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On a Class of Multiattribute Utility Functions Invariant under Shift Transformations

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We consider a class of multiattribute utility functions which are invariant with respect to the shifts having identical parameters for each attribute.

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1. Introduction

Following [1, Definition 2, p. 19], we say that a utility function U *is invariant to a continuous transformation* g, provided it satisfies the following functional equation

$$U(g(x,z)) = k(z)U(x) + \ell(z)\,, \tag{1}$$

with some functions k and ℓ. If g is a shift transformation, that is $g(x,z) = x+z$, Eq. (1) reduces to

$$U(x+z) = k(z)U(x) + \ell(z) \tag{2}$$

and a utility function U satisfying (2) is said to be *invariant under shift transformation.* Some results concerning utility functions invariant under shift transformation could be found e.g. in [1] and [3–5]. In a recent paper [2] this notion has been extended into the case of n-attribute utility functions. The problem in a natural way leads to the following generalization of (2)

$$U(x_1+z_1,\ldots,x_n+z_n)$$
$$= k(z_1,\ldots,z_n)U(x_1,\ldots,x_n) + \ell(z_1,\ldots,z_n)\,. \tag{3}$$

It turns out that in many cases (e.g. if the initial wealth of the decision maker is in the form of annuity payment which pays an amount z at every period for n successive periods) it is reasonable to assume that the utility function satisfies invariance just when the shift parameters are identical for each attribute, i.e. $z_1 = \ldots = z_n = z$ with z in an interval of positive length. It is clear that in such a case Eq. (3) reduces to

$$U(x_1+z,\ldots,x_n+z)$$
$$= k(z)U(x_1,\ldots,x_n) + \ell(z)\,. \tag{4}$$

Equation (4) has been already solved in [2] under the assumptions that D is a non-empty open set, for every $(x_1,\ldots,x_n) \in D$, the set $V_{(x_1,\ldots,x_n)} := \{z \in \mathbb{R} | (x_1+z,\ldots,x_n+z) \in D\}$ is an interval, $U : D \to \mathbb{R}$, $k,\ell : V_{\mathrm{D}} := \bigcup_{(x_1,\ldots,x_n)\in D} V_{(x_1,\ldots,x_n)} \to \mathbb{R}$ are unknown functions, and a function

$$V_{(x_1,\ldots,x_n)} \ni z \to U(x_1+z,\ldots,x_n+z) \tag{5}$$

is non-constant for at least one $(x_1,\ldots,x_n) \in D$. However, it is easy to check that [2, Theorem 4.3, p. 9] remains true if, instead of the openness of D we assume that, for every $(x_1,\ldots,x_n) \in D$, the set $V_{(x_1,\ldots,x_n)}$ is an *open* interval. In order to formulate that result in such a modified version, we need to introduce the following notation. Let

$$T := \{(x_2-x_1,\ldots,x_n-x_1) | (x_1,\ldots,x_n) \in D\}$$

and, for every $(t_1,\ldots,t_{n-1}) \in T$,

$$V^{(t_1,\ldots,t_{n-1})} := \bigcup_{(x_1,\ldots,x_n)\in D,(x_2-x_1,\ldots,x_n-x_1)=(t_1,\ldots,t_{n-1})} V_{(x_1,\ldots,x_n)}\,.$$

Furthermore, given a function $\psi : T \to \mathbb{R}$, we set

$$V_{\psi\neq 0} := \bigcup_{(x_1,\ldots,x_n)\in D,\psi(x_2-x_1,\ldots,x_n-x_1)\neq 0} V_{(x_1,\ldots,x_n)}\,.$$

