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Matter-Wave Bright Solitons of ^{7}Li Gas in an Expulsive Potential

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A Lagrangian based method is used to derive an analytical model for studying the dynamics of matter-wave bright soliton created in a harmonic potential which is attractive in the transverse direction and expulsive in the longitudinal direction. By means of sech trial functions and a Ritz optimization procedure, evolution equations are constructed for width, amplitude and nonlinear frequency chirp of the propagating soliton of the atomic condensate. Our equation for the width is an exact agreement with that of Carr and Castin, obtained by more detailed analysis. In agreement with the experiment of Paris group, the expulsive potential is found to cause the soliton to explode for $N|a_{\rm s}| = 0.98$, N being the number of atoms in the condensate and $a_{\rm s}$ — the scattering length of the atom–atom interaction.

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1. Introduction

Solitons are localized waves that can propagate over long distances with neither attenuation nor change in shape. Such waves are formed only when their dispersion is compensated by the nonlinear effects of the medium. Solitons appear in many diverse physical situations including waves in shallow water, pulse propagation in optical fibers, and plasma waves. Since the experimental observation of Bose–Einstein condensates (BEC) in 1995, it was realized that we can also have matter-wave solitons [1]. In this case, the nonlinearity is produced by binary atom–atom interactions leading to the mean field $U(\boldsymbol{r}) = 4\pi\hbar^2 a_{\rm s}|\psi|^2/m$, where $a_{\rm s}$ is the s-wave scattering length for the atom–atom scattering and m — the atomic mass. Here $\psi(\boldsymbol{r}, t)$ is the wave function or order parameter of the condensate. For $a_{\rm s} > 0$ we get dark solitons while for $a_{\rm s} < 0$ we have bright solitons. A dark soliton is a notch on the BEC with a characteristic phase step across it. On the other hand, bright soliton is a peak.

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Almost simultaneously two different groups, the one at the Ecole Normale Supériore in Paris [2] and another one at the Rice University [3], produced bright solitons from a trapped ^{7}Li BEC in the internal atomic state $|F = 1, m_F = 1\rangle$ by continuously tuning the scattering length from a positive to a negative value with the help of Feshbach resonance [4] induced by an applied magnetic field. In the Paris experiment, the spherical trapping of the BEC was adiabatically deformed into a cylindrical geometry and the condensate was finally released into a horizontal 1D waveguide such that the resulting force on the atoms can be conveniently represented by $-m\omega_z^2 z$, where ω_z is the imaginary frequency along the z direction. Understandably, this imaginary frequency is due to an offset magnetic field that produces an expulsive harmonic potential $-\frac{1}{2}m\omega_z^2 z^2$. The main effect of this expulsive term in the evolution equation is that the center of mass of the BEC accelerates along the longitudinal direction. Thus within the framework of mean-field approximation the Paris experiment can be modeled by a 3D Gross–Pitaevskii (GP) equation

$$\left[\mathrm{i}\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 - V(\boldsymbol{r}) - U(\boldsymbol{r})\right]\psi = 0 \tag{1}$$

with

$$V(\boldsymbol{r}) = \frac{1}{2}m\left[\omega_\perp^2\left(x^2+y^2\right) - \omega_z^2 z^2\right], \tag{2}$$

the confining potential. Here $\omega_\perp$ is the frequency of the radial trapping and the wave function $\psi(r,t)$ is normalized to the number of particles N in the condensate such that

$$\int |\psi|^2 \mathrm{d}\boldsymbol{r} = N. \tag{3}$$

In the case strong cylindric radial confinement it will be convenient to work with a quasi one-dimensional (Q1D) form of (1) and thereby study the dynamics of matter-wave bright solitons created in a harmonic potential which is attractive in the transverse direction but expulsive in the longitudinal direction. We shall envisage such a study with particular emphasis on the role of expulsive potential in causing explosion as observed in the Paris experiment [2].

