Vol. **113** (2008) *ACTA PHYSICA POLONICA A* No. 1

Proceedings of the CSMAG'07 Conference, Košice, July 9–12, 2007

Fracton Excitations in the Magnetic "Net Fractal" Systems

Z. Bąk

Institute of Physics, Jan Dlugosz University
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland

A localized spin system of fractal symmetry and Heisenberg exchange between nearest neighbors is considered. We define a specific class of fractals: "net fractals" and prove that in the logarithmic scale they are isomorphic with some bulk crystals. Further, with the use of logarithmic coordinates we show that the "net fractal" magnetic fractons can be presented as the conventional magnons.

PACS numbers: 61.43.Hv, 75.30.Ds

1. Introduction

The concept of fractal has become a powerful tool in analysis of common aspects of many complex processes observed in physics, biology, chemistry or earth sciences. Brownian motion, turbulence, colloid aggregation or biological pattern formation can be fully understood only when the idea of self-similarity of fractal structures is applied. The hallmark of a fractality is a hierarchical organization of its elements, described by discrete scaling laws, which makes the fractal, regardless of magnification or contraction scale, look the same. This property of fractals is called self-similarity, self-affinity or self-replicability. Although physical systems modeled by fractals are non-translation-invariant, it is a well-known fact that the self-similar fractals as well as the physical quantities on fractal systems show log-periodicities (see [1] and references therein). This opens a possibility to describe the symmetries of magnetic self-similar fractals in the way that is reminiscent of conventional formalism developed for crystalline systems. Motivated by this fact we present a study of fractal spin excitations (fractons), which is similar in spirit to the magnon approach in the solid state theory.

We say that $K \in R^3$ satisfies the scaling law S, or is an infinite-size self--similar fractal, if $S : K = K$. Let us limit our considerations to fractals in which the self-similarity can be realized only via linear maps, i.e., by transformations which point $\boldsymbol{r} = (x_1, x_2, x_3) \in K \subset R^3$ transform into the point $\boldsymbol{r}' = (x_1', x_2', x_3')$ according to the formula $x_i' = S_{i_1}x_1 + S_{i_2}x_2 + S_{i_3}x_3$; where $i = 1, 2, 3$. This transformation is represented by matrix S. If we orient coordinate axes along the eigenvectors of matrix S (i.e., $\boldsymbol{r} = (x_1, x_2, x_3) \to (\xi_1, \xi_2, \xi_3)$) then the transformation takes the form $S = S_1 S_2 S_3$. Each S_i represents multiplier along coordinate axis ξ_i. Let us consider more general transformations of the type $S^{m,n,l} = (S_1)^n (S_2)^m (S_3)^l$, where $(S_i)^n$ denotes n-tuple superposition of transformation S_i, and define a class of infinite "net fractals" G_{nf}, for which the relation $S^{m,n,l} : G_{\mathrm{nf}} = G_{\mathrm{nf}}$ is valid. It can be proven [1] that, when presented in logarithmic scale, the family of mappings $S^{(m,n,l)}$ is isomorphic with a 3D crystal lattice. This means that the isomorphism $S^{(m,n,l)} \to (ma_1, na_2, la_3)$ holds.

The purpose of this paper is to study the spin excitations over fractal subset. Most theoretical studies of the spin excitations in fractal systems are limited to considerations at universal level, without referring the specific physical model. In our study we focus our considerations on a specific model which, we believe, describes behavior of some real systems.

2. Model and discussion

Let us consider a "net fractal" cluster, consisting of N localized magnetic moments with the Heisenberg exchange interaction between nearest-neighbour sites. The Hamiltonian of the system reads

$$H = \sum_{i,j} I(r_{ij}) \sigma_i \sigma_j, \tag{1}$$

where the sum goes over all nearest neighbours sites of the fractal site n, $I(r_{ij})$ and σ_i denote the exchange integrals and localized spin, respectively. Let us discuss now the non-homogeneities of the spin density and exchange integrals on a "net fractal". We assume that the density of magnetic moments follows the averaged mass density distribution $m(r)$ which for real fractals scales on the average as $m(r) = m_0 (r/a)^d$, where d is the fractal mass dimension [2], the same refers to magnetisation. It is natural to assume that the self-similarity of the fractal is reflected also in the symmetry of exchange interactions. We assume therefore that also the exchange integrals exhibit the power law scaling with the separation, we take it in the form $I(\lambda r_{ij}) = \lambda^{-\sigma} I(r_{ij})$ (for the Ruderman–Kittel–Kasuya–Yosida (RKKY)-like interactions the σ can be approximated by the spectral dimension of the system [3, 4]). Let us now recall the (log-scale) isomorphism of "net fractals" and some crystal lattices. With the assumptions above the magnetic "net fractal" is mapped onto a spin lattice. Simultaneously, the density of energy stored in magnetic subsystem becomes (in the log-scale) uniform. We should point out here that it does not mean that system is uniform, indeed, in the fractal system the magnetic bonds exist only along directions allowed by the internal geometry. This means that we have mapped the magnetic fractal onto a crystal lattice, in which the spins bounded by the Heisenberg exchange form the percolation cluster separated from the surrounding by the "red bonds". With arbitrary boundary conditions it is impossible to gain some information about the fractal excitations, so let us consider some asymptotic cases. Let us consider first a somewhat unrealistic case when there are no broken bonds in the log-scale picture. Thus, we have a system with uniform spin density and all exchange integrals being equal.

