

# Comparison of Properties of One-Dimensional $\pm J$ Ising Models on Square Lattices

M. BŁASZYK

Institute of Physics, A. Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland

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The energies and entropies of the one-dimensional  $\pm J$  Ising models on square lattices have been compared. The method proposed in this paper explains the impossibility of getting exact results in nonzero temperatures and specifies the conditions of error minimisation.

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## 1. Introduction

Although our knowledge of the infinite-ranged spin glasses is sufficient [1], we cannot say so about the finite-ranged spin glasses. The information on the latter is mainly obtained from computer simulations, restricted to finite systems and a small number of exchange integrals and thus unable to provide reliable results. Besides this mainstream of research work, attempts have been made on development of methods describing the one-dimensional systems with disorder and frustration. In the ground state it is possible to get exact expressions, however, in nonzero temperatures we have to be satisfied with better or worse approximations. In this paper, using the method derived for the  $\pm J$  Ising model in an external magnetic field, the reasons why the exact results are unattainable besides the ground state are indicated.

## 2. Model description

We shall consider the Ising model described by the Hamiltonian

$$\mathcal{H} = - \sum_{i,j} J_{ij} S_i S_j, \quad (1)$$

where  $(ij)$  numbers the pairs of neighbouring spins,  $J_{ij} = J$  with the probability  $1 - p$  and  $J_{ij} = -J$  with the probability  $p$ . We will compare the physical properties of two one-dimensional lattices made of squares. The first of them is an infinite tube of the width of two bonds, which has been studied in detail in the ground state [2]. The second lattice is a ribbon made of squares, called also the two-leg ladder. This model has been studied in detail in [3]. The common features of the two systems of the ribbon and the tube are their infinite lengths and the width of two spins. This width in the ribbon corresponds to one square, while in the tube it corresponds to two squares. The most important difference between these two lattices is that the ribbon has an edge, while the tube does not have it.

On the example of the one-dimensional  $\pm J$  Ising model in a uniform magnetic field the method of determina-

tion of the physical quantities of the disordered systems in nonzero temperatures has been given in [4], where a possibility of application of this approximation to other lattices (in which the frustration is not a result of magnetic field but follows from the competition of exchange integrals) has been suggested. The efficiency of this approximation will be checked for the two above-mentioned lattices.

For the tube, Eq. (1) can be rewritten in the form

$$\mathcal{H} = - \sum_{i=1}^{\infty} (J_{1i} S_i S_{i+1} + J_{2i} S'_i S'_{i+1} + J_{3i} S_i S'_i + J_{4i} S_i S'_i), \quad (2)$$

where  $J_{1i}$  and  $J_{2i}$  are the exchange integrals parallel to the tube, and  $J_{3i}$  and  $J_{4i}$  are those perpendicular to the tube. Introduction of the four exchange integrals leads to the necessity of consideration of 16 transfer matrices of dimensions  $4 \times 4$ . If all four bonds are ferromagnetic, we have

$$\mathbf{ffff} = \mathbf{m}[1] = \begin{bmatrix} e^{4K} & e^{-2K} & e^{-2K} & 1 \\ e^{2K} & 1 & e^{-4K} & e^{2K} \\ e^{2K} & e^{-4K} & 1 & e^{2K} \\ 1 & e^{-2K} & e^{-2K} & e^{4K} \end{bmatrix} (1-p)^4$$

with  $K = \beta J$ ,  $\beta = 1/kT$ . If in one of the perpendicular bonds we exchange the ferromagnetic bond for the antiferromagnetic one, we get

$$\mathbf{fffa} = \mathbf{m}[2] = \mathbf{ffaf} = \mathbf{m}[3] = \begin{bmatrix} e^{2K} & 1 & 1 & e^{-2K} \\ 1 & e^{2K} & e^{-2K} & 1 \\ 1 & e^{-2K} & e^{2K} & 1 \\ e^{-2K} & 1 & 1 & e^{2K} \end{bmatrix} (1-p)^3 p.$$