There exists a number of detailed studies in respect of this. Remarkable among them are the works of Carr and Castin [5] and of Salasnich [6]. Our object in this work is to demonstrate how a simple analytical model could be used to physically realize the problem of soliton explosion. We shall show that the critical number of atoms in the soliton just before explosion is in exact agreement with the numbers predicted by the sophisticated works in Refs. [5] and [6].

2. Effective Q1D Gross–Pitaevskii equation

To obtain Q1D equation from (1) we rewrite this equation in terms of the dimensionless variables defined by

$$\tau = \nu t, \quad \rho = \frac{r}{a_0}, \quad s = \frac{z}{a_0}, \quad \psi(r,z,t) = u(\rho,s,\tau)/a_0^{\frac{3}{2}}$$

with $a_0 = \sqrt{\hbar/m\omega_\perp}$, the size of the ground state solution of the noninteracting GP equation. This gives

$$\mathrm{i}u_\tau + \frac{1}{2}\nabla^2 u - \frac{1}{2}\left(\rho^2 - \lambda_z^2 s^2\right) u - \frac{4\pi a_s}{a_0}|u|^2 u = 0, \quad \lambda_z = \frac{\omega_z}{\omega_\perp}. \tag{4a}$$

It is obvious that

$$\int |u|^2 \mathrm{d}^3\rho = N. \tag{4b}$$

Using a separable ansatz

$$u(\rho, s, \tau) = \phi(\rho)\xi(s, \tau), \tag{5}$$

(4a) can be written in the form

$$\frac{1}{\xi}\left(\mathrm{i}\xi_\tau + \frac{1}{2}\xi_{2s} - \frac{1}{2}\lambda_z^2 s^2 \xi\right) - \frac{4\pi a_\mathrm{s}}{a_0}|\xi|^2|\phi|^2$$

$$= \frac{1}{\phi}\left(-\frac{1}{2}\nabla_\rho^2\phi + \frac{1}{2}\rho^2\phi\right), \tag{6}$$

where ∇_ρ^2 stands for the Laplacian in the radial coordinate. In (6) the subscripts on ξ stands for partial derivative with respect to that particular independent variable. More specifically, $\xi_{2s} = \frac{\partial^2\xi}{\partial s^2}$. This equation shows that the presence of atom–atom interaction does not permit clearcut separation variables. However, the fourth term in Eq. (6) is quite small such that ϕ may be assumed to satisfy

$$-\frac{1}{2}\nabla_\rho^2\phi + \frac{1}{2}\rho^2\phi = \omega_\rho\phi \tag{7}$$

with ω_ρ related to $\omega_\perp$ by a scale factor. Equation (7) represents the well-known eigenvalue problem for the two-dimensional harmonic oscillator with the ground state solution given by

$$\phi_0(\rho) = \exp(-\rho^2/2).$$

Thus (6) can be written in the form

$$\mathrm{i}\xi_\tau + \frac{1}{2}\xi_{2s} + \frac{1}{2}\lambda_z^2 s^2\xi - \frac{4\pi a_\mathrm{s}}{a_0}|\xi|^2|\phi|^2\xi = \omega_\rho\xi. \tag{8}$$

The low-frequency vibration along the z direction is quite unlikely to excite the two-dimensional bosonic oscillator from its ground state. In view of this we multiply (8) by $\phi\phi^*$ and integrate over the ρ coordinate to get

$$\mathrm{i}\xi_\tau + \frac{1}{2}\xi_{2s} + \frac{1}{2}\lambda_z^2 s^2\xi - \frac{2\pi a_\mathrm{s}}{a_0}|\xi|^2\xi = \omega_\rho\xi. \tag{9}$$

Equation (9) can be written in a more convenient form by using the change of variable

$$\xi(s, \tau) = \chi(s, \tau)\mathrm{e}^{-\mathrm{i}\omega_\rho\tau}. \tag{10}$$

Using (10) in (9) we get

$$i\chi_\tau + \frac{1}{2}\chi_{2s} + \frac{1}{2}\lambda_z^2 s^2\chi - \frac{2\pi a_{\rm s}}{a_0}|\chi|^2\chi = 0 \tag{11a}$$

with

$$\int_{-\infty}^{+\infty} |\chi|^2 {\rm d}s = N/\pi. \tag{11b}$$

The initial-boundary value problem in (11a) can be converted to a variational problem in a rather straightforward manner.

