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Bayesian Forecasting of the Discounted Payoff of Options on WIG20 Index in Discrete-Time SV Models*

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In this paper the bivariate stochastic volatility models (with stochastic volatility and stochastic interest rate) and the univariate fat-tailed and correlated stochastic volatility model (with stochastic volatility and constant interest rate) are used in the Bayesian forecasting of the payoff of European call options. The basic instrument is the WIG20 index. The predictive distribution of the discounted payoff is induced by the predictive distribution of the growth rate of the WIG20 index and the WIBOR1m interest rate. The Bayesian inference about the volatilities and the predictive distribution of the discounted payoff function is based on the joint posterior distribution of the latent variables, the parameters, and the predictive distribution of future observations, which we simulate via Markov chain Monte Carlo methods (the Metropolis–Hastings algorithm is used within the Gibbs sampler). The results show that allowing interest rate to be stochastic does not significantly improve forecasting performance of the discounted payoff. The predictive distributions of the discounted payoff are characterised by huge dispersion and thick tails, thus uncertainty about the future value of the payoff was ex-ante very big.

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1. Introduction

The pricing of option with stochastic volatility (SV) and stochastic interest rate is a difficult task. The classical Black–Scholes model assumes that asset returns follow continuous diffusion process with constant conditional volatility and constant interest rate. Thus, numerous studies on option pricing have modified the Black–Scholes model to allow for stochastic volatility for the underlying assets processes or stochastic interest rates.

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The option pricing model incorporating both stochastic interest rates and stochastic volatility is not often considered in literature. Hull and White [1] assume constant interest rate and stochastic volatility. The Hull and White formula is used by Mahieu and Schotman [2] in a discrete time univariate stochastic volatility model. Merton [3], Turnbull and Milne [4] consider the cases with constant asset return volatility but stochastic interest rates. Heston [5] presents a close-form solution for options on assets with stochastic volatility, constant interest rate, and when the spot asset is correlated with volatility. Kim [6] compares the pricing performance of stock option pricing models under several stochastic interest rate processes but under constant volatility. Fouque et al. [7] present derivative pricing for a class of stochastic volatility models with constant interest rate. In [8] it is built the option pricing model which simultaneously incorporates both a stochastic interest rate and a stochastic volatility process for stock returns. Their results are used by Jiang and Sluis [9] in context of a discrete-time bivariate stochastic volatility model.

The aim of the paper is to check whether allowing interest rates to be stochastic improves forecasting performance of the discounted payoff. We compare the option pricing model under stochastic interest rate (allowing the interest rate to follow an SV process) with constant interest rate model (univariate SV model for the underlying asset).

The structure of the article is as follows: Sect. 2 consists of a short presentation of the Bayesian univariate SV model with fat-tails and correlated errors, Sect. 3 includes a brief presentation of the Bayesian bivariate SV model, Sect. 4 focuses on the Bayesian forecasting of the discounted payoff of an European call option, Sect. 5 presents the posterior results connected with the option pricing on the WIG20 index, and finally, Sect. 6 incorporates the conclusions.

2. Bayesian univariate AR(1)-FCSV model

Let x_t denote the price of the underlying asset at time t. The growth rate y_t is defined as $y_t = 100 \ln(x_t/x_{t-1})$, $t = 1, 2, \ldots, T + s$ and is modelled using the discrete-time SV model with fat-tails and correlated errors (FCSV) considered in [10]. Here T is the number of the observations used in estimation, s is the predictive horizon. The FCSV model specifies a log-normal autoregressive process for the conditional variance with correlated innovations in the conditional mean and conditional variance equations. The univariate AR(1)-FCSV model is defined as follows [‡]:

$$
\begin{aligned}
& y_t - \delta_1 = \varrho_1(y_{t-1} - \delta_1) + \varepsilon_t, \qquad (1)\\
& \varepsilon_t = u_t\sqrt{h_t/\omega_t}, \quad \ln h_t = \gamma + \phi \ln h_{t-1} + \sigma_h \eta_t,\\
& \{\omega_t\} \sim ii\chi^2(\nu)/\nu, \quad \omega_t \perp (u_t, \eta_l), \quad t, l \in \{1, 2, \ldots, T + s\},
\end{aligned}
$$

[‡]Because a positive autocorrelation of order one is quite usual for a stock market index (see [11]), we use the autoregressive structure.