Other matrices are obtained if the only antiferromagnetic bond is that perpendicular to the tube

$$\mathbf{faff} = \mathbf{m}[4] = \begin{bmatrix} e^{2K} & 1 & e^{-4K} & e^{2K} \\ e^{4K} & e^{-2K} & e^{-2K} & 1 \\ 1 & e^{-2K} & e^{-2K} & e^{4K} \\ e^{2K} & e^{-4K} & 1 & e^{2K} \end{bmatrix} (1-p)^3 p$$

and

$$\mathbf{afff} = \mathbf{m}[5] = \begin{bmatrix} e^{2K} & e^{-4K} & 1 & e^{2K} \\ 1 & e^{-2K} & e^{-2K} & e^{4K} \\ e^{4K} & e^{-2K} & e^{-2K} & 1 \\ e^{2K} & 1 & e^{-4K} & e^{2K} \end{bmatrix} (1-p)^3 p.$$

The matrix for which the bonds parallel to the tube are ferromagnetic and the perpendicular ones are anti-ferromagnetic is

$$\mathbf{ffaa} = \mathbf{m}[6] = \begin{bmatrix} 1 & e^{2K} & e^{2K} & e^{-4K} \\ e^{-2K} & e^{4K} & 1 & e^{-2K} \\ e^{-2K} & 1 & e^{4K} & e^{-2K} \\ e^{-4K} & e^{2K} & e^{2K} & 1 \end{bmatrix} (1-p)^2 p^2.$$

When the first of the bonds parallel to the tube and any of the perpendicular ones are ferromagnetic, we have

$$\mathbf{fafa} = \mathbf{m}[7] = \mathbf{faaf} = \mathbf{m}[9] =$$

$$\begin{bmatrix} 1 & e^{2K} & e^{-2K} & 1 \\ e^{2K} & 1 & 1 & e^{-2K} \\ e^{-2K} & 1 & 1 & e^{2K} \\ 1 & e^{-2K} & e^{2K} & 1 \end{bmatrix} (1-p)^2 p^2.$$

For the same situation as above but with the changed sign of the exchange integrals parallel to the tube, we get

$$\mathbf{affa} = \mathbf{m}[8] = \mathbf{afaf} = \mathbf{m}[10] =$$

$$\begin{bmatrix} 1 & e^{-2K} & e^{2K} & 1 \\ e^{-2K} & 1 & 1 & e^{2K} \\ e^{2K} & 1 & 1 & e^{-2K} \\ 1 & e^{2K} & e^{-2K} & 1 \end{bmatrix} (1-p)^2 p^2.$$

The other six matrices  $\mathbf{aaff} = \mathbf{m}[11]$ ,  $\mathbf{faaa} = \mathbf{m}[12]$ ,  $\mathbf{afaa} = \mathbf{m}[13]$ ,  $\mathbf{aafa} = \mathbf{m}[14]$ ,  $\mathbf{aaaf} = \mathbf{m}[15]$  and  $\mathbf{aaaa} = \mathbf{m}[16]$ , are obtained from the former by simultaneous exchanges  $\mathbf{f} \leftrightarrow \mathbf{a}$ ,  $K \leftrightarrow -K$ , and  $p \leftrightarrow (1-p)$ .

In order to form systems with random ferromagnetic and antiferromagnetic bonds, we construct the following matrix:

$$\mathcal{M} = \begin{bmatrix} \mathbf{m}[1] & \mathbf{m}[2] & \mathbf{m}[3] & \dots & \mathbf{m}[16] \\ \mathbf{m}[1] & \mathbf{m}[2] & \mathbf{m}[3] & \dots & \mathbf{m}[16] \\ \vdots & & & & \\ \mathbf{m}[1] & \mathbf{m}[2] & \mathbf{m}[3] & \dots & \mathbf{m}[16] \end{bmatrix}. \quad (3)$$