3. Variational formulation

For example, it is easy to see that the action principle

$$\delta \int\int \mathcal{L}(\chi, \chi^*, \chi_s, \chi_s^*, \chi_\tau, \chi_\tau^*){\rm d}s{\rm d}\tau \tag{12}$$

is equivalent to (11a) with the Lagrangian density

$$\mathcal{L} = \frac{\rm i}{2}\left(\chi\chi_\tau^* - \chi^*\chi_\tau\right) - \frac{1}{2}\lambda_z^2 s^2\chi\chi^* + \frac{\pi a_{\rm s}}{a_0}\chi^2\chi^{*2} + \frac{1}{2}\chi_s^*\chi_s. \tag{13}$$

As a solution for (11a) we introduce the trial function

$$\chi(s,\tau) = A(\tau){\rm sech}\left(s/a(\tau)\right){\rm e}^{{\rm i}b(\tau)s^2}, \tag{14}$$

with a complex amplitude $A(\tau)$. Here $a(\tau)$ stands for the width of the distribution and $b(\tau)$ — the frequency chirp. In terms of trial function in (14) we obtain a reduced variational problem.

$$\delta \int \langle\mathcal{L}\rangle {\rm d}\tau = 0 \tag{15}$$

with

$$\langle\mathcal{L}\rangle = \int_{-\infty}^{+\infty} \mathcal{L}_S {\rm d}s. \tag{16}$$

Here $\mathcal{L}_S$ stands for the values of $\mathcal{L}$ when (14) is substituted in (13). We perform the integration in (16) to get

$$\mathcal{L} = {\rm i}a\left(AA_t^* - A^*A_t\right) + \frac{\pi^2}{6}AA^*b_t a^3 - \frac{\pi^2}{12}\lambda_z^2 a^3 AA^*$$

$$+\frac{4\pi a_{\rm s}}{3a_0}aA^2A^{*2} + \frac{AA^*}{3a} + \frac{\pi^2}{3}b^2a^3AA^*. \tag{17}$$

The vanishing conditions of $\delta\langle\mathcal{L}\rangle/\delta A$, $\delta\langle\mathcal{L}\rangle/\delta A^*$, $\delta\langle\mathcal{L}\rangle/\delta a$, and $\delta\langle\mathcal{L}\rangle/\delta b$ yield

$$2{\rm i}aA_t^* + {\rm i}a_tA^* + \frac{\pi^2}{6}a^3b_tA^* - \frac{\pi^2}{12}\lambda_z^2a^3A^*$$

$$+\frac{8\pi a_{\rm s}}{3a_0}aAA^{*2} + \frac{A^*}{3a} + \frac{\pi^2}{3}b^2a^3A^* = 0, \tag{18a}$$

$$2{\rm i}aA_t + ia_tA - \frac{\pi^2}{6}a^3b_tA + \frac{\pi^2}{12}\lambda_z^2a^3A$$

$$-\frac{8\pi a_{\rm s}}{3a_0}aA^*A^2-\frac{A}{3a}-\frac{\pi^2}{3}b^2a^3A=0, \tag{18b}$$

$$\mathrm{i}\,(AA_t^*-A^*A_t)+\frac{\pi^2}{12}b_ta^2AA^*-\frac{\pi^2}{4}\lambda_z^2a^2AA^*$$

$$+\frac{4\pi a_s}{3a_0}A^2A^{*2}-\frac{AA^*}{3a^2}+\pi^2b^2a^2AA^*=0, \tag{18c}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3AA^*\right)-4ba^3AA^*=0. \tag{18d}$$

