Exact Travelling Wave Solutions for a Modified Zakharov–Kuznetsov Equation

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(Received October 6, 2007, revised version September 8, 2008)

The extended mapping method is developed to study the traveling wave solution for a modied Zakharov-Kuznetsov equation. A variety of traveling periodic wave solutions in terms of the Jacobi elliptic functions are obtained. Limit cases are studied, and solitary wave solutions are got.

PACS numbers: 05.45.Yv, 02.30.Ik, 02.30.Jr

1. Introduction

The modied Zakharov–Kuznetsov (mZK) equation,

$$u_t + u^2u_x + u_{xxx} + u_{xyy} = 0, \qquad (1)$$

represents an anisotropic two-dimensional generalization of the Korteweg–de Vries equation and can be derived in a magnetized plasma for small amplitude Alfvén waves at a critical angle to the undisturbed magnetic field, and has been studied by many authors because of its importance [1–5]. However, Eq. (1) possesses many interesting traveling wave structures which have not yet been found.

In the study of equations modeling wave phenomena, one of the fundamental objects of study is the traveling wave solution, meaning a solution of constant form moving with a fixed velocity. Traveling waves, whether their solution expressions are in explicit or implicit forms, are very interesting from the point of view of applications. These types of waves will not change their shapes during propagation and are thus easy to detect. Of particular interest are three types of traveling waves: the solitary waves, which are localized traveling waves, asymptotically zero at large distances, the periodic waves, and the kink waves, which rise or descend from one asymptotic state to another. Recently, a unified method, called the extended mapping method, is developed to obtain exact traveling wave solutions for a large variety of nonlinear partial differential equations [6–8]. By means of this method, the solitary wave, the periodic wave and the kink wave (or the shock wave) solutions can, if they exist, be obtained simultaneously to the equation in question. Thus, many of tedious and repetitive calculations may be avoided. The method is further developed in this paper to study the traveling wave solution of Eq. (1). For a given nonlinear evolution equation, say, in three independent variables

$$N(u, u_t, u_x, u_y, \ldots) = 0, \qquad (2)$$

we seek its traveling wave solution of the form

$$u = u(\xi), \qquad \xi = kx + ly - \omega t. \qquad (3)$$

Without loss of a generality, we assume that $k > 0$. Substituting Eq. (3) into Eq. (2) yields an ordinary differential equation of $u(\xi)$. Then $u(\xi)$ is expanded into a polynomial in $f(\xi)$ and $g(\xi)$:

$$u = A_0 + \sum_{i=1}^{n} f^{i-1}(A_i f + B_i g), \qquad (4)$$

where A_i and B_i are constants to be determined, and n is fixed by balancing the linear term of the highest order derivative with nonlinear term in Eq. (2), while f satises the equation

$$f' = \sqrt{pf^2 + \frac{1}{2}qf^4 + r}, \qquad (5)$$

g satisfying the same equation with different coefficients possibly. According to the feature of Eq. (1) and to compute conveniently, however, in this paper, g is restricted to fullling the following relation:

$$g'' = g(c_1 + c_2f^2), \qquad g^2 = c_3 + c_4f^2, \qquad (6)$$

where c_i are constants to be determined. Substituting Eq. (4) with Eqs. (5) and (6) into the ordinary equation, and equating the coefficients of like powers of f^ig^j $(j = 0, 1)$, these constants may be determined. If any of the parameters is left unspecified, it is regarded as being arbitrary for the solution to Eq. (1).

2. Exact traveling wave solutions to Eq. (1)

Substituting Eq. (3) into Eq. (1) and integrating once, we have

$$(k^3 + kl^2)u'' + \frac{1}{3}ku^3 - \omega u = C, \qquad (7)$$

where C is the integration constant. According to the method described above, we assume that Eq. (7) has the solution in the form

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$$u = A_0 + A_1 f + B_1 g, \tag{8}$$

where A_i and B_1 are constants to be determined, and f and g satisfy Eqs. (5) and (6). The substitution of Eq. (8) with Eqs. (5) and (6) into Eq. (7), equating the coefficients of like powers of $f^i g^j$ $(j = 0, 1)$, yields

$$(k^3 + kl^2)qA_1 + \frac{1}{3}kA_1^3 + kc_4A_1B_1^2 = 0,$$
$$kA_1^2B_1 + (k^3 + kl^2)c_2B_1 + \frac{1}{3}kc_4B_1^3 = 0,$$
$$kA_0A_1^2 + kc_4A_0B_1^2 = 0,$$
$$2kA_0A_1B_1 = 0,$$
$$(k^3 + kl^2)pA_1 - \omega A_1 + kA_0^2A_1 + kc_3A_1B_1^2 = 0,$$
$$-\omega B_1 + kA_0^2B_1 + \frac{1}{3}kc_3B_1^3 + (k^3 + kl^2)c_1B_1 = 0,$$
$$-\omega A_0 + \frac{1}{3}kA_0^3 + kc_3A_0B_1^2 = C. \tag{9}$$