It is the matrix of the size  $16 \times 16$ , whose elements are the matrices  $\mathbf{m}[1]$ ,  $\mathbf{m}[2]$ ,  $\mathbf{m}[3]$ ,  $\dots$ ,  $\mathbf{m}[16]$ . Matrix  $\mathcal{M}$  has the same rows. Matrix  $\mathcal{M}^n$  with the terms being partition functions for the systems made of  $n$  pairs of squares, will have the same property. Although the rows of the matrices  $\mathcal{M}$  and  $\mathcal{M}^n$  are identical, their interpretation imposed by the matrix notation is different. In the first

row the component matrices are preceded by the matrix  $\mathbf{m}[1]$ , while in the second row by  $\mathbf{m}[2]$ , etc. The analogous description can be introduced for the ribbon, whose Hamiltonian will be devoid of the fourth component  $J_{4i}S_iS'_i$ . In this way the number of the transition matrices (still of the size  $4 \times 4$ ) is reduced from 16 to 8.

### 3. Energy

Matrix  $\mathcal{M}$  treated as a  $64 \times 64$  matrix of scalars has four nonzero eigenvalues:  $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$  in terms of which the partition function is expressed as

$$Z_n = \text{Tr} \mathcal{M}^n = \lambda_1^n + \lambda_2^n + \lambda_3^n + \lambda_4^n \xrightarrow{n \rightarrow \infty} \lambda_1^n. \quad (4)$$

This limit has a double meaning. Preserving only  $\lambda_1$  we assume that the system is infinite. Moreover, at the same assumption the partition function undergoes factorisation and there is the same  $\lambda_1$  per each pair of squares, which ensures the intensity of the thermodynamical quantities.

Making use of the greater eigenvalue  $\lambda = \lambda_1$ , we find the right and the left eigenvectors

$$\mathcal{M}|\mathcal{Y}\rangle = \lambda|\mathcal{Y}\rangle, \quad \langle \mathcal{X}|\mathcal{M} = \lambda\langle \mathcal{X}| \quad (5)$$

with the normalisation condition

$$\langle \mathcal{X}|\mathcal{Y}\rangle = 1. \quad (6)$$

Matrix  $\mathcal{M}$  can be written as a sum of 16 matrices  $\mathcal{M}_1$  to  $\mathcal{M}_{16}$ , where  $\mathcal{M}_1$  is the matrix  $\mathcal{M}$  with only the matrices  $\mathbf{m}[1]$  left, i.e.

$$\mathcal{M}_1 = \begin{bmatrix} \mathbf{m}[1] & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{m}[1] & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & & & & \\ \mathbf{m}[1] & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \quad (7)$$

and other matrices from  $\mathcal{M}_2$  to  $\mathcal{M}_{16}$  are obtained analogously. Although the vectors  $\langle \mathcal{X}|$  and  $|\mathcal{Y}\rangle$  are the eigenvectors of matrix  $\mathcal{M}$ , they are not eigenvectors of any of the matrices  $\mathcal{M}_1$  to  $\mathcal{M}_{16}$ . Consequently,  $\lambda = \langle \mathcal{X}|\mathcal{M}|\mathcal{Y}\rangle$  is a partition function of the pair of squares but  $\langle \mathcal{X}|\mathcal{M}_1|\mathcal{Y}\rangle$  is not a partition function of the pair of ferromagnetic squares.

Writing matrix  $\mathcal{M}$  we have not established yet the type of averaging, i.e. if it is of the anneal or quench type. If it is of the annealed type, the partition function is averaged and  $[Z(J_{i,j})]_J$  is calculated, while if it is of the quenched type, the  $[\ln Z(J_{i,j})]_J$  is calculated. In both cases a set of systems differing in the realisation of individual interactions is considered. Although the above two expressions can be calculated, for the annealed averaging the procedure of determination of the thermodynamic quantities for the set of systems is reduced to determination of the thermodynamical quantities of the most probable system. It is a consequence of the fact that although a set of systems is considered it is described by only one partition function. In terms of probability, the state of the system is described by the probability with one normalising factor (partition function). The situation is different for the quench averaging. For each system a given physical

quantity is determined and then subjected to averaging over the systems, so over the interaction distribution. For such averaging there is no single representative (i.e. the most probable system) and the averaging procedure is realised in two steps. In the first step the partition function  $Z(J_{i,j})$  is found for each system and for this function the appropriate thermodynamical quantities. For each of the  $n$  systems, there is another factor normalising the probability lambda  $\lambda_n = Z(J_{i,j})$ . In the second step, the thermodynamical quantities are averaged over the systems or over the interaction distribution. According to the above, the probability distribution for the quenched system will contain the information on each of the  $n$  systems separately, and the probability will be described by all normalising factors for each of the systems.

