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Hints on the Hirota Bilinear Method*

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We discuss four stages of the Hirota bilinear method, for construction of soliton solutions to partial differential equations: the proper substitution to express the equation in the bilinear variables (1), reduction of the excess degrees of freedom (2), the perturbation scheme (3), and solution of the system of equations at the successive orders of magnitude (4). For the first stage we suggest an extension of the well-known singularity analysis. In the second stage we point out the need for caution to avoid overdetermined systems. In the third one we suggest a path to proper assumptions on the orders of magnitude of the unknown functions. Finally, we summarize the question of the choice of appropriate solutions. For the expansions at the stages (1) and (3) we suggest a "renormalization", i.e. completion of the lower order terms with higher order ones to achieve the desired form of the coefficients.

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1. Introduction

Integrable and partially integrable nonlinear partial differential equations (NPDE) have attracted much attention of mathematicians as well as physicists for the last forty years. While the former are mainly interested in new methods of solving the initial and/or boundary value problems, physicists usually look for special solutions representing phenomena. Solitons are among the most important solutions for science and technology, from ocean waves to transmission of information through optical fibres or energy transport along protein molecules. The existence of multisoliton, especially two-soliton solutions, is crucial for information technology: it makes possible undisturbed simultaneous propagation of many pulses in both directions. The Hirota bilinear method and its multilinear refinements provide simple tools for construction of such solutions, if they exist.

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Inventing the method by Hirota [1] was one of the milestones in the history of solitons, which begins with the colourful story of J. Scott riding a horse along the Union Channel [2]. The equation, which governs so discovered waves was not found until 1895, when Korteweg and de Vries (KdV) postulated an equation which has become the most famous of all soliton equations [3]

$$u_t + 6uu_x + u_{xxx} = 0. \quad (1)$$

The first completely integrable equation of the soliton type emerged much earlier, about 1840, in the context of differential geometry. It was probably known to Gauss. Much later it acquired the witty name of the sine-Gordon equation

$$u_{xt} = \sin u. \quad (2)$$

The subsequent milestones were marked in the 60s of the 20th century. Solitons as solitary waves which preserve their identities upon a collision with other waves of the same kind were discovered numerically by Zabusky and Kruskal for the KdV equation in 1965 [4]. Later Lax proved this property analytically for the KdV solitons [5].

The first multisoliton solutions were found by the inverse scattering (IS) method as special solutions of the KdV equation [6, 7]. Later on, the IS integration scheme, including the method of obtaining soliton solutions, was generalized to a large class of equations [8, 9].

Apart from rather trivial determination of single solitary wave solutions, the first method especially designed for constructing soliton solutions is due to Hirota, who developed his bilinear approach in 1971–1972.

The IS method of solving the initial value problem is more general than Hirota's approach which yields only special solutions. However, the latter has several advantages. Firstly, the IS is much more complex, mathematically demanding, and time-consuming. The Hirota method is mainly algebraic and thus almost straightforward. Secondly, the existence of two-soliton solutions is a weaker requirement than integrability by IS, hence the Hirota method encompasses a larger class of equations, including many non-integrable ones.

Hirota's method consists of several stages, each of which requires some invention and attention. First, a substitution of a rational expression(s) for the unknown function(s) in order to express the equation in the bilinear variables. The resulting equations may have extra unknowns and thus the second stage is the reduction of the excess degrees of freedom. This is achieved by postulating constraints, which eventually transform the equations into their proper bilinear form. In the third stage the bilinear equations are solved by a perturbation scheme. In many equations some terms, e.g. those of even (or odd) powers, may beforehand be assumed zero. Finally, we have to choose properly from a large family of solutions to the system of equations at their successive orders of magnitude.

In the forthcoming sections each of these stages will be discussed and illustrated with examples from equations of mathematical physics. The paper will be completed by a short discussion of extensions of Hirota's method: firstly to systems which can only be cast into a trilinear or a higher-degree multilinear form, secondly, to periodic boundary conditions.