Theorem 1. *Let D be a nonempty subset of $\mathbb{R}^n$ such that, for every $(x_1,\ldots,x_n) \in D$, $V_{(x_1,\ldots,x_n)}$ is an open interval. Assume that $U : D \to \mathbb{R}$, $k,\ell : V_{\mathrm{D}} \to \mathbb{R}$ and a function given by* (5) *is non-constant for at least one $(x_1,\ldots,x_n) \in D$. Then a triple (U,k,ℓ) satisfies Eq.* (4) *for all $(x_1,\ldots,x_n) \in D$ and $z \in V_{(x_1,\ldots,x_n)}$ if and only if one of the subsequent two conditions holds:*

(*i*) *there exist a nonconstant additive function $a : \mathbb{R} \to \mathbb{R}$ and a function $\psi : T \to \mathbb{R}$ such that*

$$\begin{cases} k(z) = 1 & \text{for } z \in V_{\mathrm{D}} \\ \ell(z) = a(z) & \text{for } z \in V_{\mathrm{D}} \\ U(x_1,\ldots,x_n) = \psi(x_2-x_1,\ldots,x_n-x_1) & \\ \quad +a(x_1) & \text{for } (x_1,\ldots,x_n) \in D; \end{cases}$$

(*ii*) *there exist a nonconstant exponential function $e : \mathbb{R} \to \mathbb{R}$, a constant $c \in \mathbb{R}$ and a not identically zero function $\psi : T \to \mathbb{R}$ such that*

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$$\begin{cases} k(z)=e(z) & \text{for } z\in V_{\psi\neq 0} \\ \ell(z)=c(1-k(z)) & \text{for } z\in V_{\mathrm{D}} \\ U(x_1,\dots,x_n)=e(x_1)\psi(x_2-x_1,\dots,x_n-x_1) & \\ +c & \text{for } (x_1,\dots,x_n)\in D. \end{cases}$$

The aim of the present paper is to complete the results of [2] by determining all solutions of (4) such that a function given by (5) is constant for every $(x_1,\dots,x_n)\in D$.

2. Results

We begin this section with the following remark

Remark 1. Since, for every $(x_1,\dots,x_n)\in D$, $V_{(x_1,\dots,x_n)}$ is an open interval containing 0, it is clear that, for every $(t_1,\dots,t_{n-1})\in T$, $V^{(t_1,\dots,t_{n-1})}$ is also an open interval containing 0. Moreover, for every $(t_1,\dots,t_{n-1})\in T$, $V^{(t_1,\dots,t_{n-1})}$ is symmetric with respect to 0. In fact, if $z\in V^{(t_1,\dots,t_{n-1})}$ then $z\in V_{(x_1,\dots,x_n)}$ for some $(x_1,\dots,x_n)\in D$ with $(x_2-x_1,\dots,x_n-x_1)=(t_1,\dots,t_{n-1})$. Thus, $(x_1+z,\dots,x_n+z)\in D$, whence $-z\in V_{(x_1+z,\dots,x_n+z)}$. As $(x_2+z-(x_1+z),\dots,x_n+z-(x_1+z))=(t_1,\dots,t_{n-1})$, this means that $-z\in V^{(t_1,\dots,t_{n-1})}$.

The next theorem is a main result of the paper

Theorem 2. *Let D be a nonempty subset of $\mathbb{R}^n$ such that, for every $(x_1,\dots,x_n)\in D$, $V_{(x_1,\dots,x_n)}$ is an open interval. Assume that $U:D\to\mathbb{R}$, $k,\ell:V_{\mathrm{D}}\to\mathbb{R}$ and a function given by* (5) *is constant for every $(x_1,\dots,x_n)\in D$. Then a triple (U,k,ℓ) satisfies Eq.* (4) *for all $(x_1,\dots,x_n)\in D$ and $z\in V_{(x_1,\dots,x_n)}$ if and only if there exist a constant $d\in\mathbb{R}$ and a function $\psi:T\to\mathbb{R}$ such that*

$$\psi(t_1,\dots,t_{n-1})=d \quad \text{whenever} \quad V^{(t_1,\dots,t_{n-1})}\setminus k^{-1}(\{1\})\neq\emptyset, \tag{6}$$

$$\ell(z)=d(1-k(z)) \quad \text{for} \quad z\in V_{\mathrm{D}} \tag{7}$$

and

$$U(x_1,\dots,x_n)=\psi(x_2-x_1,\dots,x_n-x_1) \quad \text{for} \quad (x_1,\dots,x_n)\in D. \tag{8}$$

