Vol. **113** (2008) *ACTA PHYSICA POLONICA A* No. 6

Wave-Particle Duality through a Hydrodynamic Model of the Fractal Space-Time Theory

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(Received January 16, 2008; revised version February 21, 2008)

Considering that the microparticle movements take place on fractal curves, the wave–particle duality is studied in the fractal space-time theory (scale relativity theory). The Nottale model was extended by assuming arbitrary fractal dimension, D_{F}, of the fractal curves and third-order terms in the equation of motion of a complex speed field. It results that, in a fractal fluid, the convection, dissipation, and dispersion are reciprocally compensating at any scale (differentiable or non-differentiable), whereas a generalized Schrödinger equation is obtained for an irrotational movement of the fractal fluid. The absence of the dispersion implies a generalized Navier–Stokes type equation and the usual Schrödinger equation results for the irrotational movement in $D_{\mathrm{F}} = 2$ of the fractal fluid. The absence of dissipation implies a generalized Korteweg–de Vries type equation. In such conjecture, the duality is analyzed through a hydrodynamic formulation. At the differentiable scale, the duality is achieved by the flowing regimes of the fractal fluid, while at the non-differentiable scale, a fractal potential controls, through the coherence, the duality.

PACS numbers: 05.45.Df, 47.53.+n, 03.65.–w

1. Introduction

The theoretical description of microphysical systems is generally based on Schrödinger's wave mechanics, Heisenberg's matrix mechanics, or on Feynman's path-integral mechanics. Another approach is the hydrodynamic formulation of

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quantum mechanics. The first hydrodynamic model of quantum mechanics was given in [1]. This approach has been developed and extended to non-relativistic spinning particles described by the Pauli equation [2–8]. A hydrodynamic model of relativistic quantum mechanics of the Dirac particle was given in [9]. The hydrodynamic model of the Weyl equation has been treated by Białynicki-Birula [10, 11]. This approach is similar to that given in [9] for the non-relativistic Pauli equation and has some characteristics: (i) the hydrodynamic variables comprise one scalar field — the density — and two vector fields — the velocity and momentum; (ii) the reduction in the number of variables to four requires a quantization condition — the same as in the non-relativistic case — that relates the curl of the momentum field to an axial vector built from the velocity field. The advantage of all hydrodynamic models of the quantum mechanics is that they [10, 11] "enable us to visualize quantum mechanical processes in terms of the familiar variables of classical hydrodynamics. Since the number of hydrodynamic variables always exceeds the number of variables needed to describe the wave function, it is necessary to impose an auxiliary condition on the hydrodynamic variables", i.e. a quantization condition.

On the other hand, the idea that the quantum space-time of microphysics is fractal, rather than flat and Minkowskian as assumed up to now, was suggested in [12, 13]. This proposal was itself based on earlier results [14–17], obtained at first by Feynman (see in particular [18] and references therein), concerning the geometrical structure of quantum paths. These studies have shown that the typical trajectories of quantum mechanical particles are continuous but non-differentiable, and can be characterized by a fractal dimension which jumps from $D_{\mathrm{F}} = 1$ at large length-scales to $D_{\mathrm{F}} = 2$ at small length-scales, the transition occurring about the de Broglie scale (see Refs. [19, 20]).

Now such a fractal dimension $D_{\mathrm{F}} = 2$ plays a particular role in physics. It is well known that this is the fractal dimension of Brownian motion [21], i.e. from the mathematical viewpoint, of a Markov–Wiener process. This remark leads us to recall a related attempt at understanding the quantum behavior, namely, Nelson's stochastic quantum mechanics [22, 23]. In this approach, it is assumed that any particle is subjected to an underlying Brownian motion of unknown origin, which is described by two (forward and backward) Wiener processes: when combined together, they yield the complex nature of the wave function and they transform Newton's equation of dynamics into the Schrödinger equation.

This is precisely one of the aims of the fractal space-time theory, and particularly of the scale relativity theory (SRT), to relate the fractal and stochastic approaches [12, 19, 20, 24]: the hypothesis that the space-time is non-differentiable and fractal implies that there is an infinity of geodesics between any couple of points [19] and provides us with a fundamental and universal origin for the double Wiener process of Nelson [20]. SRT is a new approach to understand quantum mechanics, and moreover physical domains involving scale laws, such as chaotic systems. It is based on a generalization of Einstein's principle of relativity to scale transformations. Namely, one redefines space-time resolutions as characterizing the state of scale of reference systems, in the same way as speed characterizes their state of motion. Then one requires that the laws of physics apply whatever the state of the reference system, of motion (principle of motion-relativity) and of scale (principle of SRT). The principle of SRT is mathematically achieved by the principle of scale-covariance, requiring that the equations of physics keep their simplest form under transformations of resolution. In such conjecture, it was demonstrated that, in the fractal dimension $D_{\mathrm{F}} = 2$, the geodesics of the space-time are given by a Schrödinger type equation [20–24]. Moreover, a hydrodynamic model was developed [25].

Recently, using the hydrodynamic model of the Weyl–Dirac theory in the non-relativistic and relativistic approach, some connections with fractal space-time through SRT are given [26, 27]. According to these papers, a non-differentiable continuum is necessarily fractal and the trajectories in such a space (or space--time) own (at least) the following properties: (i) the test particle can follow an infinity of potential trajectories: this leads one to use a fluid (fractal fluid)-like description; (ii) the geometry of each trajectory is fractal (of dimension 1 as in [26] or of dimension 2 as in [27]). Each elementary displacement is then described in terms of the sum, $\mathrm{d}\boldsymbol{X} = \mathrm{d}\boldsymbol{x} + \mathrm{d}\xi$, of a mean classical displacement $\mathrm{d}\boldsymbol{x} = \boldsymbol{v}\mathrm{d}t$ and of a fractal fluctuation $\mathrm{d}\xi$, whose behavior satisfies the principle of SRT (in its simplest Galilean version). It is such that $\langle \mathrm{d}\xi \rangle = 0$ and $\langle \mathrm{d}\xi^2 \rangle = (\hbar/m)\mathrm{d}t$ where $\hbar/2m$ defines the fractal/non-fractal transition, i.e. the transition from the explicit scale dependence to scale independence, $\hbar$ is the Planck reduced constant and m is the rest-mass of test particle. The existence of this fluctuation implies introducing new terms (first order terms as in [26] or second order terms as in [27]) in the differential equation of motion; (iii) time reversibility is broken at the infinitesimal level: this can be described in terms of a two-valuedness of the velocity vector (we denote by $\boldsymbol{v}_+$ the "forward" speed and by $\boldsymbol{v}_-$ the "backward" speed) for which we use a complex representation, $\boldsymbol{V} = (\boldsymbol{v}_+ + \boldsymbol{v}_-)/2 - \mathrm{i}(\boldsymbol{v}_+ - \boldsymbol{v}_-)/2$. These three effects were combined to construct the "fractal" time derivative operators, $\hat{O} = \partial_t + \boldsymbol{V} \cdot \nabla$ as in [26] or $\hat{O}(2) = \partial_t + \boldsymbol{V} \cdot \nabla - \mathrm{i}(\hbar/2m)\nabla$ as in [27] and then to write the Newton equations in their covariant form, $\hat{O}(1,2)\boldsymbol{V} = 0$. From here, for the irrotational movement of the quantum fluid, by separating the real and imaginary parts of the complex velocity field $\boldsymbol{V}$, the hydrodynamic model resulted.