As follows from the above, for the annealed averaging the system will assume the most favourable states not only with regard to the spin variables but also with regard to the position of the ferro- and antiferromagnetic bonds. For the quench averaging the positions of the ferro- and antiferromagnetic bonds are frozen and the averaging is performed over the spin variables only. In terms of probability, for the annealed system only one partition function  $\lambda$  is considered and this partition function determines the probability distribution, while for the quenched system we have as many probability distributions as many systems (exchange integral distributions) and each of the probability distributions corresponds to a different partition function  $\lambda_n$ . The total probability for the quenched systems is described by the probabilities of each of the systems so each  $\lambda_n$  must be used. It would seem that the only consequence of the annealed averaging is the energy competition between the ferromagnetic and antiferromagnetic bonds, which is not the case. If the energy competition between the ferro- and antiferromagnetic bonds was forbidden, i.e. if the weights of the probability of occurrence of different distributions of the ferro- and antiferromagnetic bonds were temperature independent, then the annealed system would be an approximation of the quenched one.

In analogy to the matrix  $\mathcal{M}$  described as a scalar matrix of the size  $64 \times 64$  or a matrix made by  $4 \times 4$  dimensional matrices  $\mathbf{m}[i]$  of the size  $16 \times 16$ , the vectors  $\langle \mathcal{X} |$  and  $| \mathcal{Y} \rangle$  can be understood as a combination of 16 vectors. These vector components of the vector  $\langle \mathcal{X} |$  determine which matrix preceded the pair of squares considered. For the first vector component  $\langle \mathcal{X}_1 |$  this matrix is  $\mathbf{m}[1]$ , for the second vector component  $\langle \mathcal{X}_2 |$  it is the matrix  $\mathbf{m}[2]$ , etc. Therefore,  $\langle \mathcal{X} | = \langle \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \dots, \mathcal{X}_{16} |$  and  $| \mathcal{Y} \rangle = | \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \dots, \mathcal{Y}_{16} \rangle$ . However, as the rows of the matrix  $\mathcal{M}$  of the size  $16 \times 16$  are the same, we have  $| \mathcal{Y}_1 \rangle = | \mathcal{Y}_2 \rangle = | \mathcal{Y}_3 \rangle = \dots = | \mathcal{Y}_{16} \rangle$ . In terms of the above notation the partition function  $\lambda$  can be written as

$$\lambda = \sum_{i,j} \langle \mathcal{X}_i | \mathbf{m}[j] | \mathcal{Y}_j \rangle. \quad (8)$$

Denoting by  $q[j]$  the probability of occurrence of the matrix  $\mathbf{m}[j]$  and knowing that

$$\mathbf{n}[j] = -\frac{d}{dK} \mathbf{m}[j], \quad j = 1, 2, 3, \dots, 16, \quad (9)$$

we get the expression for the energy in the form:

$$E = \sum_{i,j} q[i]q[j] \frac{\langle \mathcal{X}_i | \mathbf{n}[j] | \mathcal{Y}_j \rangle}{\langle \mathcal{X}_i | \mathbf{m}[j] | \mathcal{Y}_j \rangle}, \quad (10)$$

this value has been found numerically for the interaction constant  $K = 50$ . This value of the interaction constant corresponds to such a low temperature that the energy value differs only slightly from its value in the ground state [2], so it is possible to evaluate the quality of the approximation performed. Figure 1 presents a comparison between the results obtained with Eq. (10) (dotted line) and the exact solution in the ground state (solid line). As the lines differ significantly, similar calculations were performed for the segments of the lengths of two, three and four bonds. Although with increasing size of the segments the results improve, but the convergence of the corresponding energies is slow.

