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On the Empirical Importance of the Conditional Skewness Assumption in Modelling the Relationship between Risk and Return

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We present the results of an application of Bayesian inference in testing the relation between risk and return on the financial instruments. On the basis of the Intertemporal Capital Asset Pricing Model, proposed by Merton we built a general sampling distribution suitable in analysing this relationship. The most important feature of our assumptions is that the skewness of the conditional distribution of returns is used as an alternative source of relation between risk and return. This general specification relates to Skewed Generalized Autoregressive Conditionally Heteroscedastic-in-Mean model. In order to make conditional distribution of financial returns skewed we considered the unified approach based on the inverse probability integral transformation. In particular, we applied hidden truncation mechanism, inverse scale factors, order statistics concept, Beta and Bernstein distribution transformations and also a constructive method. Based on the daily excess returns on the Warsaw Stock Exchange Index we checked the empirical importance of the conditional skewness assumption on the relation between risk and return on the Warsaw Stock Market. We present posterior probabilities of all competing specifications as well as the posterior analysis of the positive sign of the tested relationship.

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1. Introduction

The basics of the financial economics is constituted by the relationship between risk and return. Numerous papers have investigated this fundamental

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trade-off testing linear dependence of excess return on the level of risk, both measured by conditional mean and conditional standard deviation of the value of investor's wealth. According to [1], given risk aversion among investors, when investment opportunity set is constant, there is a positive relationship between expected excess return and risk. Hence, it is possible to express the risk in terms of the expected premium generated.

Historically, authors have found mixed empirical evidence concerning the relationship. In some cases a significant positive relationship can be found, in others it is insignificant and also some authors report it as being significantly negative. For instance, using monthly U.S. data [2] and also [3] found a predominantly positive but insignificant relationship. Glosten et al. reported in [4] a negative and significant relationship on the basis of Asymmetric-Generalized Autoregressive Conditionally Heteroscedastic (GARCH) model, instead of commonly used GARCH-in-Mean framework; see [5]. Scruggs summarises the empirical evidence of considered relationship in [6].

The main goal of this paper is an application of Bayesian inference in testing the relation between risk and excess return of the financial time series. We revisited Intertemporal Capital Asset Pricing Model (ICAPM) in order to investigate the empirical importance of the skewness assumption of the conditional distribution of excess returns. On the basis of the model, we built a general sampling distribution of the observables suitable in estimating risk premium. The most important feature of our model assumptions is that the possible skewness of conditional distribution of returns is used as an alternative source of relation between risk and return. Thus pure statistical feature is equipped with economic interpretation. Our general specification fully corresponds to suggestion, that systematic skewness is economically important and governs risk premium; see [7]. In order to make conditional distribution of financial returns skewed we considered a constructive approach based on the inverse probability integral transformation presented in details in [8]. Based on the daily excess returns of index of the Warsaw Stock Exchange we checked the total impact of conditional skewness assumption on the relation between return and risk on the Warsaw Stock Market. On the basis of the posterior probabilities and posterior odds ratios, we test formally the explanatory power of competing, conditionally fat tailed and asymmetric GARCH processes.

2. Creating asymmetric distributions

The unified representation of the univariate skewed distributions that we apply in this paper is based on the inverse probability integral transformation; for details see [8]. The family of random variables $IP = \{\varepsilon_s, \varepsilon_s : \Omega \rightarrow \mathbb{R}\}$, with representative density $s(\cdot|\theta, \eta_p)$ is called the skewed version of the symmetric family I (of random variables with unimodal symmetric density $f(\cdot|\theta)$ and distribution function F , such that the only one modal value is localised at $x = 0$) if s is given by the form

$$s(x|\theta, \eta_p) = f(x|\theta)p(F(x|\theta)|\eta_p), \quad x \in \mathbb{R}. \quad (2.1)$$

The asymmetric distribution $s(\cdot|\theta, \eta_p)$, where θ is inherited from the density f and η_p groups the skewness parameters, is obtained by applying the density $p(\cdot|\eta_p)$ as a weighting function. Within the general form (2.1) several classes of distributions have been imposed on some specific families of symmetric random variables. A review of skewing mechanisms was presented in [8]. Here we apply (2.1) in order to build the set of unnested specifications, which compete in explaining the possible relationship between risk and return.