2. Stages of the Hirota method — example of the KdV equation

In order to identify the four stages of the Hirota method we will pursue an example: the KdV equation.

Ryogo Hirota [10–13] came up with the following idea: since known solitons are rational combinations of exponential functions, Padé (rational) approximants [4] rather than power series should be used for the expansion of the unknown function in a perturbation (nonlinearity) parameter. Such an approach requires a rational substitution of the unknown

$$u = \frac{G}{F} \tag{3}$$

before performing the expansion. After the substitution we have two unknown functions $G(x,t)$ and $F(x,t)$ instead of one $u(x,t)$. This is the first stage. For the KdV equation,expression (3) is substituted to the "potential" version of the KdV, i.e. Eq. (1) for ϕ defined by $u = \phi_x$. The function ϕ satisfies

$$\phi_t + 3\phi_x^2 + \phi_{xxx} = 0. \tag{4}$$

The second stage is the constraint on G, F to reduce the number of variables. A proper choice of G is $G = 2F_x$, equivalent to

$$\phi = 2(\ln F)_x, \quad u = 2(\ln F)_{xx}, \tag{5}$$

which yields the bilinear form of the KdV equation. A question arises: How to find the proper substitution and constraint(s)?

The resulting equation looks rather complex but it is bilinear in F and its derivatives. Moreover, it may be cast into a simple form by defining a new differential operator [1]

$$D_t^n D_x^m F \cdot G = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m F(x,t)G(x',t')|_{x'=x,t'=t}. \tag{6}$$

In terms of the "D" operator the bilinear form of the KdV equation reads

$$D_x(D_t + D_x^3)F \cdot F = 0. \tag{7}$$

In the third stage we assume a formal expansion [1]

$$F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots \tag{8}$$

When substituted to (7) it yields a system of equations at subsequent orders in ε, which allows for determination of its coefficients by recurrence

$$2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) f_1 = 0,$$

$$2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)f_2=-D_x(D_t+D_x^3)f_1\cdot f_1,$$
$$2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)f_3=-D_x(D_t+D_x^3)(f_1\cdot f_2+f_2\cdot f_1),\ \text{etc.} \tag{9}$$

The last stage is the solving of the equations for the perturbations. The family of possible solutions is large and the choice depends on the boundary conditions as well as on the particular physical model. Among the solutions there are functions rational in (x,t), e.g. for the KdV equation, with the boundary condition $u(0,t)=0$ we have [1]

$$f_1=a[x^3+12(t+\text{const})],\quad \text{f}_2=\text{f}_3=\ldots=0, \tag{10}$$

whence

$$u=-\frac{6x(x^3-24t)}{(x^3+12t)^2}. \tag{11}$$

The soliton solutions of the KdV are obtained from exponential solutions. A sum

$$f_1=\sum_{i=1}^{n}a_i\exp(p_ix+\Omega_it)=:\sum_{i=1}^{n}\exp(\eta_i) \tag{12}$$

solves the first of (9) if its parameters satisfy the dispersion relation

$$\Omega_i+p_i^3=0. \tag{13}$$

Then the solution of the second equation of (9) reads

$$f_2=\sum_{i>j}^{n}\exp(A_{ij}+\eta_i+\eta_j), \tag{14}$$

where

$$\exp(A_{ij})=\frac{(p_i-p_j)[(\Omega_i-\Omega_j)+(p_i-p_j)^3]}{(p_i+p_j)[(\Omega_i+\Omega_j)+(p_i+p_j)^3]} \tag{15}$$

(for $n=2$ we have only one parameter A_{21}).

