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Bounds for Value at Risk for Multiasset Portfolios

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The theory of copulas provides a useful tool for modeling dependence in risk management. In insurance and finance, as well as in other applications, dependence of extreme events is particularly important, hence there is a need for the detailed study of the tail behaviour of the multivariate copulas. In this paper we investigate the class of copulas being the weighted means of copulas having homogeneous lower tails. We show that having only such information on the structure of dependence of returns from assets is enough to get estimates on value at risk of the multiasset portfolio in terms of value at risk of one-asset portfolios.

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1. Introduction

The aim of this paper is to show the advantages of modeling the dependence between the extreme events with the help of copulas. Let us consider the following case. An investor operating on an emerging market, has in his portfolio several currencies which are highly dependent. Let s_i, $i = 1, \ldots, d$ be the quotients of the currency rates at the end and at the beginning of the investment. Let w_i be the part of the capital invested in the i-th currency, $\sum w_i = 1$, $w_i \geq 0$. Therefore the final value of the portfolio equals

$$W_1(w) = (w_1 s_1 + \ldots + w_d s_d) W_0. \tag{1}$$

For portfolio consisting of only one currency (say i-th) we put

$$w = e_i = (0, \ldots, 0, 1, 0, \ldots, 0). \tag{2}$$

Let us note that at the moment of the beginning of the investment only W_0 and w_i's are known. s_i's remain uncertain, therefore we represent them by random variables on a certain probability space $(\Omega, \mathcal{F}, P)$.

The crucial point is to estimate the risk of the investment. As a measure of risk we shall consider "value at risk" (VaR), which in last years became one of the most popular measures of risk in the "practical" quantitative finance (see for

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example [1–9]). Roughly speaking the idea is to determine the biggest amount one can lose on a certain confidence level $1 - \alpha$:

$$\mathrm{VaR}_{1-\alpha}(w) = \sup\{V \colon P(W_0 - W_1(w) \leq V) < 1 - \alpha\}. \tag{3}$$

In order to determine VaR accurately one has to deal with the complexity of the problem. The extremes hardly follow the normal distribution law. Therefore the main challenge is to describe properly the interdependences of risk factors (in our case the changes of currency rates). In this presentation, it will be based on copulas, which are scaleless dependence measures of random variables. We will show that sometimes it is enough to have only the partial information on the given copula.

The main result we would like to present is the diversification formula, i.e.. the following estimate of the value at risk of a given portfolio w in terms of value at risk of one-currency portfolios e_i (cf. [10] for two-dimensional case):

$$\sum w_i \mathrm{VaR}_{1-\alpha}(e_i) \geq \mathrm{VaR}_{1-\alpha}(w). \tag{4}$$

The above estimate is valid for sufficiently small α under the mild assumptions:

- The copula C of s_i's is a weighted mean (mixture) of copulas C_i having nonzero homogeneous lower tails,

$$C(q) = a_0 C_0(q) + a_1 C_1(q) + \ldots + a_m C_m(q), \quad a_0, \ldots, a_m \geq 0,$$

$$\sum_{i=0}^{m} a_i = 1 \tag{5}$$

  and for sufficiently small q:

$$C_i(q) = L_i(q), \quad \forall t > 0 \quad L_i(tq) = t^{k_i} L_i(q),$$

$$1 = k_0 < k_1 < \ldots < k_m. \tag{6}$$

- For $i = 1, \ldots, d$, for sufficiently small x, the function $G_i(x) = 1/F_i(x)$, where F_i is the distribution function of s_i, is convex (i.e. the hazard rate $F'(t)/F(t)^2$ is decreasing).
- For $i = 1, \ldots, d$, for a positive w and for sufficiently small α:

$$F_i(w \cdot (F_1^{-1}(\alpha) + \ldots F_d^{-1}(\alpha))) \leq \begin{cases} \delta \alpha^{\frac{1}{k_1}} & \text{if} \quad m \geq 1, \\ \delta & \text{if} \quad m = 0. \end{cases} \tag{7}$$

The first assumption is modelling the asymptotic dependence (cf. [11] Th. 2). For example it describes very well the behaviour of foreign exchange rates on an emerging market, where the *extreme* changes are usually due to the local factors (cf. [10]).

The second one is fulfilled by a wide variety of probability laws. For example it is valid if the distributions of $-\ln s_i$ have the same upper tails as normal, Pareto or Gamma distribution (i.e. if their distribution functions coincide for enough big arguments).

The last one means that the probability distributions of the extreme downfalls for different currencies "behave" in a similar way.