3. Basic model framework and competing skewed conditional distributions

Let denote by x_j the value of a stock or a market index at time j . The excess return on x_j , denoted by y_j , is defined as the difference between the logarithmic daily return on x_j in percentage points ($r_j = 100 \ln(x_j/x_{j-1})$) and the risk free short term interest rate (denoted by r_j^*), namely $y_j = r_j - r_j^*$. The voluminous literature focused on examination the relationship between risk and return bases on the Intertemporal CAPM, proposed by Merton in [1]. According to assumptions of Merton's theory there exists a set of distributions $P(\cdot|\psi_{j-1})$, conditional with respect to the information set at time j (denoted by ψ_{j-1}) such that

$$E(y_j|\psi_{j-1}) = \alpha^* D(y_j|\psi_{j-1}), \quad (3.1)$$

where symbols E and D denote expectation and standard deviation, respectively. The coefficient $\alpha^* > 0$ in (3.1) measures the relative risk aversion of the representative agent. Under assumptions of the informational efficiency of the market, the information set at time j can be reduced to the history of the process of the excess return, namely $\psi_{j-1} = (\dots, y_{j-2}, y_{j-1})$. Consequently, an econometric model of the relationship between risk and return should explain the properties of the conditional (with respect to the past of the process y_j) distribution of the excess return y_j at time j . It is also of particular interest to find any linkage between expected excess return and the measure of dispersion of the distribution of y_j , conditional to ψ_{j-1} . Following [5], [2] and [9] we consider for y_j a simple GARCH-in-Mean process, defined as follows:

$$y_j = [\alpha + E(z_j)]\sqrt{h_j} + u_j, \quad j = 1, 2, \dots, \quad (3.2)$$

where $u_j = [z_j - E(z_j)]\sqrt{h_j}$, and z_j are independently and identically distributed random variables with $E(z_j) < +\infty$. The scale factor h_j is given by the GARCH(1,1) equation; see [10]:

$$h_j = \alpha_0 + \alpha_1 u_{j-1}^2 + \beta_1 h_{j-1}, \quad j = 1, 2, \dots$$

The specific form of the conditional distribution of y_j in (3.2) is strictly determined by the type of the distribution of z_j . Initially, in model denoted by M_0 , we assumed for z_j the Student- t density with unknown degrees of freedom $\nu > 1$, zero mode and unit inverse precision

$$z_j|M_0 \sim iiSt(0, 1, \nu), \quad \nu > 1.$$

The density of the distribution of z_j in M_0 is given as follows:

$$p(z|M_0) = f_t(z|0, 1, \nu) = \frac{\Gamma[0.5(\nu + 1)]}{\Gamma(0.5\nu)\sqrt{\pi\nu}} \left(1 + \frac{z^2}{\nu}\right)^{-(\nu+1)/2}.$$

Given model M_0 , $E(z_j) = 0$, $u_j = z_j\sqrt{h_j}$, and hence (3.2) reduces to simpler form $y_j = \alpha\sqrt{h_j} + u_j$. Let us denote by $\theta = (\alpha, \alpha_0, \alpha_1, \beta_1, \nu)$ the vector of all parameters in model M_0 . Here, the conditional distribution of the error term is the Student- t distribution with degrees of freedom parameter $\nu > 1$, zero mode and inverse precision h_j . Consequently, the following density represents conditional distribution of the excess return at time j :

$$p(y_j|\psi_{j-1}, \theta, M_0) = h_j^{-0.5} f_t(h_j^{-0.5}(y_j - \alpha\sqrt{h_j})|0, 1, \nu), j = 1, 2, \dots$$

Given model M_0 the expected excess return (conditional to the whole past ψ_{j-1}) is proportional to the square root of the inverse precision h_j :

$$E(y_j|\psi_{j-1}, \theta, M_0) = \alpha\sqrt{h_j}, \quad j = 1, 2, \dots \quad (3.3)$$

The parameter $\alpha \in R$ captures the dependence between expected excess return and the level of risk, both measured by $E(y_j|\psi_{j-1}, \theta, M_0)$ and the scale parameter $\sqrt{h_j}$, respectively.

Now we want to construct a set of competing GARCH specifications $\{M_i, i = 1, \dots, k\}$ by introducing skewness into density of the conditional distribution of excess return, $p(y_j|\psi_{j-1}, \theta, M_0)$. The resulting asymmetric distributions are obtained by skewing the distribution of the random variable z_j , according to method presented in the previous section. The asymmetric density of z_j is of the general form related to the formula (2.1):

$$p(z|M_i) = f_t(z|0, 1, \nu)p[F_t(z)|\eta_i, M_i], \quad z \in R, \quad i = 1, 2, \dots, k,$$

where $p(\cdot|\eta_i, M_i)$ defines the skewing mechanism parameterised by vector η_i and $F_t(\cdot)$ is the cumulative distribution function (cdf) of the standardised Student- t distribution. This leads to the general form of the conditional distribution of daily excess return y_j in model M_i :

$$p(y_j|\psi_{j-1}, \theta, \eta_i, M_i) = h_j^{-0.5} f_t(z_j^*|0, 1, \nu)p[F_t(z_j^*)|\eta_i, M_i], \quad j = 1, \dots,$$

where $z_j^* = h_j^{-0.5}(y_j - \mu_j)$ and $\mu_j = [\alpha + E(z_j)]\sqrt{h_j}$.