Equations (18a) and (18b) can be combined to get

$$a|A|^2=\text{constant}. \tag{19}$$

From the normalization of the trial function, the constant in (19) can be identified with $N/2\pi$ such that

$$a|A|^2=\frac{N}{2\pi}. \tag{20}$$

4. Potential formulation and soliton dynamics

Equations (18a)–(18c) and (20) can be combined to write

$$a^2b_\tau-\frac{1}{2}\lambda_z^2a^2-\frac{2}{\pi^2}\frac{Na_{\rm s}}{a_0}\frac{1}{a}-\frac{2}{\pi^2a^2}+2a^2b^2=0. \tag{21}$$

From (18d) and (20) we have

$$b=\frac{1}{2a}\frac{\mathrm{d}a}{\mathrm{d}t}. \tag{22}$$

Using (22) in (21) we obtain a second-order differential equation for $a(\tau)$ given by

$$a_{2\tau}-\lambda_z^2a-\frac{4}{\pi^2}\frac{Na_{\rm s}}{a_0}\frac{1}{a^2}-\frac{4}{\pi^2a^3}=0. \tag{23}$$

Equation (23) can be integrated to get

$$\frac{1}{2}a_\tau^2+V(a)=E \tag{24}$$

with

$$V(a)=-\frac{1}{2}\lambda_z^2a^2-\frac{P}{a}+\frac{2}{\pi^2}\frac{1}{a^2}. \tag{25}$$

Here E is a constant of integration and $P=-\frac{4}{\pi^2}\frac{Na_{\rm s}}{a_0}$.

The extrema of the potential function in (25) are determined from $\frac{\mathrm{d}V(a)}{\mathrm{d}a}=0$. This gives

$$\pi^2\lambda_z^2a^4-\pi^2|P|a+4=0. \tag{26}$$

It is of interest to note that (26) is an exact agreement with Eq. (19) of Carr and Castin [5] who studied the dynamics of matter-wave bright soliton with special attention on the variation of Gross–Pitaevskii energy functional. However, as opposed to the result in Ref. [5], the width a in (26) is a function of time. This

implies that (26) holds well for any value of τ. The nature of these extrema (saddle points, minima or maxima) is determined by

$$\frac{\mathrm{d}^2V(a)}{\mathrm{d}a^2} = -\lambda_z^2 - \frac{2|P|}{a^3} + \frac{12}{\pi^2 a^4}. \tag{27}$$

In writing (26) and (27) we have introduced $P = -|P|$ to carry out the subsequent analysis by using only numerical values of a_s. We note that for bright solitons the nonlinear term is attractive and the scattering length $a_\mathrm{s} < 0$. The critical point for collapse caused by the mean field, in the case $w_z^2 < 0$, for explosion due to expulsive potential may be obtained by simultaneously solving (26) and (27) for a and $|P|$ when $\frac{\mathrm{d}^2V(a)}{\mathrm{d}a^2} = 0$. We have

$$a = \left(\frac{4}{3\pi^2\lambda_z^2}\right)^{1/4} \quad \text{and} \quad |P| = \frac{16}{3\pi^2 a}. \tag{28}$$

Equations in (28) can be combined to get

$$N|a_\mathrm{s}| = (4/3)^{3/4}\,\pi^{1/2}\lambda_z^{1/2}a_0. \tag{29}$$

Using the experimental parameters $a_0 = 1.43$ μm, $\lambda_z = 70/710$ we have $N|a_\mathrm{s}| = 0.98$ for which the expulsive potential causes the soliton to explode axially. This theoretical result is in agreement with the prediction of Khaykovich et al. [2] to within 10%. We, therefore, conclude that there are distinct advantages of viewing nonlinear dynamics of bright solitons within the framework of a Lagrangian based approach.

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