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Discrete Space-Time by Means of the Weyl–Dirac Theory

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A connection between the Weyl–Dirac theory and scale relativity theory through the hydrodynamic models (relativistic and non-relativistic approaches) is established. In such conjecture, considering that the motions of the microparticles take place on continuous but non-differentiable curves i.e. on fractals, a Weyl–Dirac type equation was found. Some correspondences with known hydrodynamic models, particularly Białynicki-Birula's approach, are analyzed. All these results reflect the fractal structure of the space-time (a concept in agreement with the new ideas on the space-time).

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1. Introduction

The way in which the geometry of space-time affects the dynamics of the particle in the Weyl–Dirac (WD) theory is given by the covariant equation [1–6]:

$$\nabla_\mu \nabla^\mu \psi - \frac{1}{6} R\psi - \frac{1}{3} \Lambda |\psi|^2 \psi = 0, \tag{1}$$

where ∇_μ is the covariant differential, R is the Ricci scalar, Λ is the cosmological constant, and ψ is the wave function associated with the particle. In fact, it is considered [1–6] "a matter shell on a cosmological background described by the field ψ which is also a source of the wave function. The law of parallel transport common to this theory requires a vector to change not only in direction but also in magnitude, after transport along a closed space-time loop. This result is given by a quantum force due to the curvature of space-time and ψ, and consequently, due to the loss of the microscopic distinguishability of the particle's trajectories. Since $|\psi|$ is taken to represent the probability density, Eq. (1) enables the quantum mechanical interpretation of the WD theory in the sense of

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Bohm [7]". Using this equation, some interesting results arise [1–6]: (i) a geometric model with conformal invariance broken in the interior space; (ii) a geometric guidance condition for the bubble at the microscopic scale; (iii) a new possibility to consider non-local effects when the interior curved space-time can have acausal properties, such as closed time-like curves; (iv) a transfer mechanism for energy-momentum between the thin shell and the Madelung fluid, etc. In Refs. [8–10] it was shown that the wave–particle duality may be associated with a phase transition (of superconducting-normal state type). Moreover, the one-dimensional and two-dimensional solutions of the WD equation in terms of the elliptic functions were obtained, and a thermodynamics of the isolated particle was developed.

On the other hand, scale relativity theory (SRT) "is a new approach to understand quantum mechanics, and moreover physical domains involving scale laws, such as chaotic systems. It is based on a generalization of Einstein's principle of relativity to scale transformations. Namely, one redefines space-time resolutions as characterizing the state of scale of reference systems, in the same way as velocity characterizes their state of motion. Then one requires that the laws of physics apply whatever the state of the reference system, of motion (principle of motion-relativity) and of scale (principle of scale-relativity). The principle of scale-relativity is mathematically achieved by the principle of scale-covariance, requiring that the equations of physics keep their simplest form under transformations of resolution" [11–15]. In such conjecture, considering that the motion of microparticles takes place on continuous but non-differentiable curves, i.e. on fractals [16], it was demonstrated that, in the topological dimension $D_{\mathrm{T}} = 2$ (for details see [16]), the geodesics of the space-time are given by a Schrödinger type equation [13–15]. Moreover, a hydrodynamic model may be developed taken into account a complex speed field [17].

In the present paper, a connection between the WD theory and SRT through the hydrodynamic models is established. Also, some correspondences with known hydrodynamic approaches are analyzed.

2. Hydrodynamic models in the Weyl–Dirac theory

The first hydrodynamic model of quantum mechanics is given by Madelung [18]. This approach has been developed and extended by Takabayasi [19–22] and others [23–25] to non-relativistic spinning particles described by the Pauli equation. A hydrodynamic model of relativistic quantum mechanics of the Dirac particle was given by Takabayasi [26]. Recently, hydrodynamic model of the WD theory has been treated in detail by Białynicki-Birula [27, 28]. This approach is similar to that given by Takabayasi for the non-relativistic Pauli equation. The hydrodynamic variables comprise one scalar field — the density — and two vector fields — the velocity and momentum. The reduction in the number of variables to four requires a quantization condition — the same as in the non-relativistic case — that relates the curl of the momentum field to an axial vector built from the velocity field.