$$\{(u_t, \eta_t)'\} \sim iiN\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & \varrho\\ \varrho & 1\end{bmatrix}\right),$$

where iiN denotes identically and independently normally distributed; $\perp$ denotes independence; $\varrho_1, \varrho, \phi \in (-1, 1)$, $\nu, \sigma_h \in (0, \infty)$, $\delta_1, \gamma \in \Re$. Let us note that the random variable $u_t/\sqrt{\omega_t}$ has a Student-t distribution with ν degrees of freedom. One interpretation for the latent variable h_t is that it represents the random, uneven and autocorrelated flow of new information into financial markets (see [12]). Here ϕ is connected with the volatility persistence, σ_h^2 is the volatility of the log--volatility. The above model can pick up the kind of asymmetric behaviour often observed in stock price movements, which is known as the leverage effect when the correlation is negative ($\varrho < 0$)[§]. In this paper we use the following prior structure:

$$p(\delta_1, \varrho_1, \gamma, \phi, \sigma_h^2, \nu, \varrho) = p(\delta_1)p(\varrho_1)p(\gamma)p(\phi)p(\sigma_h^2, \nu)p(\varrho),$$

where we use proper prior densities of the following distributions: $p(\delta_1) \sim N(0, 1)$, $\varrho_1 \sim U(-1, 1)$, $\gamma \sim N(0, 100)$, $\phi \sim N(0, 100)I_{(-1,1)}(\phi)$, $\tau \sim IG(1, 0.005)$, $\psi|\tau \sim N(0, \tau/2)$, $\nu \sim \exp(0.1)$, $\psi = \sigma_h\varrho$, $\tau = \sigma_h^2(1 - \varrho^2)$ (see [10]). The prior distribution for δ_1 is standardized normal, $U(-1, 1)$ denotes the uniform distribution over (–1,1). The prior distribution for ϕ is normal, truncated by the restriction that the absolute value of ϕ is less than one ($I_{(-1,1)}(.)$ denotes the indicator function of the interval $(-1, 1)$, which is the region of stationarity of $\ln h_t$). The symbol $IG(v_0, s_0)$ denotes the inverse Gamma distribution with mean $s_0/(v_0 - 1)$ and variance $s_0^2/[(v_0 - 1)^2(v_0 - 2)]$(thus, when $\varrho = 0$, the prior mean for σ_h^2 does not exist, but σ_h^{-2} has a Gamma prior with mean 200 and standard deviation 200). The symbol $\exp(\lambda)$ denotes the exponential distribution with mean $1/\lambda$ (thus the prior mean for ν is 10 with the standard deviation 10). The initial condition h_0 is equal to y_0^2. These assumptions reflect rather weak prior knowledge about the parameters.

3. Bayesian bivariate VAR(1)-t-TSV model

Let $x_{j,t}$ denote the price of asset j at time t for $j = 1, 2$ and $t = 1, 2, \ldots, T+s$ (in this paper $x_{1,t}$ and $x_{2,t}$ are respectively the interest rate and index level at time t). The vector of growth rates $y_t = (y_{1,t}, y_{2,t})'$, each defined by the formula $y_{j,t} = 100 \ln (x_{j,t}/x_{j,t-1})$, is modelled using the VAR(1) framework

$$y_t - \delta = R(y_{t-1} - \delta) + \xi_t, \quad t = 1, 2, \ldots, T + s. \tag{2}$$

In (2) δ is a 2-dimensional vector, R is a 2×2 matrix of parameters, and ξ_t is a bivariate SV process. More specifically

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} - \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \left(\begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} - \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \right) + \begin{bmatrix} \xi_{1,t} \\ \xi_{2,t} \end{bmatrix}.$$

[§]If ϱ is negative, then a negative innovation u_t is associated with higher contemporaneous and subsequent volatilities. On the other hand, a positive innovation u_t is associated with a decrease in volatility.