More recently, Nottale et al. [28–31] analyzed both the physical background of the scale relativity theory in connection with quantum mechanics and also the mathematical formalism using the papers of Cresson [32, 33]. In these conditions, Newton's equation was integrated in terms of a Schrödinger equation. Thus, this equation is both classical and quantum [30].

However, we note that all the treatments of Nottale are limited to the motion on fractal curves of fractal dimension $D_{\mathrm{F}} = 2$ and second order term in the equation of motion of a complex speed field.

In the present paper, assuming that the microparticle movement takes place on fractal curves (continuous but non-differentiable) of arbitrary fractal dimension D_{F}, the wave–particle duality is studied in the third-order approximation of the equation of motion, i.e. in an extended Nottale model of SRT. The paper is structured as follows: in Sect. 2 the mathematical model is developed and correspondences with known results are given. In Sect. 3 the hydrodynamic model of SRT is built by assuming that, in the fractal fluid movement, the dissipation can be neglected in comparison with the dispersion and convection. In the differentiable case, various flowing regimes of the fractal fluid were evidenced as corresponding with the dominance of one of the characters (wave or particle). In the non-differentiable case given in Sect. 4, the wave or the particle character was correlated with the self-structuring of the fractal fluid by means of a fractal potential of the Bohm type.

2. Mathematical model

Let us suppose that the motion of microparticles takes place on fractal (continuous but non-differentiable) curves of fractal dimension D_{F}. Such hypothesis is in agreement with de Broglie's theory [1–7, 13, 14]: the chaotic effect of the associated wave packet of the particle on the particle itself has as result a motion on a fractal curve. A manifold compatible with such motions will be called fractal space-time. The fractal nature of space-time implies, through non-differentiability, the breaking of differential time reflection invariance [20]. In such a context, the usual definitions of the derivative of a given function with respect to time [20],

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{\Delta t\to 0^+} \frac{f(t+\Delta t)-f(t)}{\Delta t} = \lim_{\Delta t\to 0^-} \frac{f(t)-f(t-\Delta t)}{\Delta t}$$

are equivalent in the differentiable case. One passes from one to the other by the transformation $\Delta t \to -\Delta t$ (time reflection invariance at the infinitesimal level). In the non-differentiable case two functions $(\mathrm{d}f_+/\mathrm{d}t)$ and $(\mathrm{d}f_-/\mathrm{d}t)$ are defined as explicit functions of t and of $\mathrm{d}t$ [20],

$$\frac{\mathrm{d}f_+}{\mathrm{d}t} = \lim_{\Delta t\to 0^+} \frac{f(t+\Delta t,\Delta t)-f(t,\Delta t)}{\Delta t},$$

$$\frac{\mathrm{d}f_-}{\mathrm{d}t} = \lim_{\delta t\to 0^-} \frac{f(t,\Delta t)-f(t-\Delta t,\Delta t)}{\Delta t}.$$

The sign $(+)$ corresponds to the forward process and $(-)$ to the backward process.

Then, in the spaces coordinates $\mathrm{d}\boldsymbol{X}$, we can write (for details see [12, 19, 20, 24])

$$\mathrm{d}\boldsymbol{X}_\pm = \mathrm{d}\boldsymbol{x}_\pm + \mathrm{d}\boldsymbol{\xi}_\pm = \boldsymbol{v}_\pm \mathrm{d}t + \mathrm{d}\boldsymbol{\xi}_\pm \tag{1a,b}$$

with $\boldsymbol{v}_\pm$ the forward and backward mean speeds,

$$\boldsymbol{v}_+ = \frac{\mathrm{d}\boldsymbol{x}_+}{\mathrm{d}t} = \lim_{\Delta t\to 0^+}\left\langle \frac{\boldsymbol{X}(t+\Delta t)-\boldsymbol{X}(t)}{\Delta t}\right\rangle, \tag{2a}$$

$$\boldsymbol{v}_- = \frac{\mathrm{d}\boldsymbol{x}_-}{\mathrm{d}t} = \lim_{\Delta t\to 0^-}\left\langle \frac{\boldsymbol{X}(t)-\boldsymbol{X}(t-\Delta t)}{\Delta t}\right\rangle \tag{2b}$$

and $\mathrm{d}\xi_\pm$ a measure of non-differentiability (a fluctuation induced by the fractal properties of trajectory) with the average

$$\langle \mathrm{d}\xi_\pm\rangle = 0. \tag{3}$$

While the speed concept is classically a single concept, if space-time is non-differentiable, we must introduce two speeds ($\boldsymbol{v}_+$ and $\boldsymbol{v}_-$) instead of one. This "two-valuedness" of the speed vector is a new, specific consequence of non--differentiability that has no standard counterpart (in the sense of differential physics).

However, we cannot favor $\boldsymbol{v}_+$ rather than $\boldsymbol{v}_-$. The only solution is to consider both the forward ($\mathrm{d}t > 0$) and backward ($\mathrm{d}t < 0$) processes together. Then, we can use the complex speed [20, 24]:

$$\boldsymbol{V} = \frac{\boldsymbol{v}_+ + \boldsymbol{v}_-}{2} - \mathrm{i}\frac{\boldsymbol{v}_+ - \boldsymbol{v}_-}{2} = \frac{\mathrm{d}\boldsymbol{x}_+ + \mathrm{d}\boldsymbol{x}_-}{2\mathrm{d}t} - \mathrm{i}\frac{\mathrm{d}\boldsymbol{x}_+ - \mathrm{d}\boldsymbol{x}_-}{2\mathrm{d}t}. \tag{4}$$

If $(\boldsymbol{v}_+ + \boldsymbol{v}_-)/2$ may be considered as differentiable (classical) speed, then the difference $(\boldsymbol{v}_+ - \boldsymbol{v}_-)/2$ is the non-differentiable speed.