2. Notation

2.1. Copulas

We recall that a function

$$C : [0,1]^d \longrightarrow [0,1] \tag{8}$$

is called a copula (see [12] §2.10, [13] §4.1, [14] §4.4) if for every $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$ $(u_i, v_i \in [0,1])$ and every $j \in \{1, \dots, d\}$

$$\text{(i)} \qquad u_j = 0 \Rightarrow C(u) = 0; \tag{9}$$

$$\text{(ii)} \qquad (\forall i \neq j \ u_i = 1) \Rightarrow C(u) = u_j; \tag{10}$$

$$\text{(iii)} \qquad u \preceq v \Rightarrow V_C(u,v) \geq 0, \tag{11}$$

where $u \preceq v$ denotes the partial ordering on $\mathbb{R}^d$ $(u \preceq v \Leftrightarrow \forall i \ u_i \leq v_i)$ and $V_C(u,v)$ is the C-volume of the rectangle $I(u,v)$, the one with lower vertex u and upper vertex v.

$$V_C(u,v) = \sum_{j_1=1}^{2} \dots \sum_{j_d=1}^{2} (-1)^{j_1+\dots+j_d} C(a_{1,j_1}, \dots, a_{d,j_d}), \tag{12}$$

where $a_{i,1} = u_i$ and $a_{i,2} = v_i$ for $i = 1, \dots, d$. The functions with the last property are called n-nondecreasing. Those which fulfill the first one are called grounded.

Remark **1.** *(cf. [15], Th. 12.5) Every continuous, grounded, n-nondecreasing function*

$$H : [0,a]^d \longrightarrow \mathbb{R} \tag{13}$$

is a distribution function of a Borel measure μ_H on $[0,a]^d$:

$$H(u) = \mu_H(I(0,u)), \tag{14}$$

$$\mu_H(I(u,v)) = \mu_H(\mathrm{int}(I(u,v))) = V_H(u,v). \tag{15}$$

Due to the second condition every copula is a distribution function of a probability measure on the unit rectangle $[0,1]^d$ with uniform margins (cf. [16], §1.6). Furthermore, the above remark remains true if H is defined on the whole multioctant $[0,+\infty)^d$.

Let $\mathcal{X}_i$, $i = 1, \dots, d$ be random variables defined on the same probability space $(\Omega, \mathcal{M}, \mathbb{P})$. Their joint cumulative distribution $F_{\mathcal{X}}$ can be described using an appropriate copula $C_{\mathcal{X}}$ ("Sklar Theorem" see [12], Theorem 2.10.11, [13], Theorem 4.5):

$$F_{\mathcal{X}}(x) = C_{\mathcal{X}}(F_{\mathcal{X}_1}(x_1), \dots, F_{\mathcal{X}_d}(x_d)), \tag{16}$$

where $F_{\mathcal{X}_i}$ are cumulative distributions of $\mathcal{X}_i$. Let us note that the strictly increasing transformations of random variables $\mathcal{X}_i$ do not affect the copula. Indeed, if

$$\mathcal{X}'_i = f_i(\mathcal{X}_i), \quad i = 1, \dots, d, \tag{17}$$

where f_i are strictly increasing (and so invertible), then

$$F_{\mathcal{X}'}(x) = F_{\mathcal{X}}(f_1^{-1}(x_1), \dots, f_d^{-1}(x_d)) \tag{18}$$

$$= C_{\mathcal{X}}(F_{\mathcal{X}_1}(f_1^{-1}(x)), \ldots, F_{\mathcal{X}_d}(f_d^{-1}(x_d))) = C_{\mathcal{X}}(F_{\mathcal{X}'_1}(x_1), \ldots, F_{\mathcal{X}'_d}(x_d)). \quad (19)$$

Therefore if one is interested in tail dependence of random variables rather than in their *individual* distribution, then the proper choice is to study the copula. The more so, since the copula is uniquely determined at every point u such that the equations $F_{\mathcal{X}_i}(x_i) = u_i$ have solutions.

For various applications of copulas to finance see for example [13, 17–19, 10, 20].

2.2. Model assumptions

We assume that for $t > 0$ the distribution function of each $s_i - F_i(t)$ is positive and the joint probability distribution of s_i's is continuous with respect to the Lebesgue measure and is determined by a copula C:

$$F_s(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)). \quad (20)$$

Furthermore, C is a weighted mean of copulas C_j:

$$C(q) = a_0 C_0(q) + a_1 C_1(q) + \ldots + a_m C_m(q),$$

$$a_0, \ldots, a_m \geq 0, \quad \sum_{i=0}^{m} a_i = 1, \quad (21)$$

and there is a constant $\delta \in (0,1)$ such that:

A1. For $q = (q_1, \ldots q_d)$ and $0 \leq q_i \leq \delta$, $C_j(q) = L_j(q)$, where L_j is some nonzero positive homogeneous function of degree k_j, $1 = k_0 < k_1 < \ldots < k_m$ $(\forall t > 0 \quad L_j(tq) = t^{k_j} L_j(q))$.