We assume that, conditionally on the latent variable vector Ω_t and ω_t, ξ_t follows a bivariate Gaussian distribution with mean vector $0_{[2\times 1]}$ and covariance matrix $(1/\omega_t)\Sigma_t$, i.e.

$$\xi_t|\Omega_t, \omega_t \sim N(0_{[2\times 1]}, (1/\omega_t)\Sigma_t), \quad t = 1, 2, \ldots, T+s.$$

The random variable $\omega_t, t = 1, ..., T+s$ are assumed to be $\{\omega_t\} \sim ii\chi^2(\nu)/\nu$, $\omega_t \perp \Omega_l$, for $t, l \in \{1, 2, \ldots, T+s\}$. For the matrix Σ_t the Cholesky decomposition is used (see [13]):

$$\Sigma_t = L_t G_t L_t', \tag{3}$$

where L_t is a lower triangular matrix with unitary diagonal elements, G_t is a diagonal matrix with positive diagonal elements

$$L_t = \begin{bmatrix} 1 & 0 \\ q_{21,t} & 1 \end{bmatrix}, \quad G_t = \begin{bmatrix} q_{11,t} & 0 \\ 0 & q_{22,t} \end{bmatrix},$$

$\{q_{21,t}\}$ and $\{\ln q_{jj,t}\}$ $(j = 1, 2)$, as in the univariate SV specification, are standard univariate autoregressive processes of order one, namely

$$\ln q_{jj,t} - \gamma_{jj} = \phi_{jj}(\ln q_{jj,t-1} - \gamma_{jj}) + \sigma_{jj}\eta_{jj,t}, \quad j = 1, 2,$$

$$q_{21,t} - \gamma_{21} = \phi_{21}(q_{21,t-1} - \gamma_{21}) + \sigma_{21}\eta_{21,t},$$

where $\eta_t = (\eta_{11,t}, \eta_{22,t}, \eta_{21,t})'$ and $\{\eta_t\} \sim iiN(0_{[3\times 1]}, I_3)$, $\Omega_t = (q_{11,t}, q_{22,t}, q_{21,t})'$, $\gamma_{ij} \in \Re$, $\phi_{ij} \in (-1, 1)$, $\sigma_{ij} \in (0, \infty)$, $i, j = 1, 2$, $i \geq j$. From the decomposition in (3), we have

$$\Sigma_t = \begin{bmatrix} q_{11,t} & q_{11,t}\, q_{21,t} \\ q_{21,t}\, q_{11,t} & q_{11,t}\, q_{21,t}^2 + q_{22,t} \end{bmatrix}.$$

The Cholesky decomposition of Σ_t requires no parameter constraints for the positive definiteness of Σ_t. The matrix Σ_t is positive definite if $q_{jj,t} > 0$ for $j = 1, 2$, which is achieved by modelling $\ln q_{jj,t}$ instead of $q_{jj,t}$. If $\nu > 2$ and $|\phi_{ij}| < 1$ $(i, j = 1, 2, i \geq j)$, then $\ln q_{11,t}$, $\ln q_{22,t}$, $q_{21,t}$ are stationary and the SV process is a white noise (see [14]). The conditional distribution of y_t (given the past of the process, ψ_{t-1}, and the latent variable vector Ω_t) is bivariate Student t with ν degrees of freedom, and the precision matrix Σ_t^{-1}. We make similar assumptions about the prior distributions as previously. In particular: $(\gamma_{ij}, \phi_{ij})' \sim N(0, 100I)I_{(-1,1)}(\phi_{ij})$, $\sigma_{ij}^2 \sim IG(1, 0.005)$, $\ln q_{ii,0} \sim N(0, 100)$ for $i, j \in \{1, 2\}$ and $i \geq j$; $q_{21,0} \sim N(0, 100), \nu \sim \exp(0.1)$. For all elements of δ and R we assume the multivariate standardized normal prior $N(0, I_6)$, truncated by the restriction that all eigenvalues of R lie inside the unit circle (similar to [15]).

4. Application to Bayesian forecasting of the discounted payoff

An important application of the stochastic volatility models is the option pricing. The payoff at time $T+s$ of an European call option is given by

$$V_{T+s} = \max(x_{T+s} - K, 0), \tag{4}$$

where K is the exercise price (strike price), x_{T+s} is the price of the underlying asset at time $T+s$ (no dividend being paid), s is units time before maturity. The present value of payoff considered at time T under stochastic interest rate is

$$W_{T|T+s} = \exp\left(-\int_{T}^{T+s} r_t \mathrm{d}t\right) \max(x_{T+s} - K, 0), \tag{5}$$