Prof: In order to prove that (8) holds with some function $\psi:T\to\mathbb{R}$ it is enough to show that, for every $(x_1,\dots,x_n),(y_1,\dots,y_n)\in D$, the following implication is valid:

$$(x_2-x_1,\dots,x_n-x_1)=(y_2-y_1,\dots,y_n-y_1) \implies U(x_1,\dots,x_n)=U(y_1,\dots,y_n).$$

To this end, fix $(x_1,\dots,x_n),(y_1,\dots,y_n)\in D$ and suppose that $(x_2-x_1,\dots,x_n-x_1)=(y_2-y_1,\dots,y_n-y_1)$. Then $y_j=x_j+(y_1-x_1)$ for $j=1,2,\dots,n$ and so, using the fact that a function given by (5) is constant for $(x_1,\dots,x_n)$, we get

$$U(y_1,\dots,y_n)=U(x_1+(y_1-x_1),\dots,x_n+(y_1-x_1)) = U(x_1,\dots,x_n).$$

Therefore, (8) is proved. Now, note that as a function given by (5) is constant for every $(x_1,\dots,x_n)\in D$, from (4) and (8) we derive that

$$\ell(z)=(1-k(z))\psi(x_2-x_1,\dots,x_n-x_1) \quad \text{for} \quad (x_1,\dots,x_n)\in D, z\in V^{(x_2-x_1,\dots,x_n-x_1)}. \tag{9}$$

Thus, if k is identically 1, then ℓ is identically 0. Hence, taking an arbitrary $d\in\mathbb{R}$, we get the assertion. So, assume that k is not identically 1. Put

$$V_{\mathrm{D}}^{+}:=\{z\in V_{\mathrm{D}}\cap[0,\infty)|k(z)\neq 1\}$$

and

$$V_{\mathrm{D}}^{-}:=\{z\in V_{\mathrm{D}}\cap(-\infty,0)|k(z)\neq 1\}.$$

Clearly, at least one of the sets V_{D}^{+} and V_{D}^{-} is nonempty. Assume that $V_{\mathrm{D}}^{+}\neq\emptyset$ and let $z_0^+:=\inf V_{\mathrm{D}}^{+}$. Let $\{z_n^+\}$ be a sequence of elements of V_{D}^{+} converging to z_0^+. Fix a $z\in V_{\mathrm{D}}^{+}$ and $(\tilde{x_1},\dots,\tilde{x_n})\in D$ with $z\in V_{(\tilde{x_1},\dots,\tilde{x_n})}$. Since $z_0^+\leq z$ and $V_{(\tilde{x_1},\dots,\tilde{x_n})}$ is an interval containing 0, for sufficiently large $n\in\mathbb{N}$, we have $z_n^+\in V_{(\tilde{x_1},\dots,\tilde{x_n})}$. Then, in view of (9), for sufficiently large $n\in\mathbb{N}$, we obtain

$$\ell(z_n^+)=(1-k(z_n^+))\psi(\tilde{x_2}-\tilde{x_1},\dots,\tilde{x_n}-\tilde{x_1}),$$

whence

$$\lim_{n\to\infty}\frac{\ell(z_n^+)}{1-k(z_n^+)}=\psi(\tilde{x_2}-\tilde{x_1},\dots,\tilde{x_n}-\tilde{x_1})=:d.$$

Next, taking an arbitrary $(x_1,\dots,x_n)\in D$ such that $V_{(x_1,\dots,x_n)}\cap V_{\mathrm{D}}^{+}\neq\emptyset$ and arguing as previously, we get that

$$d=\lim_{n\to\infty}\frac{\ell(z_n^+)}{1-k(z_n^+)}=\psi(x_2-x_1,\dots,x_n-x_1).$$