A2. For $i = 1, \ldots, d$ the function $G_i(t) = \frac{1}{F_i(t)}$ restricted to $t \in F_i^{-1}((0, \delta])$ is convex.

A3. For $i = 1, \ldots, d$

$$\forall w > 0 \quad \exists \alpha_0 \quad \forall 0 < \alpha \leq \alpha_0$$

$$F_i(w \cdot (F_1^{-1}(\alpha) + \ldots F_d^{-1}(\alpha))) \leq \begin{cases} \delta \alpha^{\frac{1}{k_1}} & \text{if} \quad m \geq 1, \\ \delta & \text{if} \quad m = 0. \end{cases} \quad (22)$$

The second assumption implies that the preimage of δ consists of just one point and F_i restricted to $[0, F_i^{-1}(\delta)]$ is strictly increasing. Therefore we get a simpler formula for value at risk of one-asset portfolios.

Corollary **1.** *For* $\alpha \in (0, \delta]$,

$$\mathrm{VaR}_{1-\alpha}(e_i) = W_0 \cdot (1 - F_i^{-1}(\alpha)), \quad i = 1, \ldots, d. \quad (23)$$

Let us note that the above formula is useful for the practical applications. It reduces the determination of VaR for one-asset portfolios to the estimation of the α-quantile of the quotient of the currency rates at the end and at the beginning of the investment, which can be accomplished by a standard statistical procedure.

In [21, 11] we showed that there is a large class of copulas whose tails can be approximated by a homogeneous function L of degree 1. Let us recall the basics about such L's. Comparing [11], Theorem 3, and the construction from the proof of Proposition 6 (also [11]) one gets

THEOREM **1.** *For a homogeneous of degree 1 function L, $L : [0,+\infty)^d \to \mathbb{R}$, the following conditions are equivalent:*

1. *L is equal to the lower tail of some copula C.*
2. *L is d-nondecreasing and*

$$0 \le L(u) \le \min(u_1, \ldots, u_d) \quad \text{for} \quad u \succeq 0. \tag{24}$$

3. *L is continuous, grounded, d-nondecreasing and $\mu_L = m \times \mu_\Delta$, where m is the Lebesgue measure on the real halfline and μ_Δ is a measure on the unit simplex $\Delta = \{q \in \mathbb{R}_+^d : q_1 + \ldots + q_d = 1\}$ such that*

$$\int_\Delta \frac{1}{q_i} \mathrm{d}\mu_\Delta(q) \le 1 \quad for \quad i = 1, \ldots, d. \tag{25}$$

Basing on Theorem 1 we can reduce by one the dimensionality of our problem. Indeed, the multioctant is the Cartesian product of a halfline and simplex, $\mathbb{R}_+^d = \mathbb{R}_+ \times \Delta$. Therefore, due to the Fubini Theorem, as a consequence of point 3 of the above theorem, we get the following fact.

COROLLARY **2.** *For every closed set A, $A \subset \mathbb{R}_+^d$,*

$$\mu_L(A) = \int_\Delta m(\mathbb{R}_+\xi \cap A) \mathrm{d}\mu_\Delta(\xi), \tag{26}$$

where $\mathbb{R}_+\xi$ is a halfline spanned by vector ξ.

3. The estimate

We assume, that $\forall i \quad w_i > 0$.

THEOREM **2.** *For α from $(0,1)$, such that*

$$\sum_{i=1}^d w_i F_i^{-1}(\alpha) \le \min\{w_j F_j^{-1}(\delta^*) : j = 1, \ldots, d\},$$

$$\delta^* = \begin{cases} \delta\alpha^{\frac{1}{k_1}} & if \quad m \ge 1, \\ \delta & if \quad m = 0, \end{cases} \tag{27}$$

the following inequality holds:

$$\mathrm{VaR}_{1-\alpha}(w) \le w_1 \mathrm{VaR}_{1-\alpha}(e_1) + \ldots + w_d \mathrm{VaR}_{1-\alpha}(e_d). \tag{28}$$

Let us note that due to condition A3 the set of α fulfilling the assumptions of theorem 2 is not empty. The proof of the theorem will be based on the quantile transformation and properties of the following family of sets.