where r_t is the interest rate at time t. This discounted payoff is a random variable as a measurable function of x_{T+s} and r_t, $t \in [T, T+s]$, which are random. The distribution of $W_{T|T+s}$ is induced by the predictive distributions of x_{T+s} and r_t, $t \in [T, T+s]$. The Bayesian approach naturally provides a tool to compute the predictive distribution of the discounted payoff, $W_{T|T+s}$, without finding an equivalent martingale (see [16, 17]). Thus, the predictive density of the payoff is defined by

$$p(W_{T|T+s}|y) = \int p(W_{T|T+s}|\theta, y) p(\theta|y) \mathrm{d}\theta,$$

where y is the sample of returns used for estimation, $p(\theta|y)$ is the posterior density of the parameters and latent variables of the Bayesian econometric model. In discrete time model the integral in Eq. (5) is replaced by the summation

$$W_{T|T+s} = \exp\left(-\sum_{t=T+1}^{T+s} r_{t-1}\right) \max(x_{T+s} - K, 0). \tag{6}$$

It is important to stress that the specification (1) relaxes Black and Scholes constant volatility assumption. The volatility follows a separate process. The specification (2) relaxes Black and Scholes constant volatility and constant interest rate assumptions, furthermore allows the interest rate to follow an SV process. In deterministic volatility models with constant interest rate, an investor incurs only the risk from a randomly evolving asset price. Subject to certain modelling assumptions (see [18]) it is possible to perfectly replicate the payoff of the option through dynamic trading. Thus, there is unique preference independent price for the option. This price can be calculated as the discounted expected value under the equivalent martingale measure. In the univariate AR(1)-FCSV model with constant interest rate, asset return volatility is driven by a random source that is different from the random source driving the asset returns process. Thus, the investor incurs the risk from a randomly evolving asset price and the risk of a randomly evolving volatility. In a discrete-time model with stochastic volatility, the market is incomplete and thus the equivalent martingale measure is not unique. However, in some cases (e.g. if the random source driving the asset returns process and the random source driving the asset return volatility are independent) a close-form expression for the option's price is available (see [1]). Let us note that in the bivariate VAR(1)-t-TSV model (presented above) there are four sources of risk: the risk from the asset price, the volatility of the underlying asset, the interest rate, and from the volatility of the interest rate. It is clear that this model is incomplete. As note [19], a frictionless diffusion model is incomplete if the number of sources of randomness is greater than the number of traded assets. It is well known that in an incomplete market there is no unique fair price and no universal pricing algorithm. There are several alternative methodologies which have been proposed as pricing mechanisms: writing down the dynamics of assets under a pricing measure, choosing (arbitrarily) a market price of risk for the non-traded assets, assuming that there is a call option which is liquidly traded (this approach does not explain the price of the original traded call), pricing via a hedging criteria (minimising some functional of the hedging error), minimal distance martingale measures, convex risk measures, super-replication pricing, and utility indifference pricing (see [19] for a review of the methods). In this paper we consider the original probability measure (the physical measure). This means that we assume that both the stochastic interest rate volatility and stochastic interest rate, as well as stochastic asset return volatility have zero risk premium. The Bayesian approach takes completely into account the uncertainty, which come from prediction and from the parameters, by construction of the predictive distribution of the discounted payoff. A measure of uncertainty can be attached to the option price by computing quantiles and the predictive option price may be defined as the median of $W_{T|T+s}$ (see [17]).

5. Empirical results

We use daily observations (closing quotes) of the WIG20 index and WIBOR1m (the 1-month Warsaw Interbank Offered Rate)¶ over the period from January 2, 2001 to December 31, 2004. The dataset of the daily logarithmic growth rates consists of 1005 observations (for each series). The first observation is used to construct initial conditions, thus $T = 1004$. We consider all European call options on the WIG20 index, which were quoted on Warsaw Stock Exchange (WSE) on December 31, 2004 (at the end of observed sample). The exercise dates are March 18, 2005 (i.e. $s = 55$ trading days) or June 17, 2005 (i.e. $s = 115$ trading days). As the proxy for the unobservable short rate, the 1-month WIBOR rate is used. As justified [20] and [9] the use of the 1-month WIBOR rate is a compromise between an instantaneous rate (overnight rates) and avoiding some of the associated spurious microstructure effects. In VAR(1)-t-TSV model the first component of the vector y_t is the growth rate of WIBOR1m, the second one is the growth rate of the WIG20 index.