Using the notations $\mathrm{d}x_\pm = \mathrm{d}_\pm x$, Eq. (4) becomes

$$\boldsymbol{V} = \left(\frac{\mathrm{d}_+ + \mathrm{d}_-}{2\mathrm{d}t} - \mathrm{i}\frac{\mathrm{d}_+ - \mathrm{d}_-}{2\mathrm{d}t}\right)\boldsymbol{x}. \tag{5}$$

This enables us to define the operator

$$\frac{\delta}{\mathrm{d}t} = \frac{\mathrm{d}_+ + \mathrm{d}_-}{2\mathrm{d}t} - \mathrm{i}\frac{\mathrm{d}_+ - \mathrm{d}_-}{2\mathrm{d}t}. \tag{6}$$

Let us now assume that the curve describing the movement (continuous but non-differentiable) is immersed in a 3-dimensional space, and that $\boldsymbol{X}$ of components X^i ($i = 1, 2, 3$) is the position vector of a point on the curve. Let us also consider a function $f(\boldsymbol{X}, t)$ and the following Taylor series expansion up to the third order

$$\mathrm{d}f = f(X^i + \mathrm{d}X^i, t + \mathrm{d}t) - f(X^i, \mathrm{d}t) = \left(\frac{\partial}{\partial X^i}\mathrm{d}X^i + \frac{\partial}{\partial t}\mathrm{d}t\right) f(X^i, t)$$

$$+\frac{1}{2}\left(\frac{\partial}{\partial X^i}\mathrm{d}X^i + \frac{\partial}{\partial t}\mathrm{d}t\right)^2 f(X^i, t) + \frac{1}{3!}\cdot\left(\frac{\partial}{\partial X^i}\mathrm{d}X^i + \frac{\partial}{\partial t}\mathrm{d}t\right)^3 f(X^i, t). \tag{7}$$

From here, the forward and backward average values of this relation, using notations $\mathrm{d}X^i_\pm = \mathrm{d}_\pm X^i$, take the form

$$\langle \mathrm{d}_\pm f\rangle = \left\langle \frac{\partial f}{\partial t}\mathrm{d}t\right\rangle + \langle \nabla f\cdot \mathrm{d}_\pm \boldsymbol{X}\rangle + \frac{1}{2}\left\langle \frac{\partial^2 f}{\partial t^2}(\mathrm{d}t)^2\right\rangle + \left\langle \frac{\partial^2 f}{\partial X^i \partial t}\mathrm{d}_\pm X^i \mathrm{d}t\right\rangle$$

$$+\frac{1}{2}\left\langle \frac{\partial^2 f}{\partial X^i \partial X^j}\mathrm{d}_\pm X^i \mathrm{d}_\pm X^j\right\rangle + \frac{1}{6}\left\langle \frac{\partial^3 f}{\partial X^i \partial X^j \partial X^k}\mathrm{d}_\pm X^i \mathrm{d}_\pm X^j \mathrm{d}_\pm X^k\right\rangle$$

$$+\left\langle \frac{\partial^3 f}{\partial X^i \partial X^j \partial t} \mathrm{d}_\pm X^i \mathrm{d}_\pm X^j \mathrm{d}_\pm t \right\rangle + \frac{1}{2}\left\langle \frac{\partial^3 f}{\partial X^i \partial t^2} \mathrm{d}_\pm X^i (\mathrm{d}t)^2 \right\rangle$$

$$+\frac{1}{6}\left\langle \frac{\partial^3 f}{\partial t^3} (\mathrm{d}t)^3 \right\rangle . \tag{8}$$

We make the following stipulations: the mean values of the function f and its derivates coincide with themselves, and the differentials $\mathrm{d}_\pm X^i$ and $\mathrm{d}t$ are independent, therefore the averages of their products coincide with the product of average. Thus Eq. (8) becomes

$$\mathrm{d}_\pm f = \frac{\partial f}{\partial t}\mathrm{d}t + \nabla f \langle \mathrm{d}_\pm \boldsymbol{X} \rangle + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\langle (\mathrm{d}t)^2 \rangle + \frac{\partial^2 f}{\partial X^i \partial t}\langle \mathrm{d}_\pm X^i \mathrm{d}t \rangle$$

$$+\frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^j}\langle \mathrm{d}_\pm X^i \mathrm{d}_\pm X^j \rangle + \frac{1}{6}\frac{\partial^3 f}{\partial t^3}\langle (\mathrm{d}t)^3 \rangle$$

$$+\frac{1}{2}\frac{\partial^3 f}{\partial X^i \partial X^i \partial t}\langle \mathrm{d}_\pm X^i \mathrm{d}_\pm X^j \rangle \langle \mathrm{d}t \rangle + \frac{1}{2}\frac{\partial^3 f}{\partial X^i \partial t^2}\langle \mathrm{d}_\pm X^i \rangle \langle (\mathrm{d}t)^2 \rangle$$

$$+\frac{1}{6}\frac{\partial^3 f}{\partial X^i \partial X^j \partial X^k}\langle \mathrm{d}_\pm X^i \mathrm{d}_\pm X^j \mathrm{d}_\pm X^k \rangle \tag{9}$$

or more, using Eqs. (1a,b) with the property (3),

$$\mathrm{d}_\pm f = \frac{\partial f}{\partial t}\mathrm{d}t + \nabla f \mathrm{d}_\pm \boldsymbol{x} + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(\mathrm{d}t)^2 + \frac{\partial^2 f}{\partial X^i \partial t}\mathrm{d}_\pm x^i \mathrm{d}t$$

$$+\frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^j}\left(\mathrm{d}_\pm x^i \mathrm{d}_\pm x^j + \langle \mathrm{d}\xi^i_\pm \mathrm{d}\xi^j_\pm \rangle\right) + \frac{1}{6}\frac{\partial^3 f}{\partial t^3}(\mathrm{d}t)^3$$

$$+\frac{1}{2}\frac{\partial^3 f}{\partial X^i \partial X^j \partial t}\left(\mathrm{d}_\pm x^i \mathrm{d}_\pm x^j + \langle \mathrm{d}\xi^i_\pm \mathrm{d}\xi^j_\pm \rangle\right)\mathrm{d}t + \frac{1}{2}\frac{\partial^3 f}{\partial X^i \partial t^2}\mathrm{d}_\pm x^i (\mathrm{d}t)^2$$

$$+\frac{1}{6}\frac{\partial^3 f}{\partial X^i \partial X^j \partial X^k}\left(\mathrm{d}_\pm x^i \mathrm{d}_\pm x^j \mathrm{d}_\pm x^k + \langle \mathrm{d}\xi^i_\pm \mathrm{d}\xi^j_\pm \mathrm{d}\xi^k_\pm \rangle\right) . \tag{10}$$

Since $\mathrm{d}\xi^i_\pm$ describes the fractal properties of the fractal curve which has the fractal dimension D_{F} (for details see [15, 20, 21]), it is natural to impose $(\mathrm{d}\xi^i_\pm)^{D_{\mathrm{F}}}$ to be proportional to time parameter $\mathrm{d}t$, i.e.

$$(\mathrm{d}\xi^i_\pm)^{D_{\mathrm{F}}} = \sqrt{2D}\mathrm{d}t \tag{11}$$

with D — a proportionality coefficient.