For $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\lambda_i > 0$, we put

$$Y_\lambda = \left\{ q \in \mathbb{R}_+^d : \sum_{i=1}^d \frac{\lambda_i}{q_i} \ge 1 \right\}. \tag{29}$$

LEMMA **1.**

$$\mu_{L_0}(Y_\lambda) \le \sum \lambda_i. \tag{30}$$

Proof.

We base on the fact that L_0 is homogeneous of degree 1 and

$$\mu_{L_0}(Y_\lambda) = \int_\Delta m(\mathbb{R}_+\xi \cap Y_\lambda)\mathrm{d}\mu_\Delta(\xi). \tag{31}$$

The intersection of Y_λ and the halfline given by the vector ξ is a segment of length $\sum \frac{\lambda_i}{\xi_i}$,

$$\mathbb{R}_+\xi \cap Y_\lambda = \left\{t : \sum \frac{\lambda_i}{t\xi_i} \geq 1\right\} = \left\{t : 0 \leq t \leq \sum \frac{\lambda_i}{\xi_i}\right\}. \tag{32}$$

Therefore

$$\mu_{L_0}(Y_\lambda) = \int_\Delta \sum \frac{\lambda_i}{\xi_i}\mathrm{d}\mu_\Delta(\xi) = \sum \lambda_i \int_\Delta \sum \frac{1}{\xi_i}\mathrm{d}\mu_\Delta(\xi) \leq \sum \lambda_i. \tag{33}$$

□

For $r > 0$ we put

$$V_r = \left\{q \in \mathbb{R}^d_+ : \sum_{i=1}^d w_i F_i^{-1}(q_i) \leq r\right\}. \tag{34}$$

Lemma **2.** *For positive r and $\alpha \in (0,1)$ such that*

$$r = \sum_{i=1}^d w_i F_i^{-1}(\alpha) \leq \min\{w_j F_j^{-1}(\delta) : j = 1,\ldots,d\} \tag{35}$$

the following inclusions hold:

$$V_r \subset \left[0, F_1\left(\frac{r}{w_1}\right)\right] \times \ldots \times \left[0, F_d\left(\frac{r}{w_d}\right)\right] \subset [0,\delta]^d, \quad V_r \subset Y_\lambda, \tag{36}$$

where

$$\lambda_i = \alpha \frac{w_i c_i^{-1}}{\sum w_j c_j^{-1}}; \quad c_j = F_j'(F_j^{-1}(\alpha)). \tag{37}$$

Proof.

If q belongs to V_r then

$$\sum_{i=1}^d w_i F_i^{-1}(q_i) \leq r = \sum_{i=1}^d w_i F_i^{-1}(\alpha) \leq \min\{w_j F_j^{-1}(\delta)\}. \tag{38}$$

Therefore for each i:

$$w_i F_i^{-1}(q_i) \leq r \leq w_i F_i^{-1}(\delta) \tag{39}$$

and

$$q_i \leq F_i\left(\frac{r}{w_i}\right) \leq F_i(F_i^{-1}(\delta)) = \delta. \tag{40}$$

To proof the second inclusion, we use the convexity of $G_i = 1/F_i$.

$$\begin{aligned}\frac{1}{q_i} - \frac{1}{\alpha} &= \frac{1}{F_i(F_i^{-1}(q_i))} - \frac{1}{F_i(F_i^{-1}(\alpha))} = G_i(F_i^{-1}(q_i)) - G_i(F_i^{-1}(\alpha)) \\ &\geq G_i'(F_i^{-1}(\alpha))\left(F_i^{-1}(q_i) - F_i^{-1}(\alpha)\right) \\ &= \frac{-F_i'(F_i^{-1}(\alpha))}{(F_i(F_i^{-1}(\alpha)))^2}\left(F_i^{-1}(q_i) - F_i^{-1}(\alpha)\right)\end{aligned}$$

$$= -\frac{c_i}{\alpha^2}\left(F_i^{-1}(q_i) - F_i^{-1}(\alpha)\right) \tag{41}$$

thus

$$F_i^{-1}(q_i) - F_i^{-1}(\alpha) \geq -\frac{\alpha^2}{c_i}\left(\frac{1}{q_i} - \frac{1}{\alpha}\right). \tag{42}$$

If q belongs to V_r then we obtain

$$\begin{aligned} 0 &\geq \sum_{i=1}^{d} w_i F_i^{-1}(q_i) - r = \sum_{i=1}^{d} w_i F_i^{-1}(q_i) - \sum_{i=1}^{d} w_i F_i^{-1}(\alpha) \\ &\geq -\sum_{i=1}^{d} \frac{w_i \alpha^2}{c_i}\left(\frac{1}{q_i} - \frac{1}{\alpha}\right) \\ &= -\alpha\left(\sum_{i=1}^{d} \frac{\lambda_i}{q_i} \sum_{j=1}^{d} \frac{w_j}{c_j} - \sum_{i=1}^{d} \frac{w_i}{c_i}\right) = -\alpha \sum_{j=1}^{d} \frac{w_j}{c_j}\left(\sum_{i=1}^{d} \frac{\lambda_i}{q_i} - 1\right). \end{aligned} \tag{43}$$

Therefore

$$0 \leq \sum \frac{\lambda_i}{q_i} - 1, \tag{44}$$

and therefore q belongs to Y_λ.