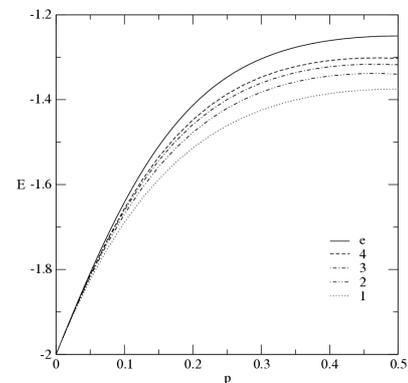


Fig. 1. The internal energy per spin  $E$  versus the concentration of the antiferromagnetic exchange integrals  $p$  for the tube, for the temperature determined by the interaction constant  $K = 50$ , for four sizes of the segment considered: 1 — one bond segment, 2 — two bonds, 3 — three bonds, 4 — four bonds and the solid line represents the exact result for the ground state.

Analogous calculations of the ground state energy and at temperatures corresponding to the interaction constant  $K = 50$  were made for the ribbon composed of the squares. Figure 2 presents the dependence of the energy of the ribbon in the ground state on the concentration of the admixed bonds, for the temperature corresponding to  $K = 50$  for the length of four segments. As follows from the figure, starting from the two-bond approximation the results are a good approximation of the exact results for the ground state. Despite the similarity of the two systems, i.e. the ribbons and tubes, the approximations of their internal energies are significantly different. In order to obtain the same degree of approximation of the internal energy as that obtained for the ribbon when the segments of the lengths of two or three bonds are considered for the tube it would be necessary to consider the lengths of a few or a few ten bonds. Therefore, it seems that there is no other possible reason explaining these

differences but the dependence of the entropy contributions on the size of the degenerated blocks in the ground state; the degenerated blocks are the smallest parts of the system bringing additive contributions to entropy. The dependence is shown in Fig. 3, where the squares correspond to the contributions of the frustrated blocks for the ribbon and circles correspond to the analogous contributions for the tube. For the ribbon the blocks bringing the greatest contribution (of the greatest entropy per spin) are those of the length smaller than 5 bonds, and the contributions to entropy decrease with increasing length of the block. For the tube, a comparable entropy per spin is brought by the blocks of the lengths from 1 to 18 bonds, and the maximum contribution is brought by a block of 5 bonds. If for example the approximation is made with a block of the length of 4 bonds, the probability that we do not cross a degenerated block is much higher for the ribbon than for the tube.

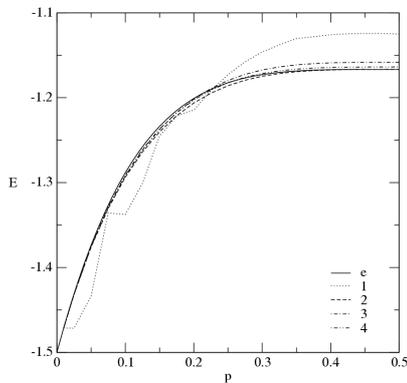


Fig. 2. The internal energy for spin  $E$  versus the concentration of the antiferromagnetic exchange integrals  $p$  for the ribbon, for the temperature determined by the interaction constant  $K = 50$ , for the four sizes of the segment considered: 1 — one bond, 2 — two bonds, 3 — three bonds, 4 — four bonds and the solid line — exact result for the ground state.

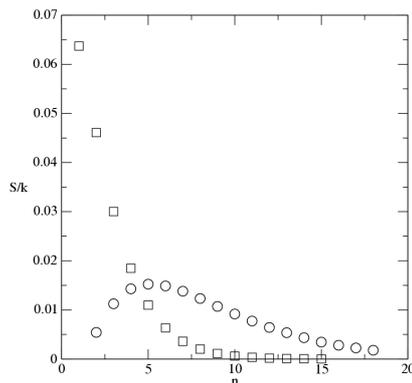


Fig. 3. The contributions to the entropy per spin  $S$  corresponding to the degenerated blocks of the length  $n$  for the tube (circles) and ribbon (squares).