Even the average value of the fractal coordinate, $\mathrm{d}\xi^i_\pm$, is null (see (3)) for the high order of the fractal coordinate average the situation can be different. First, let us focus now on the mean $\langle \mathrm{d}\xi^i_\pm \mathrm{d}\xi^j_\pm \rangle$. If $i \neq j$ this average is zero due to the independence of $\mathrm{d}\xi^i$ and $\mathrm{d}\xi^j$. Therefore, using (11) we can write

$$\langle \mathrm{d}\xi_\pm^i \mathrm{d}\xi_\pm^j \rangle = \pm\delta^{ij} 2D(\mathrm{d}t)^{2/D_\mathrm{F}-1}\mathrm{d}t \tag{12a}$$

with

$$\delta^{ij} = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j, \end{cases}$$

and we had considered that

$$\begin{cases} \langle \mathrm{d}\xi_+^i \mathrm{d}\xi_+^j \rangle > 0 \quad \text{and} \quad \mathrm{d}t > 0, \\ \langle \mathrm{d}\xi_-^i \mathrm{d}\xi_-^j \rangle > 0 \quad \text{and} \quad \mathrm{d}t < 0. \end{cases}$$

Then, let us consider the mean $\langle \mathrm{d}\xi_\pm^i \mathrm{d}\xi_\pm^j \mathrm{d}\xi_\pm^k \rangle$. If $i \neq j \neq k$ this average is zero due to the independence of $\mathrm{d}\xi^i$ on $\mathrm{d}\xi^j$ and $\mathrm{d}\xi^k$. Now, using Eq. (11), we can write

$$\langle \mathrm{d}\xi_\pm^i \mathrm{d}\xi_\pm^j \mathrm{d}\xi_\pm^k \rangle = \delta^{ijk}(2D)^{3/2}(\mathrm{d}t)^{3/D_\mathrm{F}-1}\mathrm{d}t \tag{12b}$$

with

$$\delta^{ijk} = \begin{cases} 1, & \text{if} \quad i = j = k, \\ 0, & \text{if} \quad i \neq j \neq k, \end{cases}$$

and we considered that

$$\begin{cases} \langle \mathrm{d}\xi_+^i \mathrm{d}\xi_+^j \mathrm{d}\xi_+^k \rangle > 0 \quad \text{and} \quad \mathrm{d}t > 0, \\ \langle \mathrm{d}\xi_-^i \mathrm{d}\xi_-^j \mathrm{d}\xi_-^k \rangle > 0 \quad \text{and} \quad \mathrm{d}t < 0. \end{cases}$$

Then Eq. (10) may be written under the form

$$\begin{aligned} \mathrm{d}_\pm f = {} & \frac{\partial f}{\partial t}\mathrm{d}t + \nabla f \mathrm{d}_\pm \boldsymbol{x} + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(\mathrm{d}t)^2 + \frac{\partial^2 f}{\partial X^i \partial t}\mathrm{d}_\pm x^i \mathrm{d}t \\ & + \frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^j}\mathrm{d}_\pm x^i \mathrm{d}_\pm x^j \pm \frac{\partial^2 f}{\partial X^i \partial X^j}\delta^{ij} D(\mathrm{d}t)^{2/D_\mathrm{F}-1} \\ & + \frac{1}{6}\frac{\partial^3 f}{\partial t^3}(\mathrm{d}t)^3 + \frac{1}{2}\frac{\partial^3 f}{\partial X^i \partial X^j \partial t}\mathrm{d}_\pm x^i \mathrm{d}_\pm x^j \mathrm{d}t + \frac{1}{2}\frac{\partial^3 f}{\partial X^i \partial t^2}\mathrm{d}_\pm x^i (\mathrm{d}t)^2 \\ & + \frac{1}{6}\frac{\partial^3 f}{\partial X^i \partial X^j \partial X^k}\mathrm{d}_\pm x^i \mathrm{d}_\pm x^j \mathrm{d}_\pm x^k \\ & + \frac{\partial^3 f}{\partial X^i \partial X^j \partial X^k}\delta^{ijk}\frac{\sqrt{2}}{3}(\mathrm{d}t)^{3/D_\mathrm{F}-1}\mathrm{d}t. \end{aligned} \tag{13}$$

If we divide by $\mathrm{d}t$ and neglect the terms which contain differential factors (for details on the method see [19, 20, 24]), Eq. (13) is reduced to

$$\frac{\mathrm{d}_\pm f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \boldsymbol{v}_\pm \nabla f \pm D(\mathrm{d}t)^{2/D_\mathrm{F}-1}\Delta f + \frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_\mathrm{F}-1}\nabla^3 f \tag{14}$$

with

$$\nabla^3 f = \frac{\partial^3 f}{(\partial X^1)^3} + \frac{\partial^3 f}{(\partial X^2)^3} + \frac{\partial^3 f}{(\partial X^3)^3}.$$

This relation also allows us to give the definition of the operator

$$\frac{\mathrm{d}_\pm}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{v}\nabla \pm D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta + \frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3. \tag{15}$$