In this way we have proved that there is a $d\in\mathbb{R}$ such that

$$V_{(x_1,\dots,x_n)}\cap V_{\mathrm{D}}^{+}\neq\emptyset \implies \psi(x_2-x_1,\dots,x_n-x_1)=d. \tag{10}$$

Analogously, one can prove that, if $V_{\mathrm{D}}^{-}\neq\emptyset$, then there exists a $\tilde{d}\in\mathbb{R}$ such that

$$V_{(x_1,\dots,x_n)}\cap V_{\mathrm{D}}^{-}\neq\emptyset \implies \psi(x_2-x_1,\dots,x_n-x_1)=\tilde{d}. \tag{11}$$

Now, in order to show (6) it remains to prove that if both sets V_{D}^{+} and V_{D}^{-} are nonempty, then $d=\tilde{d}$. So, suppose that $V_{\mathrm{D}}^{+}\neq\emptyset$ and $V_{\mathrm{D}}^{-}\neq\emptyset$. Let $z_0^-:=\sup V_{\mathrm{D}}^{-}$ and, as previously, $z_0^+:=\inf V_{\mathrm{D}}^{+}$. Assume that $z_0^-\geq -z_0^+$ (in the case where $z_0^-\leq -z_0^+$, the proof runs analogously). Then $0\leq z_0^-+z_0^+\leq z_0^+$ whence, taking $(x_1,\dots,x_n)\in D$ such that $V_{(x_1,\dots,x_n)}\cap V_{\mathrm{D}}^{+}\neq\emptyset$, we get $z_0^-+z_0^+\in V_{(x_1,\dots,x_n)}$ and so $z_0^-\in V_{(x_1+z_0^+,\dots,x_n+z_0^+)}$. Since $V_{(x_1+z_0^+,\dots,x_n+z_0^+)}$ is an open interval, this means that $V_{(x_1+z_0^+,\dots,x_n+z_0^+)}\cap V_{\mathrm{D}}^{-}\neq\emptyset$. Thus, by (11), $\psi(x_2-x_1,\dots,x_n-x_1)=\tilde{d}$. On the other hand, as $V_{(x_1,\dots,x_n)}\cap V_{\mathrm{D}}^{+}\neq\emptyset$, by (10), we get $\psi(x_2-x_1,\dots,x_n-x_1)=d$. Hence $d=\tilde{d}$ and so (6) is proved. Finally, we show that (7) holds. Since k is not identically 1, there is $z_0\in V_{\mathrm{D}}$ with $k(z_0)\neq 1$. Let

$(x_1^0, \dots, x_n^0) \in D$ be such that $z_0 \in V^{(x_2^0 - x_1^0, \dots, x_n^0 - x_1^0)}$. Then $V^{(x_2^0 - x_1^0, \dots, x_n^0 - x_1^0)} \setminus k^{-1}(\{1\}) \neq \emptyset$ and so, in view of (6), we get $\psi(x_2^0 - x_1^0, \dots, x_n^0 - x_1^0) = d$. Thus, making use of (9), we conclude that

$$\ell(z) = d(1 - k(z)) \quad \text{for} \quad z \in V^{(x_2^0 - x_1^0, \dots, x_n^0 - x_1^0)}. \quad (12)$$