The condition of the indivisibility of blocks includes the condition of the accuracy of entropy determination and hence the condition of the probability of states. In this way we get the image of degenerated blocks with well localised excitations inside. The division of the degenerated blocks is equivalent to the approximation leading to restriction of the excitation to one of the block fragments, although actually they are localised in the whole block. In general, the internal energy is composed of two components: the localised and nonlocalised ones. In the system in which the former brings a greater contribution (ribbon) a description with the help of segments of the length of a few bonds is satisfactory, while when the second component is more important (tube) this description is a rougher approximation.

#### 4. Entropy

Analogous calculations of entropy have been performed for the tube (the ribbon). If  $p(j)$  denotes the probability of occurrence of four (a pair of) squares and  $w(i, j)$  is the density of probability of occurrence of the four spins of the second pair of squares (second square), then the entropy per spin can be written as

$$S/k = -\frac{1}{4} \sum_j p(j) \sum_i [w(i, j) \ln w(i, j)]. \quad (11)$$

For  $K = 50$  this expression has been tabulated and the relevant plot is shown in Fig. 4 for the tube and in Fig. 5 for the ribbon (line 1) and compared with the results obtained for the ground state (line e). Apart from the approximation with the use of a one bond segment the calculations have been made for the segments of the lengths of two, three and four (for the ribbon) bonds.

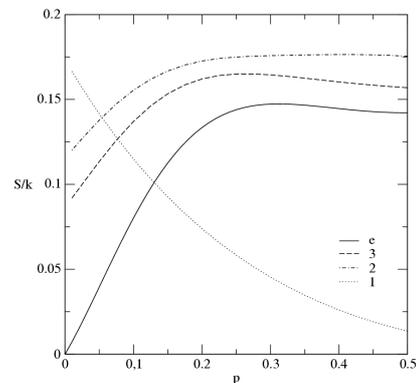


Fig. 4. The entropy per spin  $S$  versus the concentration of the antiferromagnetic exchange integrals  $p$  for the tube, for the temperature determined by the interaction constant  $K = 50$  and for the three sizes of the segment considered: 1 — one bond, 2 — two bonds, 3 — three bonds and the solid line — exact results for the ground state.

The errors in the calculations follow from three reasons. The first reason is that the edge of the ribbon introduces irregularities, similarly as it was for the internal

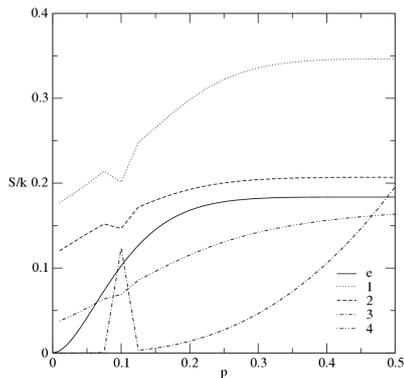


Fig. 5. The entropy per spin  $S$  versus the concentration of the antiferromagnetic exchange integrals  $p$  for the ribbon, for the temperature determined by the interaction constant  $K = 50$  and for the four sizes of the segment considered: 1 — one bond, 2 — two bonds, 3 — three bonds, 4 — four bonds and the solid line — exact results for the ground state.

energy. The second reason are the errors in determination of the probability following from the division of the degenerated blocks, also discussed for the internal energy. The greatest errors come from consideration of the finite fragments of the lattice, Eq. (11). However, these

errors can be easily determined when the concentration  $p$  is low.

## 5. Summary

The paper presents a comparison of physical properties and the possibility of their determination for the two systems differing in the presence of the edge. The presence of the edge increases the accuracy of the approximation considered. For the tube the approximation involves more frequent division of degenerated blocks introducing inaccuracies in the probability. The latter is accompanied by the appearance of the delocalised character of the measurable quantities (in this case the internal energy). In determination of entropy the error following from a finite size of the fragment considered is greater.

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