Let us calculate, under the circumstances $(\delta f/\mathrm{d}t)$. Taking into account Eqs. (15) and Eq. (6), we obtain

$$\begin{aligned}
\frac{\delta f}{\mathrm{d}t} &= \frac{1}{2}\left[\frac{\mathrm{d}_+ f}{\mathrm{d}t} + \frac{\mathrm{d}_- f}{\mathrm{d}t} - \mathrm{i}\left(\frac{\mathrm{d}_+ f}{\mathrm{d}t} - \frac{\mathrm{d}_- f}{\mathrm{d}t}\right)\right] \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_+\nabla f + D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta f + \frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3 f\right) \\
&+\frac{1}{2}\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_-\nabla f - D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta f + \frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3 f\right) \\
&-\frac{\mathrm{i}}{2}\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_+\nabla f + D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta f + \frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3 f\right) \\
&+\frac{\mathrm{i}}{2}\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_+\nabla f - D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta f + \frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3 f\right) \\
&= \frac{\partial f}{\partial t} + \left(\frac{\boldsymbol{v}_+ + \boldsymbol{v}_-}{2} - \mathrm{i}\frac{\boldsymbol{v}_+ - \boldsymbol{v}_-}{2}\right)\nabla f - \mathrm{i}D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta f \\
&+\frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3 f
\end{aligned} \tag{16}$$

or using Eq. (5):

$$\frac{\delta f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \boldsymbol{V}\cdot\nabla f - \mathrm{i}D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta f + \frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3 f. \tag{17}$$

This relation also allows us to give the definition of the fractal operator

$$\frac{\delta}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla - \mathrm{i}D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta + \frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3. \tag{18}$$

We now apply the principle of scale covariance, and postulate that the passage from classical (differentiable) mechanics to the "fractal" mechanics which is considered here can be implemented by replacing the standard time derivative $\mathrm{d}/\mathrm{d}t$ by the complex operator $\delta/\mathrm{d}t$ (this results in a generalization of the principle of scale covariance given by Nottale in [19, 20, 24]). As a consequence, we are now able to write the equation of geodesics (a generalization of the first Newton principle) in a fractal space-time under its covariant form

$$\frac{\delta \boldsymbol{V}}{\mathrm{d}t} = \frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\nabla\boldsymbol{V} - \mathrm{i}D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta\boldsymbol{V}$$

$$+\frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3\boldsymbol{V}=0. \tag{19}$$

This means that the global complex acceleration field, $\delta\boldsymbol{V}/\mathrm{d}t$, depends on the local complex acceleration field, $\partial_t\boldsymbol{V}$, on the non-linearity (convective) term, $\boldsymbol{V}\cdot\nabla\boldsymbol{V}$, on the dissipative term, $\Delta\boldsymbol{V}$, and on the dispersive one, $\nabla^3\boldsymbol{V}$. Moreover, the behavior of a "fractal fluid" is of viscoelastic or of hysteretic type which means that the fractal space-time has memory. Such a result is in agreement with the opinion given in [34–36]: the fractal fluid can be described by the Kelvin–Voight or Maxwell rheological model with the aid of complex quantities e.g. the complex speed field, the complex acceleration field, etc.

Particularizing Eq. (19) interesting results arise. Thus, if in the fractal space-time the dissipative and convective effects are dominant in comparison with the dispersive ones, then the microparticle movement is described by a generalized Navier–Stokes (GNS) type equation,

$$\frac{\delta\boldsymbol{V}}{\mathrm{d}t}=\frac{\partial\boldsymbol{V}}{\partial t}+\boldsymbol{V}\cdot\nabla\boldsymbol{V}-\mathrm{i}D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta\boldsymbol{V}=0 \tag{19a}$$

with an imaginary viscosity coefficient, $\eta=\mathrm{i}D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}$. If the dissipative effects can be neglected by comparison with the convective and dispersive ones, then the microparticle movement is described by a generalized Korteweg–de Vries (GKdV) type equation,

$$\frac{\partial\boldsymbol{V}}{\partial t}+\boldsymbol{V}\cdot\nabla\boldsymbol{V}+\frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3\boldsymbol{V}=0. \tag{19b}$$

From Eq. (19) and through the operational relation $\boldsymbol{V}\cdot\nabla\boldsymbol{V}=\nabla(\boldsymbol{V}^2/2)-\boldsymbol{V}\times(\nabla\times\boldsymbol{V})$ we obtain the equation

$$\frac{\delta\boldsymbol{V}}{\mathrm{d}t}=\frac{\partial\boldsymbol{V}}{\partial t}+\nabla\left(\frac{\boldsymbol{V}^2}{2}\right)-\boldsymbol{V}\times(\nabla\times\boldsymbol{V})-\mathrm{i}D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta\boldsymbol{V}$$

$$+\frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3\boldsymbol{V}=0. \tag{20}$$

If the motions of the fractal fluid are irrotational, i.e. $\boldsymbol{\Omega}=\nabla\times\boldsymbol{V}=0$ we can choose $\boldsymbol{V}$ of the form

$$\boldsymbol{V}=\nabla\phi \tag{21}$$

with ϕ — a complex speed potential. Then, Eq. (20) becomes

$$\frac{\delta\boldsymbol{V}}{\mathrm{d}t}=\frac{\partial\boldsymbol{V}}{\partial t}+\nabla\left(\frac{\boldsymbol{V}^2}{2}\right)-\mathrm{i}D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta\boldsymbol{V}$$

$$+\frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3\boldsymbol{V}=0 \tag{22}$$

and more, by substituting Eq. (21) in Eq. (22), we shall have by integration

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}(\nabla\phi)^2-\mathrm{i}D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\Delta\phi$$

$$+\frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3\phi = F(t), \tag{23}$$

with $F(t)$ being a function of time only. We realize that Eq. (22) has been reduced to a single scalar relation (23), i.e. a generalized Bernoulli (GB) type equation.

Let us choose the complex speed potential in the form

$$\phi = -2\mathrm{i}D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\ln\psi. \tag{24}$$

Then, ψ by means of Eq. (23) satisfies a generalized Schrödinger (GS) type equation

$$D^2(\mathrm{d}t)^{4/D_{\mathrm{F}}-2}\Delta\psi + \mathrm{i}D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\partial_t\psi$$

$$+\mathrm{i}\frac{\sqrt{2}}{3}D^{5/2}(\mathrm{d}t)^{5/D_{\mathrm{F}}-2}(\nabla^3\ln\psi)\psi = F(t). \tag{25}$$

In such context, if in the fractal space-time the dissipative and convective effects dominate the dispersive ones, with the restriction $F(t) = 0$, Eq. (25) can be reduced to a Schrödinger type equation in the fractal dimension D_{F},

$$D^2(\mathrm{d}t)^{4/D_{\mathrm{F}}-2}\Delta\psi + \mathrm{i}D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\partial_t\psi = 0. \tag{26}$$

For $D = \hbar/2m$ and $D_{\mathrm{F}} = 2$ (e.g. the Peano type curve which completely covers a two-dimensional surface — see Nottale's approach of SRT [20]), Eq. (26) is reduced to standard Schrödinger equation.