For arbitrary index k each f_k is a sum of exponents, each of the exponents — a sum of k symbols η and $k(k-1)/2$ symbols A_{ij}, e.g.

$$f_3=\sum_{i>j>k}^{n}\exp(A_{ij}+A_{ik}+A_{jk}+\eta_i+\eta_j+\eta_k). \tag{16}$$

The scheme closes with f_n, due to properties of D. Eventually the n-soliton solution may be expressed as

$$F=\sum_{\mu_1=0}^{1}\ldots\sum_{\mu_n=0}^{1}\exp\left(\sum_{i,j=1,j<i}^{n}A_{ij}\mu_i\mu_j+\sum_{i=1}^{n}\mu_i\eta_i\right), \tag{17}$$

where $(\mu_1,\ldots,\mu_n)$ spans all possible n-element sequences of zeroes and ones [1].

Other solutions may also be of some interest. In principle f_1 may be an infinite sum of exponents or even an integral over p, while $\Omega=-p^3$. A special case in which the exponents sum up to Riemann's θ functions will be discussed later.

The fact that the recurrence closes in a finite number of steps is strictly connected with properties of the operator D. It is worthwhile to look up some of these properties.

Let z be any spatial or time coordinate, $a,\ b$ — its functions. We have [1]

$$D_z^m a \cdot 1 = (\partial/\partial z)^m a, \tag{18a}$$

$$D_z^m a \cdot b = (-1)^m D_z^m b \cdot a, \tag{18b}$$

whence $D_z^m a \cdot a = 0$ for odd m.

$$D_z a \cdot b = a_z b - a b_z, \tag{18c}$$

$$\frac{\partial}{\partial z}\left(\frac{a}{b}\right) = \frac{1}{b^2} D_z a \cdot b, \tag{18d}$$

$$\frac{\partial}{\partial z} \ln f = \frac{1}{f} D_z f \cdot 1, \tag{18e}$$

$$\frac{\partial^2}{\partial z^2} \ln f = \frac{1}{2f^2} D_z^2 f \cdot f. \tag{18f}$$

The properties crucial for the closure of the infinite system of the recurrence relations is the action of D on exponential functions

$$D_x^m \exp(p_1 x) \cdot \exp(p_2 x) = (p_1 - p_2)^m \exp\left((p_1 + p_2)x\right). \tag{18g}$$

If P is a polynomial, then

$$\begin{aligned} &P(D_x, D_t, \ldots) \exp(p_1 x + \Omega_1 t + \ldots) \cdot \exp(p_2 x + \Omega_2 t + \ldots) \\ &\quad = \frac{P(p_1 - p_2, \Omega_1 - \Omega_2, \ldots)}{P(p_1 + p_2, \Omega_1 + \Omega_2, \ldots)} \\ &\quad \times P(D_x, D_t, \ldots) \exp\left((p_1 + p_2)x + (\Omega_1 + \Omega_2)t + \ldots\right) \cdot 1. \end{aligned} \tag{18h}$$

The D symbol as well as its generalizations to trilinear and multilinear operators have one property in common, called "gauge invariance" [15]

$$\begin{aligned} &P(D_x, D_t, \ldots)[\exp(px + \Omega t + \ldots)a] \cdot [\exp(px + \Omega t + \ldots)b] \\ &\quad = \exp\left(2(px + \Omega t + \ldots)\right) P(D_x, D_t, \ldots) a \cdot b. \end{aligned} \tag{18i}$$

3. 1st and 2nd stages — substitution and the constraint

It is generally believed that finding the proper substitution is an art rather than a systematic method. Here we suggest, how the substitution can be determined by means of singularity analysis. Our approach is an extension of the known application of the singularity analysis for this purpose [16]. It is based on the fact that the choice of G and F in (3) (or, more general, $u = G/F^n$) can usually be done in such a way that movable singularities of u are zeroes of F. If the equation has the generalized Painlevé property [17] or at least the partial Painlevé property [18], u has the Laurent expansion in the complex (x, t) space like in the Painlevé test [17–19]. Then the principal part of the expansion may be used for the substitution [16].

For the KdV equation (1) the Laurent expansion in the variable F reads

$$u = \sum_{j=0}^{\infty} u_j F^{j+p}, \quad p < 0. \tag{19}$$

Compatibility of the dominant terms requires

$$p = -2, \quad u_0 = -2F_x^2. \tag{20}$$

In the next order we get $u_1 = 2F_{xx}$. Hence we substitute $u = u_0/F^2 + u_1/F = 2(-F_x^2/F^2 + F_{xx}/F) = 2(\ln F)_{xx}$ as in (5).