In Tables I and II we report the main characteristics of the predictive distributions of the discounted payoff for the European call option on WIG20 index‖. In the univariate AR(1)-FCSV model with constant interest rate, according to the

¶The data was downloaded from www.money.pl.

‖Our posterior results are obtained using MCMC methods: Metropolis–Hastings within the Gibbs sampler.

TABLE I

The predictive characteristics of the discounted payoff with $s = 55$.

Model	Quantil of order	$K = 1700$	$K = 1800$	$K = 1900$	$K = 2000$	$K = 2100$
	0.05	8.68	0	0	0	0
	0.25	172.05	73.47	0	0	0
FCSV	0.50	284.27	185.69	87.11	0	0
	0.75	401.76	303.18	204.60	106.02	7.44
	0.95	595.20	496.62	398.04	299.46	200.88
	IQR	229.71	229.71	204.08	106.02	7.44
	$P(W_{T\|T+s} = 0\|y)$	0.045	0.132	0.302	0.527	0.737
	0.05	29.45	0	0	0	0
	0.25	180.73	82.15	0	0	0
t-TSV	0.50	286.44	187.86	89.28	0	0
	0.75	397.73	299.15	200.57	101.99	3.41
	0.95	580.63	482.05	383.47	284.89	186.31
	IQR	217.00	217.00	200.57	101.99	3.41
	$P(W_{T\|T+s} = 0\|y)$	0.034	0.114	0.284	0.523	0.744
true value of discounted payoff		271.11	172.53	73.94	0	0
quotations on December 31, 2004/ January 3, 2005		285/265	200/190	105/94	50/44	15/13.5

recommendation of Warsaw Stock Exchange and Polish National Depository for Securities it was assumed that the risk-free interest rate is 6.5 percent per annum (i.e. $r = 6.5$ percent on annual base, see [21]). The predictive distributions of the discounted payoff have such huge dispersions that in practice the differences are negligible. The observed market prices of the options are located between the medians and the quantiles of order 0.75 or between the quantiles of order 0.25 and the medians, but in the close neighbourhood of medians. Also, the true values of the discounted payoffs were close to the medians. It is worth to stress that the inter-quartile range (IQR) indicates huge uncertainty of the future payoff. When we use the median of the predictive distribution of the discounted payoff as the objective option price, we see that all models overprice the option. The overpricing may be due to our assumption that the risk premiums in both interest rate and asset return processes as well as the conditional volatility processes are zero.

In Fig. 1 we present histograms of the predictive distributions of the discounted payoff of the European call options with the exercise price K equal to 1800 index's points. The first bars of graphs denote probabilities of non-exercise of the options. The little grey points represent the true values of the discounted payoff. They are located between the first quartiles and the medians of the predictive distributions of the discounted payoff. The predictive histograms are characterised by

TABLE II

The predictive characteristics of the discounted payoff with $s = 115$

Model	Quantil of order	$K = 1800$	$K = 1900$	$K = 2000$	$K = 2100$
FCSV	0.05	0	0	0	0
	0.25	36.58	0	0	0
	0.50	211.11	114.08	17.05	0
	0.75	401.14	304.11	207.08	110.05
	0.95	728.50	631.47	534.44	437.10
	IQR	364.56	304.11	207.08	110.05
	$P(W_{T\|T+s} = 0\|y)$	0.207	0.331	0.474	0.616
t-TSV	0.05	0	0	0	0
	0.25	54.56	0	0	0
	0.50	216.69	119.35	22.32	0
	0.75	393.70	296.67	199.33	102.30
	0.95	698.12	601.09	503.75	406.72
	IQR	339.14	296.67	199.33	102.30
	$P(W_{T\|T+s} = 0\|y)$	0.182	0.311	0.464	0.616
true value of discounted payoff		217.43	120.36	23.30	0
quotations on December 31, 2004/ January 3, 2005		189/228	119/119	78/78	45/31