Since usually the wave–particle duality is analyzed in the standard model of quantum mechanics (which from our perspective involves Navier–Stokes type equation — see Eq. (19a)), in the followings this aspect will be discussed from the perspective of Eq. (19b).

3. Wave–particle duality at differentiable scale

Let us consider the relation (4) in the form

$$\boldsymbol{V} = \boldsymbol{v} + \mathrm{i}\boldsymbol{u}. \tag{27}$$

According to our previous paragraph, $\boldsymbol{v}$ will correspond to the classical speed given by the differential part of $\boldsymbol{V}$, and $\boldsymbol{u}$ will correspond to the fractal speed given by the non-differential part of $\boldsymbol{V}$. By replacing (27) in Eq. (19b) and separating the real part from the imaginary one, we obtain the following system:

$$\frac{\partial\boldsymbol{v}}{\partial t} + \nabla\left(\frac{\boldsymbol{v}^2}{2} - \frac{\boldsymbol{u}^2}{2}\right) + \frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3\boldsymbol{v} = 0, \tag{28a}$$

$$\frac{\partial\boldsymbol{u}}{\partial t} + \nabla(\boldsymbol{v}\cdot\boldsymbol{u}) + \frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3\boldsymbol{u} = 0. \tag{28b}$$

Substituting in (24) and then in (21), $\psi = \sqrt{\rho}\exp(\mathrm{i}S)$, with $\sqrt{\rho}$ the amplitude and S the phase, the components of the complex speed $\boldsymbol{V}$ are

$$\boldsymbol{v} = 2D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\nabla S, \qquad \boldsymbol{u} = -D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\nabla\ln\rho. \tag{29a,b}$$

Then, Eq. (28a) corresponds to the momentum conservation law

$$m\left[\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3\boldsymbol{v}\right] = -\nabla(Q) \tag{29c}$$

with $Q = -m\boldsymbol{u}^2/2$ and Eq. (28b), up to an arbitrary phase factor which may be set to zero by a suitable choice of the phase of ψ, to the probability conservation law

$$\partial_t\rho + \nabla\cdot(\rho v) = \rho\Delta\cdot v - \rho\frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\nabla^3\ln\rho. \tag{29d}$$

Moreover, the compatibility between the scale relativity hydrodynamic model and wave mechanics implies the quantization condition

$$\oint m\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{r} = 2D(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}\oint\nabla S\cdot\mathrm{d}\boldsymbol{r} = 4\pi nD(\mathrm{d}t)^{2/D_{\mathrm{F}}-1}, \qquad n = 0,1,2\ldots$$

Particularly, for $D = 2m\hbar$ and $D_{\mathrm{F}} = 2$, the previous relation takes the standard form

$$\oint m\boldsymbol{v}\mathrm{d}\boldsymbol{r} = nh.$$

In the differentiable case, $\boldsymbol{u} = 0$ or $\rho = \mathrm{const}$, Eqs. (29c,d), in one--dimensional case, takes the standard form of the KdV equation [37]:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial X} + \frac{\sqrt{2}}{3}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}\frac{\partial^3 v}{\partial X^3} = 0. \tag{30}$$

Using the dimensionless parameters, $\bar{\phi} = v/v_0$, $\tau = \omega_0 t$, $\xi = k_0 X$, and the normalizing conditions $(k_0 v_0/2) = \sqrt{2}D^{3/2}(\mathrm{d}t)^{3/D_{\mathrm{F}}-1}k_0^3 = 3\omega_0$, Eq. (30) becomes

$$\partial_\tau\bar{\phi} + 6\bar{\phi}\partial_\xi\bar{\phi} + \partial_{\xi\xi\xi}\bar{\phi} = 0. \tag{31}$$

Through the substitutions, $w(\theta) = \bar{\phi}(\xi,\tau)$, $\theta = \xi - u\tau$, where (ω_0, k_0, v_0) are the specific parameters of speed field, Eq. (31), by double integration, becomes

$$\frac{1}{2}w'^2 = F(w) = -\left(w^3 - \frac{u}{2}w^2 - gw - h\right) \tag{32}$$

with g, h two integration constants. If $F(w)$ has real roots, they are of the form

$$e_1 = 2a\frac{E(s)}{K(s)}, \qquad e_2 = 2a\left[\frac{E(s)}{K(s)} - 1\right], \qquad e_3 = 2a\left[\frac{E(s)}{K(s)} - \frac{1}{s^2}\right] \tag{33a–c}$$

with

$$a = \frac{e_1 - e_2}{2}, \qquad s^2 = \frac{e_1 - e_2}{e_1 - e_3}, \qquad K(s) = \int_0^{\pi/2}(1 - s^2\sin^2\varphi)^{-1/2}\mathrm{d}\varphi,$$

$$E(s) = \int_0^{\pi/2}(1 - s^2\sin^2\varphi)^{1/2}\mathrm{d}\varphi \tag{34a–d}$$

and $K(s)$, $E(s)$ the complete elliptic integrals [38]. Then, the solution of Eq. (31) has the expression

$$\begin{aligned}\bar{\phi}(\xi,\tau) = {} & 2a\left(\frac{E(s)}{K(s)} - 1\right) \\ & + 2a\cdot\mathrm{cn}^2\left\{\frac{\sqrt{a}}{s}\left[\xi - 2a\left(\frac{3E(s)}{K(s)} - \frac{1+s^2}{s^2}\right)\tau + \xi_0\right]; s\right\},\end{aligned} \tag{35}$$

where cn is the Jacobi elliptic function of s modulus [38] and ξ_0 constant of integration. As a result, the wave–particle duality is achieved by one-dimensional cnoidal oscillation modes of the speed field — see Fig. 1. This process is characterized through the normalized wave length,

$$\lambda = \frac{2sK(s)}{\sqrt{a}} \tag{36}$$

— see Fig. 2, and the normalized phase speed

$$u = 4a\left[3\frac{E(s)}{K(s)} - \frac{1+s^2}{s^2}\right] \tag{37}$$

— see Fig. 3. Then, the following results:

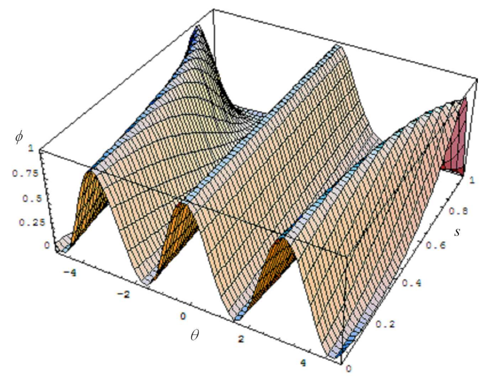

Fig. 1. One-dimensional cnoidal oscillation modes.