The modified KdV equation (MKdV)

$$u_t + 24u^2 u_x + u_{xxx} = 0 \tag{21}$$

requires a bit more effort. The Laurent expansion (19) yields for the dominant terms

$$p = -1, \quad u_0 = \pm(1/2)\mathrm{i}F_x. \tag{22}$$

In this case the solution may have two families of singularities, corresponding to the plus or minus sign in the dominant term $\pm(\mathrm{i}/2)F_x/F$. The substitution must set it free from both [16]. Hence we substitute

$$u = (\mathrm{i}/2)\,(G_x/G - F_x/F) = (\mathrm{i}/2)[\ln(G/F)]_x. \tag{23}$$

This is indeed equivalent to the classical substitution of Hirota: for real solutions G is the complex conjugate of F. Setting

$$F = f + \mathrm{i}g, \quad G = f - \mathrm{i}g, \tag{24}$$

we obtain the substitution as

$$u = \frac{\mathrm{i}}{2}\left[\ln\frac{f - \mathrm{i}g}{f + \mathrm{i}g}\right]_x = \left(\arctan\frac{g}{f}\right)_x, \tag{25}$$

exactly as in [12].

Still the substitutions (23) or (25) leave us with more unknown functions (two) than equations (just one). Let us proceed with our version (23). As in the KdV case (4), it is more convenient to start with the potential version of Eq. (21)

$$\phi_t + 8\phi_x^3 + \phi_{xxx} = 0, \tag{26}$$

which yields terms with lower order derivatives. The substitution of $\phi = (\mathrm{i}/2)\ln(G/F)$ to (26) yields

$$\frac{FG_t - F_tG + G_{xxx}F - F_{xxx}G}{2FG} + \frac{6(FF_xG_x^2 - F_x^2GG_x) + 3(F_xF_{xx}G^2 - F^2G_xG_{xx})}{2F^2G^2} = 0. \tag{27}$$

We are tempted to split Eq. (27) into two after the first line. However, expression $FG_{xxx} - F_{xxx}G$ does not have the "D" form. The second line looks even worse: it has a quadrilinear numerator. The usual procedure is the completion of the above expression to the "D" form, which reads

$$D_x^3 G \cdot F = FG_{xxx} - GF_{xxx} - 3F_xG_{xx} + 3F_{xx}G_x. \tag{28}$$

The added components are subtracted from the second line in the hope that we obtain a D-like expression. Indeed, the remaining terms sum up to $3(D_x^2 G\cdot F)(D_x G\cdot F)/(2F^2G^2)$.

Eventually the whole equation is split into

$$(D_t + D_x^3)G\cdot F = 0, \tag{29}$$

$$D_x^2 G\cdot F = 0, \tag{30}$$

which are equivalent to (but even simpler than) the original Hirota form [12].

In both previous cases the numerator of the principal part already had the "D" form: $\pm(\mathrm{i}/2)(\ln F)_x = \pm(\mathrm{i}/2)D_x F\cdot 1/F$ for the MKdV and $(\ln F)_{xx} = D_x^2 F\cdot F/(2F^2)$ for the KdV equation. We here suggest an extension of this approach: To get the proper substitution *take the principal part of the expansion and complete its numerator to the bilinear "D" form with expression of a higher order in the expansion variable F*. An example where it is necessary is the nonlinear Schrödinger equation (NLS)

$$\mathrm{i}u_t + u_{xx} + |u|^2 u = 0. \tag{31}$$

In its expansion into a Laurent series u and its complex conjugate $v = u^*$ are no more conjugated, when x and t leave the real values. Therefore, we have distinct Laurent expansions for these functions

$$u = \sum_{j=0}^{\infty} u_j F^{j+p}, \quad v = \sum_{j=0}^{\infty} v_j F^{j+q}. \tag{32}$$

Compatibility of the dominant terms requires

$$p = q = -1, \quad u_0 v_0 = -2F_x^2 \tag{33}$$

with one of $u_0,\ v_0$ arbitrary.