□

Corollary **3.** *Under the assumptions of theorem 2 and lemma 2, for each i*

$$\mu_{L_i}(V_r) \leq \alpha. \tag{45}$$

Proof.

Case L_0. We base on the inclusion $V_r \subset Y_\lambda$ (lemma 2) and the estimated measure of Y_λ (lemma 1).

$$\mu_{L_0}(V_r) \leq \mu_{L_0}(Y_\lambda) \leq \sum \lambda_i = \alpha. \tag{46}$$

Case L_i, $i = 1, \dots, m$. We base on the inclusion $V_r \subset \times_{i=1}^{d}[0, F_i(\frac{r}{w_i})]$ from lemma 2 and the quasihomogeneity of L_i.

$$\begin{aligned} \mu_{L_i}(V_r) &\leq \mu_{L_i}\left(\times_{i=1}^{d}\left[0, F_i\left(\frac{r}{w_i}\right)\right]\right) = L_i\left(F_1\left(\frac{r}{w_1}\right), \dots, F_d\left(\frac{r}{w_d}\right)\right) \\ &\leq L_i(\delta\alpha^{\frac{1}{k_1}}, \dots, \delta\alpha^{\frac{1}{k_1}}) = \alpha^{\frac{k_i}{k_1}} L_i(\delta, \dots, \delta) = \alpha^{\frac{k_i}{k_1}} C_i(\delta, \dots, \delta) \leq \alpha. \end{aligned} \tag{47}$$

□

Proof of theorem 2.

In order to estimate $\mathrm{VaR}_{1-\alpha}(w)$ we consider

$$\begin{aligned} 1 - P&\left(W_0 - W_1(w) \leq \sum w_i \mathrm{VaR}_{1-\alpha}(e_i)\right) \\ &= P\left(W_0 - W_1(w) \geq \sum w_i \mathrm{VaR}_{1-\alpha}(e_i)\right) \\ &= P\left(1 - \sum w_i s_i \geq \sum w_i(1 - F_i^{-1}(\alpha))\right) \\ &= P\left(\sum w_i s_i \leq \sum w_i F_i^{-1}(\alpha)\right) \end{aligned}$$

$$= P(\sum w_i s_i \le r) = \mu_C(V_r)$$

$$= \sum a_i \mu_{C_i}(V_r) = \sum a_i \mu_{L_i}(V_r) \le \sum a_i \alpha = \alpha. \tag{48}$$

Therefore

$$P\left(W_0 - W_1(w) \le \sum w_i \mathrm{VaR}_{1-\alpha}(e_i)\right) \ge 1 - \alpha. \tag{49}$$

Since

$$\mathrm{VaR}_{1-\alpha}(w) = \sup\{V : P(W_0 - W_1(w) \le V) < 1 - \alpha\}, \tag{50}$$

we obtain the estimate

$$\mathrm{VaR}_{1-\alpha}(w) \le \sum w_i \mathrm{VaR}_{1-\alpha}(e_i). \tag{51}$$

□

REMARK **2.** *If $a_0 = 1$ then condition A3 is not necessary (cf. [20]), on the other hand if $a_0 = 0$ then we may omit condition A2.*

4. Conclusions

We proved in this paper that under the mild assumptions, for sufficiently small α the value at risk of a portfolio such that w_i part of the capital is invested in i-th currency ($w_i \ge 0$) is smaller the w-weighted sum of values at risk of one-asset portfolios

$$\mathrm{VaR}_{1-\alpha}(w) \le w_1 \mathrm{VaR}_{1-\alpha}(e_1) + \ldots + w_d \mathrm{VaR}_{1-\alpha}(e_d). \tag{52}$$

Since the supervision institutions, like the Basle Committee on Banking Supervision (cf. [1]), impose the rules that the value of risk should not exceed certain threshold, such an estimation simplifies the task of a risk manager. Indeed, if the w-weighted sum of VaR's of one-asset portfolios does not exceed the threshold so does the VaR of the portfolio w.

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