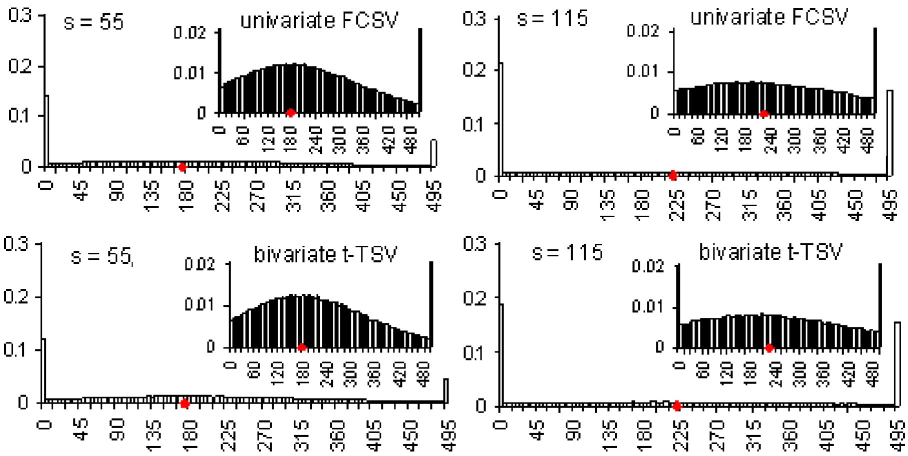

Fig. 1. Histograms of the predictive distributions of the discounted payoff ($K = 1800$).

huge dispersion and thick tails, thus uncertainty about the future value of payoff was ex-ante very big. We see that the right tails grow with forecast horizon.

TABLE III

The predictive median of $W_{T|T+s}$ minus the true value of the discounted payoff.

Model	$s = 55$ $K = 1700$	$s = 55$ $K = 1800$	$s = 55$ $K = 1900$	$s = 55$ $K = 2000$	$s = 55$ $K = 2100$	MFE
FCSV	13.16	13.16	13.17	0.00	0.00	–
t-TSV	15.37	15.36	15.35	0.00	0.00	–
Model	$s = 115$ $K = 1700$	$s = 115$ $K = 1800$	$s = 115$ $K = 1900$	$s = 115$ $K = 2000$	$s = 115$ $K = 2100$	–
FCSV	–	–6.32	–6.28	–6.25	0.00	2.29
t-TSV	–	–1.10	–1.21	–1.01	0.00	4.75

The options, for which the probability of non-execution is above 0.5, were not exercised. The settlement prices for derivative securities were equal to 1975 (for $s = 55$) and 2024 (for $s = 115$)**. The options with the strike price 2100 index's points (in the case of OW20C5210 and OW20F5210) were not executed. Also the call option with exercise price $K = 2000$ and $s = 55$ (i.e. OW20C200) was not exercised. In the last column in Table III we have the average (mean) forecasting errors (MFE)††. The level of MFE in the bivariate VAR(1)-t-TSV model (with stochastic interest rate) is a bit higher than in univariate AR(1)-FCSV model (with constant interest rate). The empirical results allow us to infer that stochastic interest rates are not important for the forecasting of the discounted payoff. Let us note that in the case of $s = 115$ the VAR-t-TSV model turns out better (in term of MFE) than the AR(1)-FCSV model. But, it seems that stochastic interest rate has minimal impact on option prices. Surprisingly, in the univariate AR(1)-FCSV model the uncertainty of the future value of payoff (measured by IQR) is bigger.

6. Conclusions

In this paper the bivariate stochastic volatility models (with stochastic volatility and stochastic interest rate) and the univariate fat-tailed and correlated stochastic volatility model (with stochastic volatility and constant interest rate) are used in Bayesian forecasting of the payoff of the European call options. The basic instrument is the WIG20 index. The empirical results indicate that allowing interest rates to be stochastic does not significantly improve forecasting performance of the discounted payoff. The VAR(1)-t-TSV model (with stochastic interest rate) does not dominantly outperform the AR(1)-FCSV model (with constant interest rate). The predictive distributions of the discounted payoff are

**The last value of the WIG20 index was equal to 1960.57.

††The average pricing error is defined as: $MFE = (1/n)\sum_{i=1}^{n} \hat{C}_i - C_i$, where n is the number of options used in the comparison, C_i and $\hat{C}_i$ represent respectively the true value of discounted payoff and the predictive median of the discounted payoff.

characterised by huge dispersion and thick tails, thus uncertainty about the future value of the payoff was ex-ante very big. The true values of the discounted payoff (observed ex-post) are located between the first quartile and the median of the predictive distribution of the discounted payoff, but the predictive distributions of the discounted payoff have such huge dispersion that they are hardly informative for the purpose of option pricing. On the other hand, the financial markets are characterised by a very high risk and uncertainty thus the huge dispersion and heavy tails of the predictive distributions of the discounted payoff are quite understandable.

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