When looking for the substitution which leads to the Hirota bilinear form, we can remain at real (x,t). Let us name $u_0 = G$, then $v_0 = G^*$. We substitute $u = G/F$, to the NLS (without loss of generality F may be assumed real). The resulting equation has too many unknown functions, and we may use the coefficient (33) of the Laurent expansion to link G with F. However $G^*G = -2F_x^2$ lacks the "D" form. Therefore, we "renormalize" G^*G with a term of a higher order in the expansion variable F, namely $2FF_{xx}$. This way we obtain

$$G^*G = -2F_x^2 + 2FF_{xx} = D_x^2 F\cdot F. \tag{34}$$

Indeed NLS is equivalent to

$$F(\mathrm{i}D_t + D_x^2)G\cdot F - G(D_x^2 F\cdot F - G^*G) = 0. \tag{35}$$

Our assumption (34) is equivalent to the requirement that both components of Eq. (35) vanish. Thus the bilinear equations consist of (34) and

$$(\mathrm{i}D_t + D_x^2)G\cdot F = 0. \tag{36}$$

This equation has N-soliton solutions for arbitrary N [14].

The reduction of the excess variables may unnecessarily limit the set of possible soliton solutions if too many constraints are imposed. Though counting equations seems to be a trivial task, such an error has occurred in many papers.

In some of them splitting the equations was forced for the sake of solvability. In [20] the authors solve an equation for propagation of optical pulses along optical fibres

$$\mathrm{i}u_z + u_{tt} + |u|^2 u - \mathrm{i}\left(u_{ttt} + \gamma_1(|u|^2 u)_t + \gamma_2(|u|^2)_t u\right) = 0 \tag{37}$$

for the case $3\gamma_1 + 2\gamma_2 = 3$. This equation is a generalization of NLS (31) with reversed roles of the spatial and time coordinates; u is a rescaled slowly-varying envelope of the pulse. It is known as the higher-order nonlinear Schrödinger equation (abberviated to HNLS or HONSE). While the NLS takes into account merely the group velocity dispersion (term A_{tt}), and the self-modulation of the phase (the $|u|^2u$ term), Eq. (37) includes third order dispersion (term u_{ttt}), self-steepening (Kerr dispersion — term $\gamma_1(|u|^2u)_t$), and frequency shifting via stimulated Raman scattering (term $\gamma_2(|u|^2)_t u$).

Substitution $u = G/F$, where F is assumed real, leads to a multilinear equation with one excess unknown. In [20] the reduction is performed by assuming a relation between G and F

$$(\mathrm{i}D_z + D_t^2 - \mathrm{i}D_t^3)G \cdot F = 0, \quad D_t^2 F \cdot F - G^* G = 0, \quad D_t G \cdot G^* = 0. \tag{38}$$

The system, which the authors obtain, is evidently overdetermined. As a result, all the soliton solutions that they can get are reduced to a single NLS-type soliton already known from [14].

This too far going reduction was later criticized in [21]. In that work a new system of Hirota-like equations was obtained on the basis of separation of linear and nonlinear terms

$$\begin{aligned}
&(D_z - \mathrm{i}D_t^2 - D_t^3)G \cdot F = 0, \\
&(G \cdot F)[-\mathrm{i}D_t^2(F \cdot F) + \mathrm{i}(G^* \cdot G) + (\gamma_1 + \gamma_2)D_t(G^* \cdot G)] \\
&\qquad + D_t(G \cdot F)[-3{D_t}^2(F \cdot F) + (3\gamma_1 + 2\gamma_2)(G^* \cdot G)] = 0.
\end{aligned} \tag{39}$$

This looks better than (38). Apparently (39) is a system of 2 equations with 2 unknowns. However, if we count the equations a bit more thoroughly, taking each complex equation for two real ones, we see that (39) consists of two complex equations, while they only describe the propagation of one complex and one real unknown. Although the higher order NLS is more difficult to solve than the "normal" NLS, its bilinear reduction (39) is not: the present author provided the general solution of the reduced system [22]. The general solution of (39) depends on the values of the parameters γ_1 and γ_2. It is an envelope wave with the envelope given by either Jacobi elliptic functions or any solution of the MKdV equation (21).