The advantage of all hydrodynamic models of the quantum mechanics is that they "enable us to visualize quantum mechanical processes in terms of the familiar variables of classical hydrodynamics. Since the number of hydrodynamic variables always exceeds the number of variables needed to describe the wave function, it is necessary to impose an auxiliary condition on the hydrodynamic variables" [27, 28] i.e. a quantization condition.

In such a conjecture, our hydrodynamic models of WD theory differs from the one developed by Białynicki-Birula [27, 28], by introducing a complex speed field, both in non-relativistic and relativistic approach. Then, a connection with the fractal space-time can be established.

In the weak-field approach (WFA) of the WD theory, Eq. (1) takes the form

$$\Box\,\psi_0 - \frac{1}{6}R\psi_0 - \frac{1}{3}\Lambda_0|\psi_0|^2\psi_0 = 0, \tag{2}$$

where R_0, Λ_0, and ψ_0 are the new parameters in the WFA approach (for more details — the WFA method, the physical significations of R_0, Λ_0, and ψ_0 parameters, etc. see Refs. [5, 6]). With $\psi_0 = A\mathrm{e}^{(\mathrm{i}S/\hbar)}$, Eq. (2) leads to the system

$$\partial_\nu(A^2\partial_\nu S) = 0, \tag{3a}$$

$$\partial_\nu S\partial_\nu S + \frac{\hbar^2}{6}(R_0 + 2\Lambda_0 A^2) - \hbar\left(\frac{\Box\,A}{A}\right) = 0, \quad \nu = \overline{1,4} \tag{3b}$$

with A and S the amplitude and the phase of the wave function, respectively, and $\hbar$ the reduced Planck constant.

One cannot define the velocity quadri-vector u_ν through the relation

$$u_\nu = m_0^{-1}\partial_\nu S, \tag{4}$$

because the condition

$$u_\mu u_\mu = -c^2 \tag{5}$$

is not generally verified. However, introducing the proper fluctuation mass M_0 through the relation

$$\left(\frac{M_0 c}{\hbar}\right)^2 = \frac{1}{6}(R_0 + 2\Lambda_0 A^2) - \left(\frac{\Box\,A}{A}\right) \tag{6}$$

and using the quadri-vector speed,

$$u_\mu = M_0^{-1}\partial_\nu S, \tag{7}$$

the condition (5) through the Hamilton–Jacobi type Eq. (3b)

$$\partial_\nu S\partial_\nu S + M_0^2 c^2 = 0 \tag{8}$$

is always satisfied.

Now, through the quadri-vector u_ν (7) and the density

$$\rho = A^2\frac{M_0}{m_0} \tag{9}$$

from Eqs. (3a,b) a hydrodynamics in the WD theory can be built.

First Eq. (3a) with $j_\nu = \rho u_\nu$, i.e.

$$\partial_\nu j_\nu = 0 \tag{10}$$

corresponds to the law of conservation of the current density j_ν.

Equation (8) under the form

$$\partial_\nu S \partial_\nu S = M_0 u_\nu \partial_\nu S = M_0 \partial_\tau S = -M_0^2 c^2 = -E_0 M_0, \tag{11a}$$

$$E_0 = M_0 c^2, \tag{11b}$$

where E_0 is the proper fluctuation energy and τ the proper time, by applying the 4-dimensional gradient, ∂_ν, becomes

$$\partial_\tau (M_0 u_\nu) = -\partial_\nu (M_0 c^2). \tag{12}$$

From this point, through multiplication with ρ and considering the identities

$$\rho \partial_\tau (M_0 u_\nu) = \partial_\mu (\rho M_0 u_\mu u_\nu), \tag{13a}$$

$$-\rho \partial_\nu (M_0 c^2) = \partial_\mu \left[\frac{\hbar^2 \rho}{2M_0} \partial_\mu \partial_\nu \ln A - \frac{m_0 \Lambda_0 \hbar^2 \rho^2}{12 M_0^2} \delta_{\mu\nu} \right], \tag{13b}$$

we obtain the conservation law of energy-momentum tensor,

$$\partial_\mu T_{\mu\nu} = 0 \tag{14}$$

with

$$T_{\mu\nu} = \rho M_0 u_\mu u_\nu - \frac{\hbar^2 \rho}{2M_0} \partial_\mu \partial_\nu \ln A + \frac{m_0 \Lambda_0 \hbar^2 \rho^2}{12 M_0^2} \delta_{\mu\nu}. \tag{15}$$