From here, one gets two sets of solutions

$$A_0 = B_1 = 0,$$
$$A_1 = \pm\sqrt{-3(k^2 + l^2)q}, \quad \omega = (k^3 + kl^2)p, \tag{10}$$

and

$$A_1 = \pm\sqrt{-\frac{(2c_2c_3 + c_1c_4 - c_4p)(k^2 + l^2)}{2c_3}},$$
$$B_1 = \pm\sqrt{\frac{3(c_1 - p)(k^2 + l^2)}{2c_3}},$$
$$A_0 = 0, \quad \omega = \frac{1}{2}(3c_1 - p)(k^3 + kl^2),$$
$$3qc_3 + 4c_4(c_1 - p) - c_2c_3 = 0. \tag{11}$$

Thus, one obtains two kinds of exact solutions for Eq. (1) as follows:

$$u = \pm\sqrt{-3(k^2 + l^2)q}\,f(\xi), \tag{12}$$

with $\xi = kx + ly - (k^3 + kl^2)pt$, and $f(\xi)$ satisfying Eq. (5), and

$$u = \pm\sqrt{-\frac{(2c_2c_3 + c_1c_4 - c_4p)(k^2 + l^2)}{2c_3}}\,f(\xi)$$
$$\pm\sqrt{\frac{3(c_1 - p)(k^2 + l^2)}{2c_3}}\,g(\xi), \tag{13}$$

where $\xi = kx + ly - \frac{1}{2}(3c_1 - p)(k^3 + kl^2)t$, and $f(\xi)$ and $g(\xi)$ satisfy Eqs. (5) and (6), with the constraint among the parameters

$$3qc_3 + 4c_4(c_1 - p) - c_2c_3 = 0.$$

The specific expressions of solutions according to the value of constants c_i, p, q and r is a trivial exercise due to Ref. [8]. In what follows, we study a new type of solution to Eq. (1).

3. Rational traveling wave solutions of Eq. (1)

Generally speaking, one can only obtain the traveling wave solutions of polynomial form by the method used in previous section. In this section, in order to get rational traveling wave solutions of Eq. (1), we introduce a new expansion of the form

$$u = \frac{bf}{d + f}, \tag{14}$$

instead of Eq. (8) for Eq. (7), where f satises Eq. (5) and b and d are constants to be determined. Multiplying $(d + f)^3$ on the two sides of Eq. (7), then substituting Eq. (4) with Eq. (5) into it, and collecting the coefficients of like powers of f, we obtain

$$-3C + b^3k + 3b[d^2(k^3 + kl^2)q - \omega] = 0,$$
$$3C + b[(k^3 + kl^2)p + 2\omega] = 0,$$
$$-3C + b[(k^3 + kl^2)p - \omega] = 0,$$
$$Cd^2 + 2b(k^3 + kl^2)r = 0. \tag{15}$$

It follows from here that

$$C = \pm\sqrt{-3k^2(k^2 + l^2)^3p(p^2 - 2qr)},$$
$$\omega = -2(k^3 + kl^2)p,$$
$$b = \frac{C}{(k^3 + kl^2)p} = \pm\frac{\sqrt{-3(k^2 + l^2)p(p^2 - 2qr)}}{p},$$
$$d = \pm\sqrt{-\frac{2r}{p}}. \tag{16}$$

Therefore, we obtain a new exact solution of Eq. (1):

$$u = \frac{\pm\sqrt{-3(k^2 + l^2)p(p^2 - 2qr)}\,f(\xi)}{p\left[\pm\sqrt{-2r/p} + f(\xi)\right]}, \tag{17}$$

with $\xi = kx + ly + 2(k^3 + kl^2)pt$, where $f(\xi)$ satisfies Eq. (5), and the choice of signs is arbitrary. Thus, the rational traveling wave solutions of Eq. (1) obtained by Eq. (17) are as follows:

$$u_1 = \frac{\pm\sqrt{3(k^2 + l^2)(1 + m^2)}(1 - m^2)\mathrm{sn}\xi}{\pm\sqrt{2(1 + m^2)} + (1 + m^2)\mathrm{sn}\xi}, \tag{18}$$

with $\xi = kx + ly - 2(k^3 + kl^2)(1 + m^2)t$, where $\mathrm{sn}\xi = \mathrm{sn}(\xi|m)$, and below $\mathrm{cn}\xi = \mathrm{cn}(\xi|m)$ and $\mathrm{dn}\xi = \mathrm{dn}(\xi|m)$, are the Jacobi elliptic functions with m $(0 < m < 1)$ being the modulus of the elliptic function. Detailed explanations about the Jacobi elliptic functions can be found in Refs. [9, 10]. Figure 1 illustrates its spatial structure (taking two plus signs in Eq. (18)):