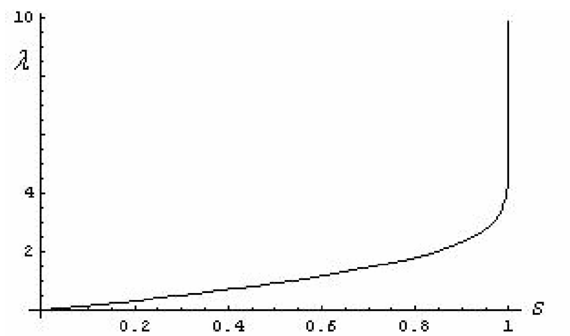

Fig. 2. The dependence of the normalized wave length λ with s.

i) through the D coefficient, the parameter s becomes a measure of the wave–particle coupling. Thus, for $s \to 0$, the normalized phase speed $|u|$ is high and the normalized wave length λ is small — see Figs. 2 and 3. On the contrary, for $s \to 1$, $|u|$ is small and λ is high — see Figs. 2 and 3;

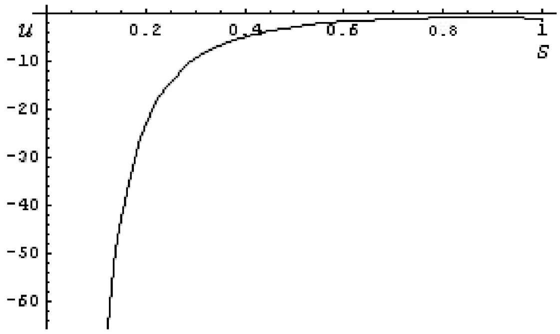

Fig. 3. The dependence of the normalized phase speed u with s.

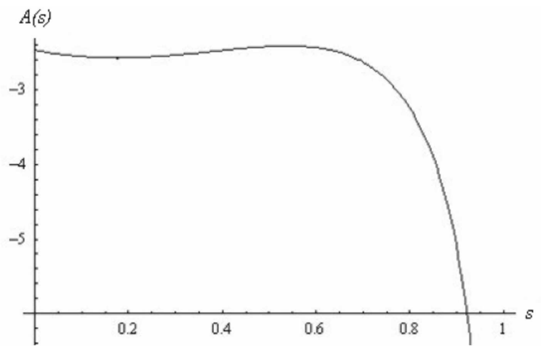

Fig. 4. The dependence $A = A(s)$.

ii) the one-dimensional cnoidal speed oscillation modes contain as subsequences: (ii$_1$) for $s = 0$ the one-dimensional speed harmonic waves and (ii$_2$) for $s \to 0$ the one-dimensional speed waves packet. These two subsequences describe the wave–particle duality in a non-autonomous regime and in this regime the wave character is dominant. (ii$_3$) For $s = 1$, the solution (35), with the substitutions $\bar{\phi}_0 = e_3$ and $k^2 = (e_1 - e_3)/2$, becomes the one-dimensional speed soliton

$$\bar{\phi}(\xi, \tau) = \bar{\phi}_0 + 2k^2 \mathrm{sech}^2 \left[k(\xi - (4k^2 + 3\bar{\phi}_0)\tau + \xi_0)\right] \tag{38}$$

of amplitude $2k^2$, width k^{-1} and phase velocity $u = 4k^2 = 3\bar{\phi}_0$, while (ii$_4$) for $s \to 1$ the one-dimensional speed solitons packet results. These last two subsequences describe the wave–particle duality in a quasi-autonomous regime and in this regime the particle character is dominant;

iii) by eliminating the parameter a from relations (36) and (37), one obtains

$$u\lambda^2 = A(s), \qquad A(s) = 16\left[3s^2 E(s)K(s) - (1 + s^2)K^2(s)\right], \tag{39a,b}$$

where the quantity $A(s)$ is numerically evaluated. For $s = 0 \div 0.7$, $A(s) \approx \mathrm{const}$ — see Fig. 4, and Eq. (39a) takes the form

$$u\lambda^2 = \mathrm{const} \tag{40}$$

and the value 0.7 is separating the wave character by the particle one.

In conclusion, the wave–particle duality is controlled through the flowing regimes of the fractal fluid, and the separation between them is given by the 0.7 value of the coupling parameter s.

4. Wave–particle duality at non-differentiable scale

In Eq. (32) the passage from the differentiable scale to the non-differential one is achieved by the substitutions $w = (u/4)f^2$, $\mathrm{i}\eta = (u/4)^{1/2}\theta$. Assuming the restriction, $h = 0$, Eq. (32) becomes a Ginzburg–Landau type equation [37]:

$$\partial_{\eta\eta} f = f^3 - f. \tag{41}$$

It results:

i) The η coordinate has dynamic significations, e.g. complex time [39–41], and the variable f acquires probabilistic significance. The space-time becomes a fractal one (for details see [19, 20, 24]) and the fluid acquires fractal properties;

ii) According to [42] we can build a field theory with spontaneous symmetry breaking. Indeed, Eq. (41) is obtained from the variational principle $\delta \int L\mathrm{d}V = 0$ applied to the Lagrangian density

$$L = \frac{1}{2}(\partial_\eta f)^2 - V(f) \tag{42}$$

with the potential

$$V(f) = \frac{f^4}{4} - \frac{f^2}{2} \tag{43}$$

and $\mathrm{d}V$ the elementary volume.

Equation (41) has the solutions $f_1 = 0$, $f_{2,3} = \pm 1$. By calculating the second derivative with respect to f of the potential entering (43) and substituting the extreme values into the result of this differentiation, we find $V_{ff}(0) = -1$, $V_{ff}(\pm 1) = 2 > 0$, i.e. the solution $f = \pm 1$ is associated with the minimum energy. The physical states associated with the minimum of energy define the vacuum states. Hence, the model under consideration has double degenerated vacuum states (for details see [42]).

From (42) there result both the energy

$$\varepsilon(f) = \int_{-\infty}^{\infty} \mathrm{d}\eta \left[\frac{1}{2}(\partial_\eta f)^2 + V(f)\right] \tag{44}$$

and the energy relative to the vacuum

$$\varepsilon(f) - \varepsilon(f_\nu) = \int_{-\infty}^{\infty} \mathrm{d}\eta \left[\frac{1}{2}(\partial_\eta f)^2 + \frac{1}{4}(f^2 - 1)^2\right]. \tag{45}$$

Because all the terms in (45) are positive and in view of the infinite limits of integration, the finiteness of the energy implies that at $\eta \to \pm\infty$:

$$\partial_\eta f = 0, \qquad (f^2 - 1)^2/4 = 0. \tag{46}$$

From this, it follows that for $\eta \to \pm\infty$ the function $f(\eta)$ tends to its vacuum values $f_\nu \to \pm 1$.