The following conclusions can be drawn. (i) Any particle is in a permanent interaction with the "subquantum level" characterized by the ρ density (9) and the quantum potential $Q \equiv M_0 c^2$. (ii) As a result of this interaction, the particle acquires the proper fluctuation mass M_0 and the proper fluctuation energy, respectively. (iii) The "subquantum level" is identified with a quantum fluid (WD fluid) described by Eqs. (10) and (14). (iv) The correspondence with Klein–Gordon fluid is obtained for

$$\Lambda_0 = 0, \quad \left(\frac{m_0 c}{\hbar} \right)^2 = \frac{R_0}{6}. \tag{16a,b}$$

(v) Applying to the "scalar potential" [29]

$$I = -\ln \psi_0 \tag{17}$$

the 4-operator $\mathrm{i}\hbar M_0^{-1} \partial_\nu$, the 4-generalized speed field is obtained

$$V_\nu = u_\nu + \mathrm{i}\nu_\nu \tag{18}$$

with

$$\nu_\nu = -\frac{\hbar}{M_0} \partial_\nu \ln A. \tag{19}$$

Then, the energy-momentum tensor (15) takes the form

$$T_{\mu\nu} = \rho M_0 u_\mu u_\nu + \frac{\hbar \rho}{2M_0} \partial_\mu (M_0 \nu_\nu) + \frac{m_0 \Lambda_0 \hbar^2 \rho^2}{12 M_0^2} \delta_{\mu\nu}. \tag{20}$$

Therefore, the potential (17) becomes not only a source for the energy-momentum tensor through the 4-generalized speed field, but also for the wave–particle duality.

The correspondence with the non-relativistic hydrodynamic model is accomplished for $S = S' - m_0 c^2 t$, where S' is the classical action and m_0 the rest mass of the test particle. Then, using the identities,

$$\begin{aligned}
&\partial_t S = \partial_t S' - m_0 c^2, \quad \partial_{tt} S = \partial_{tt} S', \\
&\partial_\nu S \partial_\nu S = (\nabla S')^2 - c^{-2}(\partial_t S' - m_0 c^2)^2 \approx (\nabla S')^2 + 2m_0 \partial_t S' - m_0^2 c^2, \\
&\Box\, S \cong \mathit{\Delta} S', \quad \Box\, A \cong \mathit{\Delta} A, \\
&\partial_\nu (A^2 \partial_\nu S) = \nabla (A^2) \nabla S' - c^{-2}(\partial_t S' - m_0 c^2) \partial_t (A^2) + A^2 \Delta S' \\
&\qquad \approx (\nabla A^2) \nabla S' + m_0 \partial_t (A^2) + A^2 \Delta S,
\end{aligned} \tag{21a–f}$$

it follows that

$$\partial_t (m_0 A^2) + \nabla (A^2 \nabla S') = 0, \tag{22a}$$

$$2m_0 \partial_t S' + (\nabla S')^2 - m_0^2 c^2 + \frac{\hbar^2}{6}(R_0 + 2\mathit{\Lambda}_0 A^2) - \hbar^2 (\Delta A/A) = 0, \tag{22b}$$

or, through the speed fields

$$\boldsymbol{\nu} = m_0^{-1} \nabla S', \quad \boldsymbol{u} = -(\hbar/2m_0) \nabla \ln \rho, \quad \rho = A^2 \tag{23a–c}$$

and the application of ∇ operator to Eq. (22b),

$$m_0 (\partial_t \boldsymbol{\nu} + \boldsymbol{\nu} \cdot \nabla \boldsymbol{\nu}) = -\nabla (U + Q), \quad \partial_t \rho + \nabla \cdot (\rho \boldsymbol{\nu}) = 0 \tag{24a,b}$$

with

$$U = \frac{\mathit{\Lambda}_0 \hbar^2}{6m_0} \rho + U_0, \quad U_0 = \frac{1}{2m_0} \left(\frac{R_0 \hbar^2}{6} - m_0^2 c^2 \right), \tag{25a,b}$$

$$Q = -\frac{\hbar^2}{2m_0} \frac{\Delta \rho^{1/2}}{\rho^{1/2}} = -(h/2m_0) \nabla \cdot \boldsymbol{u} - (m_0 \boldsymbol{u}^2/2). \tag{25c}$$