Similarly, limited classes of solutions were obtained by direct solving the overdetermined bilinear equations for other models describing the propagation of laser pulses. Interaction of the fundamental mode with its 3rd harmonic in planar waveguides was treated in [23], while a model for interaction of the fundamental mode with the 2nd harmonic was solved in [24].

Let us note that the excess constraints may also be introduced on purpose, in order to choose special solutions or make the model solvable.

4. 3rd stage: formal expansion

In the 3rd stage the bilinear variables, F, G, etc. are formally expanded into

$$F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots, \tag{40}$$

which is substituted to equations of the form $P(D_x, D_t)F \cdot F = 0$, where P denotes a polynomial. In the lowest order we obtain

$$P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) f_1 = 0, \tag{41}$$

the next order equations have the form

$$P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) f_2 = -P(D_x, D_t) f_1 \cdot f_1,$$

$$P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) f_3 = -P(D_x, D_t)(f_1 \cdot f_2 + f_2 \cdot f_1), \quad \text{etc.} \tag{42}$$

In the above, the expansion requires $P(1, 1) = 0$ in the zero order, i.e. it works when P is free of zero order terms in D_x, D_t.

If $P(1, 1) \neq 0$ there is no constant term and the expansion starts with the 1st order in ε.

Let us note that for odd-order terms $P_{\text{odd}}(D_x, D_t)F \cdot F = 0$, whence the action of $P(D_x, D_t)$ on $F \cdot F$ consists of even order terms only.

There are cases in which even-order or odd-order terms may be omitted. For the NLS in its bilinear form (34, 36) the bilinear variables G, G^*, and F are substituted with

$$G = \varepsilon g_1 + \varepsilon^3 g_3 + ..., \; G^* = \varepsilon g_1 *^{+} \varepsilon^3 g_3^* + ..., \; F = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + ... \tag{43}$$

without loss of generality. Let us give a closer look at the expansion to see when such a simplification is possible.

Certainly, we may assume $f_0 = 1$. What if we retain all the odd and even powers of ε from 0 to ∞?

In the 0th order we have

$$\mathrm{i}g_{0,t} + g_{0,xx} = 0, \quad -g_0 g_0^* = 0, \tag{44}$$

whence $g_0 = 0$.

Equations in the 1st order have the form

$$\mathrm{i}g_{1,t} + g_{1,xx} = 0, \quad 2f_{1,xx} = 0. \tag{45}$$

We obtain a system consisting of a nontrivial equation for g_1 and a trivial one for f_1. The f_1 is either linear in x or zero.

In the 2nd order

$$\mathrm{i}g_{2,t} + g_{2,xx} = 0, \quad 2f_{2,xx} = g_1 g_1^*, \tag{46}$$

we obtain the first nontrivial equation for f. On the other hand, the equation for

g_2 is identical with that for g_1. Hence we may include g_2 into g_1 thus performing a kind of renormalization.

In the same way odd terms in the expansion of F may be included into the even ones, while even terms in the expansion of G may be included into the odd terms.

5. 4th stage — solutions

The stage 3 has left us with an infinite system of simple linear differential equations. Solving them does not make a great problem. A greater one is: how to obtain a solution for which the procedure ends after a finite number of steps.

We have two kinds of "good" solutions

- Polynomials in (x,t) (and other independent variables, if there are more of them).
- Sums of exponents which are linear in the independent variables, e.g. $\exp(px + \Omega t)$.