$$u_2 = \frac{\pm\sqrt{3(k^2 + l^2)(1 + m^2)}(1 - m^2)\mathrm{cn}\xi}{\pm\sqrt{2(1 + m^2)}\mathrm{dn}\xi + (1 + m^2)\mathrm{cn}\xi}, \tag{19}$$

with the same ξ as that of u_1. Its typical spatial structure is plotted in Fig. 2:

$$u_3 = \frac{\pm\sqrt{3(k^2 + l^2)}(1 - m^2)}{\pm\sqrt{2}m\mathrm{sn}\xi + \sqrt{(1 + m^2)}}, \tag{20}$$

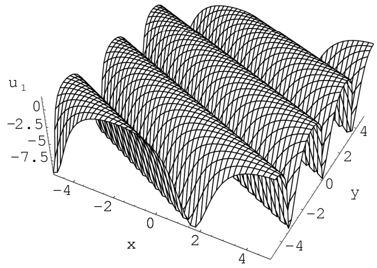

Fig. 1. The spatial structure graph of Eq. (18).

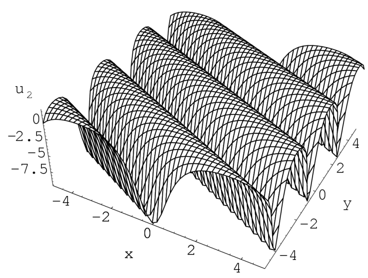

Fig. 2. The spatial structure graph of Eq. (19).

with the same ξ as that of u_1.

$$u_4 = \frac{\pm\sqrt{3(k^2+l^2)}(1-m^2)\mathrm{dn}\xi}{\pm\sqrt{2}m\mathrm{cn}\xi + \sqrt{(1+m^2)}\mathrm{dn}\xi}, \tag{21}$$

with the same ξ as that of u_1.

$$u_5 = \frac{\pm\sqrt{-3(k^2+l^2)(2m^2-1)(1-8m^2)}\mathrm{cn}\xi}{\pm\sqrt{-2(1-m^2)(2m^2-1)} + (2m^2-1)\mathrm{cn}\xi}, \tag{22}$$

with $\xi = kx + ly + 2(k^3+kl^2)(2m^2-1)t$. Equation (22) is a real solution under the condition $0 < m \leq \sqrt{2}/4$:

$$u_6 = \frac{\pm\mathrm{i}\sqrt{3(k^2+l^2)(2-m^2)}m^2}{\pm\sqrt{2(2-m^2)}\mathrm{dn}\xi + (2-m^2)}, \tag{23}$$

a complex periodic wave solution, with $\xi = kx + ly + 2(k^3+kl^2)(2-m^2)t$. As $m \to 1$, Eq. (23) degenerates to

$$u_7 = \frac{\pm\mathrm{i}\sqrt{3(k^2+l^2)}}{\pm\sqrt{2}\mathrm{sech}\xi + 1}, \tag{24}$$

a complex solitary wave solution, with $\xi = kx + ly + 2(k^3+kl^2)t$.

$$u_8 = \frac{\pm\mathrm{i}\sqrt{3(k^2+l^2)}m^2\mathrm{cn}\xi}{\pm\mathrm{i}\sqrt{2(1-m^2)}\mathrm{sn}\xi + \sqrt{2-m^2}\mathrm{cn}\xi}, \tag{25}$$

a new complex periodic wave solution, with the same ξ as that of u_6.

$$u_9 = \frac{\pm\sqrt{-3(k^2+l^2)(2m^2-1)}}{\pm\sqrt{2(2m^2-1)}m\mathrm{cn}\xi + (2m^2-1)}, \tag{26}$$

with the same ξ as that of u_5.

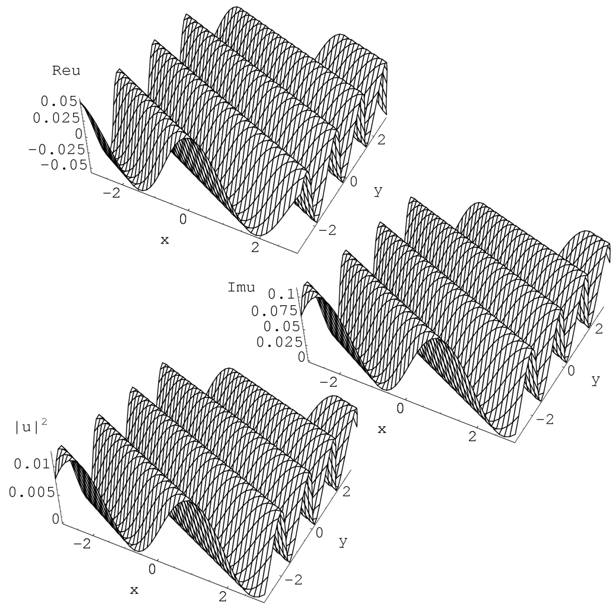

Fig. 3. The spatial structure graph of Eq. (29).