Now, let $z \in V_{\mathrm{D}}$ be arbitrary and $(x_1, \dots, x_n) \in D$ be such that $z \in V^{(x_2 - x_1, \dots, x_n - x_1)}$. According to Remark 1, $V^{(x_2^0 - x_1^0, \dots, x_n^0 - x_1^0)}$ and $V^{(x_2 - x_1, \dots, x_n - x_1)}$ are intervals symmetric with respect to 0. Thus, either $V^{(x_2 - x_1, \dots, x_n - x_1)} \subset V^{(x_2^0 - x_1^0, \dots, x_n^0 - x_1^0)}$ or $V^{(x_2^0 - x_1^0, \dots, x_n^0 - x_1^0)} \subset V^{(x_2 - x_1, \dots, x_n - x_1)}$. In the first case, in view of (12), we get $\ell(z) = d(1 - k(z))$. In the second one, we have $V^{(x_2 - x_1, \dots, x_n - x_1)} \setminus k^{-1}(\{1\}) \neq \emptyset$ which, together with (6) and (9), gives again $\ell(z) = d(1 - k(z))$. Therefore (7) holds and the proof is completed.

Example 1. Let $D = D_1 \cup D_2$, where

$$D_1 = \{(x_1, x_2) \in \mathbb{R}^2 | 1 \le x_2 - x_1 < 2, 1 < x_1 + x_2 < 2\}$$

and

$$D_2 = \{(x_1, x_2) \in \mathbb{R}^2 | 0 < x_2 - x_1 < 1, 1 < x_1 + x_2 < 3\}.$$

Then, for every $(x_1, x_2) \in D$, the set $V_{(x_1, x_2)}$ is an open interval. In fact, we have

$$V_{(x_1,x_2)} = \begin{cases} \left(\frac{1-(x_1+x_2)}{2}, \frac{2-(x_1+x_2)}{2}\right) & \text{for } (x_1, x_2) \in D_1 \\ \left(\frac{1-(x_1+x_2)}{2}, \frac{3-(x_1+x_2)}{2}\right) & \text{for } (x_1, x_2) \in D_2 . \end{cases}$$

Thus $V_{\mathrm{D}} = (-1, 1)$. Furthermore, we have $T = (0, 2)$ and

$$V^t = \begin{cases} (-1, 1) & \text{for } t \in (0, 1) \\ (-\frac{1}{2}, \frac{1}{2}) & \text{for } t \in [1, 2) . \end{cases}$$

Fix $d \in \mathbb{R}$ and $\Delta \in [0, \infty)$. Let $k : (-1, 1) \to \mathbb{R}$ be an arbitrary function such that $k^{-1}(\{1\}) = [-\Delta, \Delta]$ and let $\ell(z) = d(1 - k(z))$ for $z \in (-1, 1)$. Note that:

— if $\Delta \ge 1$ then $V^t \setminus k^{-1}(\{1\}) = \emptyset$ for every $t \in (0, 2)$;

— if $\Delta \in [\frac{1}{2}, 1)$ then $V^t \setminus k^{-1}(\{1\}) \neq \emptyset$ for every $t \in (0, 1)$;

— if $\Delta \in [0, \frac{1}{2})$ then $V^t \setminus k^{-1}(\{1\}) \neq \emptyset$ for every $t \in (0, 2)$.

Therefore, applying Theorem 2, we conclude that a triple (k, ℓ, U), where $U : D \to \mathbb{R}$, satisfies (4) if and only if

$$U(x_1, x_2) = \psi(x_2 - x_1) \quad \text{for} \quad (x_1, x_2) \in D,$$

where $\psi : (0, 2) \to \mathbb{R}$ is such that:

a) $\psi_{|(0,1)} \equiv d$ in the case where $\Delta \in [\frac{1}{2}, 1)$;

b) $\psi \equiv d$ in the case where $\Delta \in [0, \frac{1}{2})$.

3. Conclusion

In several cases, the initial wealth of the decision maker is in the form of annuity payment which pays an amount z at every period for n successive periods. In such cases, a multiattribute utility function is invariant under shift transformation, provided Eq. (2) holds with some functions k and ℓ. This equation has been considered in [2] under the assumption that a function given by (5) is non-constant for at least one $(x_1, \dots, x_n) \in D$. In the present paper we complete the results in [2] by determining the functional form of utility function in the case where a function given by (5) is constant for every $(x_1, \dots, x_n) \in D$.

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