If we solve $P(D_x, D_t)F \cdot F = 0$, where P is even, then the class of solutions of the first equation $P((\partial/\partial x), (\partial/\partial t))f_1 = 0$ encompasses f_1 in a form of a polynomial of the same degree in x, t as a degree of P. Moreover, the polynomial solution for f_1 has a sufficient number of free parameters to satisfy the second equation with $f_2 = 0$ (and consequently $f_i = 0$ for all $i > 2$). Such a solution for the KdV was given above (10).

The polynomial solutions hardly ever represent a physical reality. Therefore, they are not very useful for applications.

The typical n-soliton solution is obtained by assuming f_1 in a form of a sum of exponents. Let

$$f_1 = \sum_{i=1}^{n} a_i \exp(p_i x + \Omega_i t) =: \sum_{i=1}^{n} \exp(\eta_i). \tag{47}$$

Then the lowest order equation (41) yields

$$P(p_i, \Omega_i) = 0 \quad \text{for } i = 1, \ldots, n, \tag{48}$$

which is a dispersion relation.

If the original equation is integrable ("in the Hirota sense") the perturbative system (42) closes in the nth order of the expansion due to the property (18h). However the closure is not automatic: the necessary condition is vanishing of the r.h.s in the equation for f_{n+1}

$$\sum_{\sigma_1=\pm 1} \cdots \sum_{\sigma_n=\pm 1} P\left(\sum_{i=1}^{n} \sigma_i \partial_i, \sum_{i=1}^{n} \sigma_i \Omega_i\right)$$
$$\times \prod_{i>j}^{m} P\left(\sigma_i \partial_i - \sigma_j \partial_j, \sigma_i \Omega_i - \sigma_j \Omega_j\right) \sigma_i \sigma_j = 0 \tag{49}$$

for all $m = 1, 2, \ldots, n$.

This condition is always satisfied for $n = 2$ (two-soliton solution). It is the 3-soliton condition, which distinguishes between "Hirota integrable" (i.e. solvable with the Hirota method) and nonintegrable in the Hirota sense [25]. Hence, if an equation can be cast into the bilinear form $P(D_x, D_t)F \cdot F = 0$, at least its two-soliton solution exists. However, other bilinear forms of the equation do not ensure the existence of such solutions.

On the other hand the existence of two-soliton solutions may sometimes be excluded by means of the singularity analysis. Some partially integrable equations like the Zakharov equations or Maxwell–Bloch equations have special solutions which may be expanded into the Laurent series about a movable singularity in the (x, t) plane iff the singularity consists merely of straight lines [26‡, 27]. The rectilinearity condition means that $\xi_{tt} = 0$ for any solution $x = \xi(t)$ of the equation $F(x, t) = 0$, where $F(x, t)$ is the expansion variable in the Laurent series (19). The rigid shape of the trajectory of the singularity is incompatible with the picture of soliton collision: interacting solitons slow down or accelerate. This results in a curvature of the trajectories $\xi(t)$ of any fixed point in the soliton, including its singularities in the complex plane. Therefore, neither the Zakharov nor the Maxwell–Bloch equations can have two-soliton solutions.

6. Extensions of the Hirota method

6.1. Multilinear operators

One of the properties of the Hirota bilinear D is its "gauge invariance", i.e. commutativity with multiplication of the arguments by $\exp(px + \Omega t + \ldots)$ like in (18i). This property was proved to be crucial in [15] as it defined the bilinear operator.

The authors of [15] considered trilinear and higher multilinear operators having the same property. The trilinear operators D_{12}, D_{23}, D_{31} act on trilinear forms according to

$$D_{12}^m F \cdot G \cdot H = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^m F(x_1)G(x_2)H(x_3)|_{x_3=x_2=x_1=x} \tag{50}$$

(with the appropriate permutation of respective indices for D_{23} and D_{31}). Only two of the operators D_{12}, D_{23}, D_{31} are independent. The operators D_{ij} may be expressed in terms of a symmetric operator T and its complex conjugate

$$T = \frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + j^2\frac{\partial}{\partial x_3}, \tag{51}$$

where j is a complex cubic root of unity.