To find the explicit form of the solution of (41), we multiply it by $\partial_\eta f$ and integrate over η. It results

$$\frac{1}{2}(\partial_\eta f)^2 = -\frac{f^2}{2} + \frac{f^4}{4} + \frac{1}{2}f_0, \tag{47}$$

where f_0 is an integration constant. From this we have

$$\eta - \eta_0 = \tau \int_0^{\bar{f}} \frac{\mathrm{d}f}{\sqrt{\frac{f^4}{2} - f^2 + f_0}}, \tag{48}$$

where η_0 is another constant of integration. For an arbitrary f_0, to this general solution there corresponds an infinite value of the energy $\varepsilon(f)$. To obtain the solution with finite energy, we make use of the boundary conditions $f_\nu = \pm 1$. From (47) it results that $f_0 = 1/2$. Replacing this value of f_0 into (48), the solution $f_k(\eta)$ of the field Eq. (47) with a finite energy is

$$f_k(\eta) = f(\eta - \eta_0) = \tanh\left[\frac{1}{\sqrt{2}}(\eta - \eta_0)\right]. \tag{49}$$

This is called the fractal kink solution (a kink solution in a fractal space-time).

Combining (45) with the expression $f_\nu = 1$ and the expression for f_k, we obtain the energy of the fractal kink relative to the vacuum

$$\varepsilon(f_k) - \varepsilon(f_\nu) = \frac{2\sqrt{3}}{3}. \tag{50}$$

Therefore, the fractal kink solution was obtained by a spontaneous symmetry breaking.

A topological method [42] can be further applied because the admissible number of fractal kinks is determined by the topological properties of the symmetry group of Eq. (41). In this context, the following problems will be solved: (i) the number of admissible fractal kink solutions determined by the topological properties of Eq. (41); (ii) the topological charge.

The fractal kink solution can be considered as mapping of a spatial zero--sphere S^0, taken at infinity onto the vacuum manifold model of (41). The homotopy group for this model is $\Pi_0(Z_0) = Z_2$, i.e. the model gives rise to two solutions: a constant solution f_ν and the fractal kink solution.

The associated topological charge is

$$Q = \frac{1}{2}\int_{-\infty}^{\infty} j(\eta)\mathrm{d}\eta = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}f}{\mathrm{d}\eta}\mathrm{d}\eta = \frac{1}{2}[f(+\infty) - f(-\infty)]. \tag{51}$$

The vacuum solution (absence of spatial gradients) and the fractal kink solution can be characterized by attributing the $Q = 0$ and $Q = 1$, respectively (the result is obtained by an adequate normalization of f);

iii) The fractal kink spontaneously breaks the "vacuum symmetry" of the fractal fluid by tunneling and generates coherent structures. This mechanism is similar to the one of superconductivity [43];

iv) By an analogy with the Bohm potential [44], the normalized fractal potential [27] describes the "vacuum states" of the fractal fluid. Then, it has a very simple expression which is directly proportional with the states density of the fractal fluid, i.e.

$$Q = -\frac{1}{f}\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} = 1 - f^2. \tag{52}$$

When the states density, f^2, becomes zero, the normalized fractal potential takes a finite value, $Q = 1$. The fractal fluid is normal and there are no coherent structures in it. When f^2 becomes 1, the normalized fractal potential is equal to zero, i.e. the entire quantity of energy of the fractal fluid is transferred to its coherent structures. Then the fractal fluid becomes coherent through self-structuring. Therefore, one can assume that the energy from the fractal fluid can be stocked by transforming all the environment's entities into coherent structures and then "freezing" them. The fractal fluid acts as an energy accumulator through the normalized fractal potential;

v) substituting (49) in (52) the fractal potential becomes a fractal soliton (a soliton in a fractal space-time)

$$Q = \mathrm{sech}^2\left[\frac{1}{\sqrt{2}}(\eta - \eta_0)\right]. \tag{53}$$

Having in view the conclusion of the previous paragraph, the fractal potential controls the wave–particle duality through the coherence of the fluid. Thus, if the "fluid" is incoherent (non-quasi-autonomous flowing regime), the wave character is dominant, while if the "fluid" is coherent (quasi-autonomous flowing regime) the particle character is dominant.

5. Conclusions

The main conclusions of the present paper are the following:

i) Considering that the microparticle movements take place on fractal curves of fractal dimension D_{F}, the wave–particle duality is studied in an extension of scale relativity theory;

ii) An equation of motion is deduced for the complex speed field, where the local complex acceleration, convection, dissipation and dispersion are reciprocally compensating;

iii) The absence of the dispersion implies a generalized Navier–Stokes type equation, and from here, for the irrotational movement and fractal dimension $D_{\mathrm{F}} = 2$, the usual Schrödinger equation resulted. This is the Nottale standard model of scale relativity;

iv) The absence of dissipation implies a generalized Korteweg–de Vries type equation. In such conjecture, the wave–particle duality is analyzed. It resulted: (iv_1) through the SR hydrodynamic model in the differentiable case, the wave–particle duality is achieved by one-dimensional cnoidal oscillation modes of the speed field; (iv_2) for different degrees of the wave–particle coupling, the one--dimensional cnoidal speed oscillation modes contain the one-dimensional speed harmonic waves, the one-dimensional speed waves packet, the one-dimensional speed solitons packet and the one-dimensional speed soliton. The first two subsequences describe the non-autonomous regime of the wave–particle coupling, i.e. a situation in which the wave character is dominant, while the last ones describe the quasi-autonomous regime of the wave–particle coupling, i.e. a situation in which the particle character is dominant; (iv_3) in the non-autonomous regime, a relation between the normalized wave length and the normalized phase speed, i.e. a dispersion type relation, is obtained; (iv_4) these two regimes are separated by "0.7 structure"; (iv_5) in the non-differentiable case we build a field theory with spontaneous symmetry breaking. The fractal kink spontaneously breaks the "vacuum symmetry" of the fractal fluid by tunneling and generates coherent structures. Moreover, the fractal fluid acts as an energy accumulator through the fractal potential (fractal soliton). Then, the wave–particle duality is controlled through the fractal potential: if the fluid is incoherent, the wave character is dominant, while if the fluid is coherent the particle character is dominant.

Acknowledgments

We wish to thank the referees for helpful observation on the manuscript of this paper.

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