Let us suppose that some magnetic "net fractal" system is perturbed and consider the allowed excitations of it. In real space the excitations of a fractal system are called fractons and their spatio-temporal variations depend strongly on the fractal symmetry and exchange constants [2]. Fortunately, our "net fractal" system possesses a specific symmetry which allows us to draw more general conclusions that refer to the nature of magnetic fractons. As we know, the only allowed excitations in a uniform magnetic system are the spin waves. This means that fracton excitations in the magnetic "net fractal", when pictured in the logarithmic scale mimic the spin waves and fracton appears to be the logarithmic-scale magnon. The scenario presented above refers to the ideal system, in real ones we should account for the percolation effect. As the starting point for our analysis of the percolation effect we will take the fact that magnetisation oscillation $\mu(r,t)$ associated with the spin wave satisfies the classical wave equation $\nabla^2\mu - (1/c^2)(\partial^2\mu/\partial t^2) = 0$.

As we have shown above, the "net fractals" can be mapped onto percolating network. The fact that such relation holds for real system has been confirmed experimentally in site diluted ferromagnets [5]. To account for the existence of "red bonds" we will use the fact that linearized equation of motion for ferromagnetic spins has the form of diffusion Eq. [5]. This allows the evolution of the local spin perturbation to be modeled by the diffusion process. The diffusion on the fractal system as the rule involves the possibility of fractional dynamics [6] medium. The effect of percolation (due to the "red bonds") is accounted for by introduction of fractional derivatives into the conventional wave equation. Within this approach the generalized wave equation for the amplitude of local oscillation $u(x,t)$ can be formally written as [6]:

$$(\mathit{\Delta})^{2\beta}\mu - \frac{1}{c^{2\alpha}}\,{}_tD^{2\alpha}\mu = 0. \tag{2}$$

The fractional time derivative reflects the damping effect while fractional space derivatives describe the reduced dimensionality of the system. The ratio of α and β in Eq. (2) determines the fractional spectral (fracton) dimension, which in turn governs the thermodynamical behavior of the spin system [1, 4]. Since there are many definitions of fractional derivatives in any approach, which involves the fractional calculus techniques, one should define, which definition of fractional pseudo--differential is used. Following the approaches of [6] we assume that the fractional time derivatives ${}_tD^{a*}$ are these of Caputo, while the space ${}_xD^b$ ones are these of Riemann–Liouville [7]. Finding the solution of Eq. (2) for arbitrary values α and β in its most general form is impossible, however, under additional assumptions we can find some specific solutions. Indeed, let us suppose that we can separate the variables x and t. This means that we assume that $u(x,t) = u_1(x)\,u_2(t)$. Provided that these our solutions fulfill such an assumption, we can rewrite Eq. (2) as

$$\frac{1}{u_1(x)}{}_xD^{2\beta}u_1(x) - \frac{1}{c^{2\alpha}u_2(t)}{}_tD^{2\alpha} * u_2(t) = 0. \tag{3}$$

Equation (3) is equivalent to the two independent differential equations of single variable x or t, which give us the oscillation amplitude in the form [7]:

$$u(x,t) = u_0|x|^{\beta-1}tE_{\beta,\beta}(\mathrm{i}K|x|^{\beta})E_{\alpha,\alpha}(\mathrm{i}Kc^{\alpha}t^{\alpha}). \tag{4}$$

The Mittag–Leffler function $E_{\alpha,\alpha}(x)$ [7] behaves like a stretched-exponential $\exp(-x^{\alpha})$, at short times and like x^{α} as $x \to \infty$. Evidently real fractals are always finite so the boundary conditions have to be accounted for. Let us suppose that our finite system extended over continuous manifold $\mathcal{M}$ is tethered at the boundary $\partial\mathcal{M}$. Let us assume (independently of the initial conditions which ensure uniqueness of the solution) typical tethering of the form $u(x,t)|_{\partial\mathcal{M}} = 0$. It can be easily seen that in the case of symmetrical $\mathcal{M}$ (e.g. $x \in [-L, L]$) solution (4) can satisfy this condition. Indeed, the tethering is equivalent to $E_{\alpha,\alpha}(\mathrm{i}K|L|) = 0$. This means that the number of allowed vibrational eigenmodes is equal to the number of zeroes $\tilde{x}_n$, i.e. $E_{\alpha,\alpha}(x_n) = 0$, of the generalized Mittag–Leffler functions. Thus, the allowed values of K (K is the counterpart of wave vector k in conventional systems) become quantized, $K_n = x_n/L$. As we know, the Mittag–Leffler functions have a finite number of zeroes so the condition $u(x,t)|_{\partial\mathcal{M}} = 0$ implies that only a finite number of vibration eigenmodes within finite fractal system is possible.

References

[1] Z. Bak, *Phase Transit.* **80**, 79 (2007).

[2] S. Alexander, *Phys. Rev. B* **40**, 7953 (1989).

[3] Z. Bak, R. Jaroszewicz, W. Gruhn, *J. Magn. Magn. Mater.* **213**, 340 (2000).

[4] Z. Bak, *Phys. Rev. B* **68**, 064511 (2003).

[5] T. Nakayama, K. Yakubo, R.L. Orbach, *Rev. Mod. Phys.* **66**, 381 (1994).

[6] Y.Z. Povstenko, *Int. J. Eng. Sci.* **43**, 977 (2005).

[7] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, in: *Theory and Applications of Fractional Differential Equations*, Ed. J. van Mill, North-Holland, Amsterdam 2006, Ch. 4.