By imposing skewness, the expectation $E(z_j)$ is no longer equal to zero. Consequently, given M_i , the expectation of the excess return (conditional to ψ_{j-1}) is still proportional to $\sqrt{h_j}$, but the coefficient of proportionality changes

$$E(y_j|\psi_{j-1}, \theta, \eta_i, M_i) = [\alpha + E(z_j)]\sqrt{h_j}.$$

Hence, the skewness of conditional distribution of y_j is treated in M_i as another source of information about the strength of tested relationship. As the first specification, denoted by M_1 , we consider GARCH model with skewed Student- t distribution obtained by the method proposed in [11], or equivalently in [12]. The model M_2 is the result of skewing conditional distribution according to the hidden

TABLE I

The conditional skewing mechanisms $p(\cdot|\eta_i)$, defined for $u \in (0,1)$, the skewness parameters η_i and conditional symmetry restrictions in all competing specifications M_i , $i = 1, 2, 3, 4, 5, 6$.

<p style="text-align: center;">M_5</p> <p>Bernstein density (2 parameters) $(\omega_1, \omega_2) \in (0, 1)^2$, $\omega_3 = 1 - \omega_1 - \omega_2$ $p(u \omega_1, \omega_2) = \sum_{j=1}^3 \omega_j Be(u j, 4 - j)$ symmetry: $\omega_1 = \omega_2 = \frac{1}{3}$</p>	<p style="text-align: center;">M_1</p> <p>Inverse scale factors; see [11] and [12] parameterisation as in [11] $\gamma_1 > 0$, $C = \frac{2}{\gamma_1 + \gamma_1^{-1}}$ $I_1(u) = I_{(0,0.5)}(u)$, $I_2(u) = I_{(0.5,1)}(u)$ $p(u \gamma_1) = C \frac{f(\gamma_1 F^{-1}(u))I_1(u) + f(\gamma_1^{-1} F^{-1}(u))I_2(u)}{f(F^{-1}(u))}$ symmetry: $\gamma_1 = 1$</p>
<p style="text-align: center;">M_2</p> <p>Hidden truncation; see [13] $\gamma_2 \in \mathbb{R}$ $p(u \gamma_2) = 2F(\gamma_2 F^{-1}(u))$ symmetry: $\gamma_2 = 0$</p>	<p style="text-align: center;">M_3</p> <p>Beta one parameter; see [14] $\gamma_3 > 0$ $p(u \gamma_3) = Be(u \gamma_3, \gamma_3^{-1})$ symmetry: $\gamma_3 = 1$</p>
<p style="text-align: center;">M_6</p> <p>The construct proposed in [15] $\gamma_4 \in \mathbb{R}$ $p(u \gamma_4) = 1 + l(\gamma_4)[g(u \gamma_4) - 1]$ symmetry: $\gamma_4 = 0$</p>	<p style="text-align: center;">M_4</p> <p>Beta two parameters; see [14] and [16] $a > 0, b > 0$ $p(u a, b) = Be(u a, b)$ symmetry: $a = b = 1$</p>

truncation idea; see [13]. In models M_3 and M_4 we apply Beta skewing mechanism; see [14]. In model M_5 we apply Bernstein density based skewing mechanism with $m = 2$ free parameters, while model M_6 is built on the basis of the skewing construct defined in [15]. All competing specifications, together with analytical forms of skewing mechanisms and model specific parameters are presented in Table I. For some details concerning sampling densities and prior specifications in each model see [8].

4. Empirical results for WSE index

In this part we present an empirical example of Bayesian comparison of all competing specifications. We also discuss the posterior analysis of the total impact of the conditional skewness assumption on the relationship between risk and return on the Warsaw Stock Exchange (WSE). Our dataset y was constructed on the basis of $t = 2144$ observations of daily growth rates, r_j , of the WSE index (WIG) from 06.01.98 till 31.07.06. The risk free interest rate, r_j^* , used to calculate excess return y_j , was approximated by the WIBOR overnight interest rate (WIBORo/n instrument). Our empirical results remained practically unchanged for r_j^* calculated on the basis of the middle and long term WIBOR Polish Zloty interest rates and also in the case $r_j^* = 0$.

