R}\) such that

\[
V(x_1, \ldots, x_n) \cap V_D^- = \emptyset\]

\[
\iff \psi(x_2 - x_1, \ldots, x_n - x_1) = \tilde{d}. \tag{11}
\]

Now, in order to show (6) it remains to prove that if both sets \(V_D^+\) and \(V_D^-\) are nonempty, then \(d = \tilde{d}\). So, suppose that \(V_D^+ \neq \emptyset\) and \(V_D^- \neq \emptyset\). Let \(\tilde{z}_0^+ = \sup V_D^+\) and, as previously, \(\check{z}_0^+ = \inf V_D^-\). Assume that \(\check{z}_0^{-} \geq -\tilde{z}_0^{+}\) (in the case where \(\tilde{z}_0^{-} \leq -\check{z}_0^{+}\), the proof runs analogously). Then 0 \(\leq \tilde{z}_0^{+} = \check{z}_0^{-}\) whence, taking \((x_1, \ldots, x_n) \in D\) such that \(V(x_1, \ldots, x_n) \cap V_D^- = \emptyset\), we get \(\tilde{z}_0^{+} - \check{z}_0^{-} = V(x_1, \ldots, x_n)\) and so \(\check{z}_0^{-} \in V(x_1, \ldots, x_n)\), and so \(\check{z}_0^{+} \in V(x_1, \ldots, x_n)\). Since \(V(x_1, \ldots, x_n) \cap V_D^- = \emptyset\) is an open interval, this means that \(V_D^+ \cap V_D^- = \emptyset\). Thus, by (10), \(\psi(x_2 - x_1, \ldots, x_n - x_1) = \tilde{d} = \check{d}\). On the other hand, as \(V(x_1, \ldots, x_n) \cap V_D^+ = \emptyset\), by (10), we get \(\psi(x_2 - x_1, \ldots, x_n - x_1) = d\). Hence \(d = \tilde{d}\).
Let \((x_1^0, \ldots, x_n^0) \in D\) be such that \(z_0 \in V(x_1^0 - x_1^0, \ldots, x_n^0 - x_n^0)\). Then \(V(x_1^0 - x_1^0, \ldots, x_n^0 - x_n^0 \setminus k^{-1}(\{1\}) \neq \emptyset\) and so, in view of (6), we get \(\psi(x_1^0 - x_1^0, \ldots, x_n^0 - x_n^0) = d\). Thus, making use of (9), we conclude that
\[
\ell(z) = d(1 - k(z)) \quad \text{for } z \in V(x_1^0 - x_1^0, \ldots, x_n^0 - x_n^0). \quad (12)
\]
Now, let \(z \in V_D\) be arbitrary and \((x_1, \ldots, x_n) \in D\) be such that \(z \in V(x_2-x_1, \ldots, x_n-x_1)\). According to Remark 1, \(V(x_2-x_1, \ldots, x_n-x_1)\) and \(V(x_2-x_1, \ldots, x_n-x_1)\) are intervals symmetric with respect to 0. Thus, either \(V(x_2-x_1, \ldots, x_n-x_1) \subset V(x_2-x_1, \ldots, x_n-x_1)\) or \(V(x_2-x_1, \ldots, x_n-x_1) \subset V(x_2-x_1, \ldots, x_n-x_1)\). In the first case, in view of (12), we get \(\ell(z) = d(1 - k(z))\). In the second one, we have \(V(x_2-x_1, \ldots, x_n-x_1 \setminus k^{-1}(\{1\}) \neq \emptyset\) which, together with (6) and (9), gives again \(\ell(z) = d(1 - k(z))\). Therefore (7) holds and the proof is completed.

Example 1. Let \(D = D_1 \cup D_2\), where
\[
D_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 \leq x_2 - x_1 < 2, 1 < x_1 + x_2 < 2\}
\]
and
\[
D_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_2 - x_1 < 1, 1 < x_1 + x_2 < 3\}.
\]
Then, for every \((x_1, x_2) \in D\), the set \(V(x_1, x_2)\) is an open interval. In fact, we have
\[
V(x_1, x_2) = \left\{ \frac{1-(x_1+x_2)}{2}, \frac{2-(x_1+x_2)}{2} \right\} \quad \text{for } (x_1, x_2) \in D_1
\]
and
\[
V(x_1, x_2) = \left\{ \frac{1-(x_1+x_2)}{2}, \frac{2-(x_1+x_2)}{2} \right\} \quad \text{for } (x_1, x_2) \in D_2.
\]
Thus \(V_D = (-1, 1)\). Furthermore, we have \(T = (0, 2)\) and
\[
V^t = \begin{cases} (0, 1) & \text{for } t \in (0, 1) \\ (-1/2, 1/2) & \text{for } t \in [1, 2]. \end{cases}
\]
Fix \(d \in \mathbb{R}\) and \(\Delta \in [0, \infty)\). Let \(k : (-1, 1) \to \mathbb{R}\) be an arbitrary function such that \(k^{-1}(\{1\}) = [-\Delta, \Delta]\) and let \(\ell(z) = d(1 - k(z))\) for \(z \in (-1, 1)\). Note that:
- if \(\Delta \geq 1\) then \(V^t \setminus k^{-1}(\{1\}) = \emptyset\) for every \(t \in (0, 2)\);
- if \(\Delta \in (0, 1)\) then \(V^t \setminus k^{-1}(\{1\}) \neq \emptyset\) for every \(t \in (0, 1)\);
- if \(\Delta \in (1, 2)\) then \(V^t \setminus k^{-1}(\{1\}) \neq \emptyset\) for every \(t \in (0, 2)\).

Therefore, applying Theorem 2, we conclude that a triple \((k, \ell, U)\), where \(U : D \to \mathbb{R}\), satisfies (4) if and only if
\[
U(x_1, x_2) = \psi(x_2 - x_1) \quad \text{for } (x_1, x_2) \in D,
\]
where \(\psi : (0, 2) \to \mathbb{R}\) is such that:
- \(\psi(0, 1) \equiv d\) in the case where \(\Delta \in [1/2, 1]\);
- \(\psi \equiv d\) in the case where \(\Delta \in [0, 1/2]\).

3. Conclusion

In several cases, the initial wealth of the decision maker is in the form of annuity payment which pays an amount \(z\) at every period for \(n\) successive periods. In such cases, a multiattribute utility function is invariant under shift transformation, provided Eq. (2) holds with some functions \(k\) and \(\ell\). This equation has been considered in [2] under the assumption that a function given by (5) is non-constant for at least one \((x_1, \ldots, x_n) \in D\). In the present paper we complete the results in [2] by determining the functional form of utility function in the case where a function given by (5) is constant for every \((x_1, \ldots, x_n) \in D\).

References