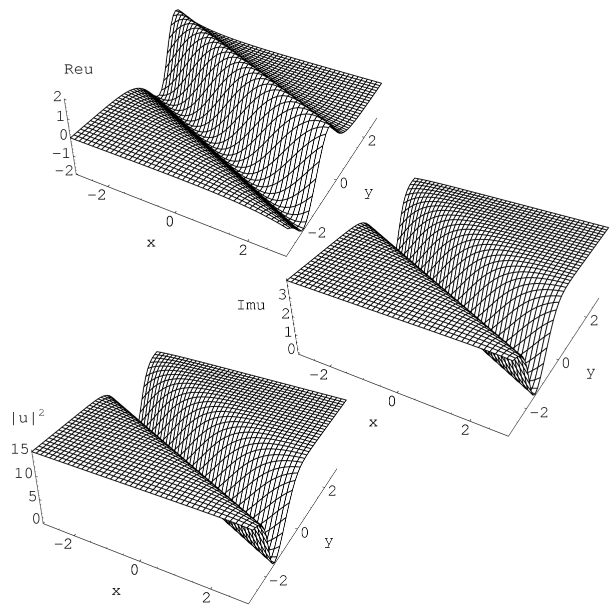

Fig. 4. The spatial structure graph of Eq. (30).

$$u_{10} = \frac{\pm\sqrt{-3(k^2+l^2)}m^2\mathrm{dn}\xi}{\pm\sqrt{2(1-m^2)} + \sqrt{2-m^2}\mathrm{dn}\xi}, \tag{27}$$

with the same ξ as that of Eq. (23).

$$u_{11} = \frac{\pm\sqrt{-3(k^2+l^2)(2m^2-1)}\mathrm{dn}\xi}{\pm\sqrt{2(1-m^2)(2m^2-1)}m\mathrm{sn}\xi + (2m^2-1)\mathrm{dn}\xi}, \tag{28}$$

with the same ξ as that of Eq. (22).

$$u_{12} = \frac{\pm \mathrm{i}\sqrt{3(k^2+l^2)}m^2 \mathrm{sn}\xi}{\pm \mathrm{i}\sqrt{2}\mathrm{cn}\xi + \sqrt{2-m^2}\mathrm{sn}\xi}, \tag{29}$$

with the same ξ as that of Eq. (23). This is a complex periodic wave solution. The structures of its real part, imagine part (with plus signs) and square of module are illustrated in Fig. 3. As $m \to 1$, Eq. (29) degenerates to

$$u_{13} = \frac{\pm \mathrm{i}\sqrt{3(k^2+l^2)}\tanh\xi}{\pm \mathrm{i}\sqrt{2}\mathrm{sech}\xi + \tanh\xi}, \tag{30}$$

with $\xi = kx + ly + 2(k^3 + kl^2)t$, a complex line solitary wave solution to Eq. (1), and the corresponding structure is shown in Fig. 4.

$$u_{14} = \frac{\pm\sqrt{-3(k^2+l^2)(2m^2-1)}\mathrm{sn}\xi}{\pm\sqrt{-2(2m^2-1)}m\mathrm{dn}\xi + (2m^2-1)\mathrm{sn}\xi}, \tag{31}$$

with the same ξ as that of Eq. (22). Equation (31) is a real solution under the condition $0 < m \le \sqrt{2}/2$.

4. Conclusion and discussion

By further developing the extended mapping method, the exact traveling wave solutions to the modified Zakharov–Kuznetsov equation have been studied. Abundant periodic wave solutions, real or complex, polynomial form or rational form, in terms of the Jacobi elliptic functions are obtained. Limit cases are studied and exact solitary wave solutions are got. Some of the solutions obtained in this paper develop singularity at a finite point, i.e. for any fixed $t = t_0$, there exists x_0 at which the solutions blow up. There is much interest in the formation of so-called hot spots or blow up of solutions [11, 12]. It appears that these singular solutions will model these physical phenomena. It can be easily seen that the method used in this paper may further be improved to solve more nonlinear partial differential equations.

Acknowledgments

The author would like to acknowledge the anonymous referees for their valuable comments. This work is supported by Research Fund for the Doctoral Program of Higher Education of China (No. 20070486094).

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