Table II presents posterior probabilities $P(M_i|y)$ calculated for each of competing models M_i , $i = 0, 1, \dots, 6$. The initial specification M_0 , built on the basis of the conditional symmetric Student- t distribution, receives a little data support, as the posterior probability $P(M_0|y)$ is slightly greater than 8%. All remaining posterior probability mass is attached to specifications which allow for conditional skewness. It is clear that the modelled dataset of excess returns of WIG index do not support decisively superiority of any of the competing skewing mechanism. The greatest value of $P(M_i|y)$ receives conditionally skewed Student- t GARCH model generated by the Beta distribution transformation with two free parameters. In this case the value of posterior probability is equal to about 40%. The dataset also support conditionally skewed Student- t GARCH model with hidden truncation mechanism (M_2) and Beta distribution transformation with one free parameter (M_3). Those three models cumulate more than 90% of the posterior probability mass, making all remained conditionally skewed specifications improbable in the view of the data. Thus, inverse scale factors, the Bernstein density transformation and construct proposed in [15], namely models M_1 , M_5 and M_6 , lead to very doubtful explanatory power. Those specifications are strongly rejected by the data, as the values of posterior probabilities are much smaller than posterior probability of symmetric GARCH model.

TABLE II
Decimal logarithms of the marginal data density values, posterior probabilities of all competing specifications M_i and posterior probabilities of the positive sign of the relative risk aversion coefficient $\alpha + E(z_j)$.

	M_1	M_2	M_3	M_4	M_5	M_6	M_0
$\log p(y M_i)$	-1559.45	-1558.50	-1558.78	-1558.41	-1560.82	-1560.10	-1559.06
$P(M_i y), i = 0, \dots, 6$	0.0353	0.3152	0.1654	0.3878	0.0015	0.0079	0.0868
$P(M_i y), i = 1, \dots, 6$	0.0387	0.3452	0.1811	0.4246	0.0017	0.0087	-
$P(\alpha + E(z_j) > 0 M_i, y)$	0.9102	0.9894	0.9528	0.9972	0.9893	0.9230	0.9201

In Table II we also compare the total impact of the conditional skewness effect on the tested relation between risk and return. According to our assumptions, the conditional expectation of the excess return is proportional to the square root of the inverse precision h_j . Since we parameterise the market risk by a more general dispersion measure than standard deviation, we report the information about the relative risk aversion by the posterior probability of the positive sign of the function $\alpha + E(z_j)$. According to (3.2) it enables to test the positive sign of the relative risk aversion coefficient. Initially we checked the strength of the relation in model M_0 , which does not allow for conditional skewness. Given M_0 , $E(z_j) = 0$ and the whole information about relative risk aversion is reflected in parameter α ; see (3.3). Just like many researchers we obtained positive, but rather weak, relation between expected excess return and risk, given model with symmetric conditional distribution. The posterior probability $P(\alpha > 0|M_0, y)$,

equal to about 0.92, leaves considerable level of uncertainty about true strength of the tested relation. Consequently, model M_0 does not confirm our hypothesis strongly. Imposing unreasonable (in the view of the data) skewness into conditional distribution of excess returns also may not strengthen our inference. In case of models with weak data support (M_1 and M_6) the assumption of asymmetry of the density $p(y_j|\psi_{j-1}, \theta, \eta_i, M_i)$ does not improve posterior inference about the sign of $\alpha + E(z_j)$. In case of M_1 and M_6 posterior probability of positive relationship is very close to the value generated within M_0 . Only in case of the skewing mechanisms with the greatest data support, namely Beta transformation with two parameters and hidden truncation, the WIG excess returns yield decisive support of the positive sign of the relative risk aversion coefficient. In case of model M_3 , the posterior probability of positive sign of $\alpha + E(z_j)$ is greater than 0.99, leaving no doubt about the significance of the relationship between risk and return postulated by Merton in [1]. Hence, we confirmed positive sign of $\alpha + E(z_j)$ strongly only in case of imposing specific skewing mechanism into conditional distribution of excess returns. Beta distribution transformation with two free parameters was able to detect additional source of information about risk premium in the WIG dataset. Also, hidden truncation mechanism and the Bernstein density transformation strongly confirm positive sign of the risk aversion coefficient, as posterior probability $P(\alpha + E(z_j) > 0|M_i, y)$ is greater than 0.98, for $i = 2$ and 5.

5. Concluding remarks

We checked the impact of the conditional skewness assumption on the strength of the relationship between risk and expected return. On the basis of the Intertemporal CAPM model, proposed in [1], we built a GARCH-in-Mean type sampling distribution suitable in modelling such relationship. Our approach, which fully relates to the model proposed in [9], treats the skewness of the conditional distribution of excess returns as an alternative source of information about risk aversion. Based on the daily excess returns of the Warsaw Stock Exchange index we checked the empirical importance of the conditional skewness assumption on the relation between risk and return. Posterior inference about skewing mechanisms showed positive and decisively significant value of the coefficient of the relative risk aversion once a specific skewing mechanism was imposed in conditional Student- t distribution. The greatest data support, and also very strong support of the relation postulated by Merton in [1], received skewness generated by Beta distribution transformation with two free parameters.

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