In a similar way multilinear operators may be introduced, both in their "D" form like (50) or in the symmetrized form, like (51).

‡The misleading title of this paper *The Zakharov equations: a non-Painlevé system with exact n-soliton solutions*, was based on an erroneous claim of a previous author (Ref. [8] in [26]) that he had found multisoliton solutions to the Zakharov equations.

This multilinear formalism is very useful for the integrable equations for which bilinear form is either not known or requires unnatural introduction of auxiliary functions. Examples of equations which require such a form are [15]:

- 5th order Lax equation [28]

$$u_t + 30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{xxxxx} = 0. \tag{52}$$

After the transformation $u = 2(\ln F)_{xx}$ it may be written as

$$T_x(27T_t + 20T_x^2 T_x^{*3} + 7T_x^5)F \cdot F \cdot F = 0. \tag{53}$$

- The Monge–Ampère equation

$$w_{xy}^2 - w_{xx}w_{yy} = 0. \tag{54}$$

Through $w = \ln F$ equivalent to [15]

$$(T_x T_x^* T_y T_y^* - T_x^2 T_y^{*2})F \cdot F \cdot F = 0. \tag{55}$$

Solving the trilinear or quadrilinear equations may be more complex than the procedure applied to the bilinear equations. In [29] an example of solving a trilinear system was given for the equation

$$T_x(T_x^3 + 8T_x^3 T_x^* T_z + 9T_x T_t)F \cdot F \cdot F = 0, \tag{56}$$

which is a trilinear form of a (2+1)-dimensional generalization of the KdV equation (similar to the Kadomtsev–Petviashvili equation)

$$\phi_{xt} + 4\phi_x\phi_{xz} + 2\phi_{xx}\phi_z + \phi_{xxxz} = 0, \tag{57}$$

where ϕ is a "potential" of u, i.e. $u = \phi_x$.

On the line $z = x$ the equation turns into the KdV. Details of the solution may be found in the quoted paper [29].

6.2. Periodic boundary conditions

By using Riemann's θ functions it is possible to obtain periodic counterparts to one and many-soliton solutions. The θ function is defined by

$$\theta(z|M) = \sum_{\boldsymbol{m}} \exp\left(2\pi \mathrm{i}(\tfrac{1}{2}\boldsymbol{m}^{\mathrm{T}} M \boldsymbol{m} + \boldsymbol{m}^{\mathrm{T}} z)\right) \tag{58}$$

(summation is performed over all n-element sequences of integers $\boldsymbol{m} \in \mathcal{Z}^n$, M is a $n \times n$ matrix with a positive definite imaginary part).

Riemann's θ functions, which are multidimensional generalizations of the Jacobi θ functions, replace finite combinations of exponents when we construct periodic counterparts to one-soliton and multisoliton solutions. The authors of [30] have this way constructed wave trains travelling in one direction as well as interacting two-wave-train solutions which tend to two-soliton solutions in the proper limit.

7. Conclusions

- The Hirota method is definitely a useful tool for obtaining soliton solutions.
- The Laurent (Painlevé, Kovalevskaya) expansion may be used to make the equation bilinear by determining its principal part and completing it to the "D" form.
- *(trivial but necessary)* When counting equations and unknown functions, do not forget that a complex one is worth two real!
- If an equation may be cast into a bilinear form of the type $P(D_x, D_t)F{\cdot}F = 0$, it has two-soliton solutions. They need not exist for bilinear equations of other shape.
- If the necessary condition for the existence of the Laurent expansion (19) is rectilinear shape of the singularity manifold then the equation does not have two or multisoliton solutions.
- If we cannot cast an equation into a bilinear form and we suspect that it may have soliton solutions (e.g. on the basis of the Painlevé test), we may try to make it trilinear or multilinear.
- Once we have the bilinear equations, we may also look for their periodic and multiperiodic solutions by means of the Riemann θ functions.

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