The following conclusions can now be drawn. (i) Any particle is in a permanent interaction with the "subquantum level" through the quantum potential Q. (ii) The "subquantum level" is identified with a non-relativistic quantum fluid (non-relativistic WD fluid) described by the hydrodynamic Eqs. (24a,b). (iii) The quantum potential depends only by the imaginary part of the complex velocity,

$$\boldsymbol{V} = -\frac{\mathrm{i}\hbar}{m_0} \nabla (\ln \psi_0) = \boldsymbol{\nu} + \mathrm{i}\boldsymbol{u}. \tag{26}$$

(iv) The wave function of $\psi_0(\boldsymbol{r}, t)$ is invariant when its phase changes by an integer multiple of 2π. Indeed, Eq. (23a) gives

$$\oint m_0 \boldsymbol{\nu} \mathrm{d}\boldsymbol{r} = \oint \mathrm{d}S' = \hbar \oint \mathrm{d}s' = nh, \quad n = 1, 2, \ldots \tag{27}$$

a condition of compatibility between the WD non-relativistic hydrodynamic model and the wave mechanics (WM). This result is in agreement with the opinions of Białynicki-Birula [27, 28].

(v) In the ground states, i.e. for the quantum numbers $n = 1$, $l = m = 0$, the state density is

$$\rho(r) = \frac{1}{\pi r_0^3} \mathrm{e}^{-2r/r_0}$$

— for details see [15, 17]. Substituting this result in the quantum potential expression (25c), i.e.

$$Q(r) = -\frac{\hbar^2}{2m_0\sqrt{\rho}}\left(\frac{\mathrm{d}^2\sqrt{\rho}}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\sqrt{\rho}}{dr}\right) = -\frac{\hbar^2}{2m_0 r_0}\left(\frac{1}{r_0} - \frac{2}{r}\right),$$

we obtain the attractive force

$$F(r) = -\frac{\partial Q}{\partial r} = -\frac{\hbar^2}{m_0 r_0}\frac{1}{r^2}.$$

Consequently, the "subquantum level" becomes a force field source of an attractive type.

3. A connection between the Weyl–Dirac theory and scale relativity theory

The complex speed fields given through the relativistic and the non-relativistic hydrodynamic approach (see the relations (18) and (26)) indicate (in our opinion), according to [11–14], a possible connection between the WD theory and SRT. Then, the motions of the microparticles take place on continuous but non-differentiable curves, i.e. on fractals. The consequences of this assumption will be analyzed.

Let $P(x^1, x^2)$ be a point of the fractal curve and let us consider a line which starts from this point and let Q be the first intersection of this line with the fractal curve. We denote by $x^1 + \mathrm{d}X^1$, $x^2 + \mathrm{d}X^2$ the coordinates of Q, thus $\boldsymbol{PQ}$ is a vector of components $\mathrm{d}X^1$, $\mathrm{d}X^2$. We denote by $\mathrm{d}X^i_+$ the components of the vector $\boldsymbol{PQ}$ for which $\mathrm{d}X^1 > 0$, hence they are at the right of the dotted line and by $\mathrm{d}X^i_-$ the case when $\mathrm{d}X^1 < 0$, such as is the case for the vector $\boldsymbol{PQ'}$ — see Fig. 1. Considering all the lines (segments) which start from P, we denote the average of these vectors by $\mathrm{d}x^i_\pm$, i.e.

$$\langle \mathrm{d}X^i_\pm \rangle = \mathrm{d}x^i_\pm \quad (i = 1,\ 2). \tag{28}$$

Therefore we can write

$$\mathrm{d}X^i_\pm = \mathrm{d}x^i_\pm + \mathrm{d}\xi^i_\pm, \tag{29}$$

where

$$\langle \mathrm{d}\xi^i_\pm \rangle = 0. \tag{30}$$

Details on the averaging process are given in Refs. [10, 12–14]. Here $\mathrm{d}x^i_\pm$ are the left and right differentials of the classical variables, and $\mathrm{d}\xi^i_\pm$ describe the fractal characteristics. From (29) we obtain the speed field

$$\frac{\mathrm{d}\boldsymbol{X}_\pm}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{x}_\pm}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{\xi}_\pm}{\mathrm{d}t}. \tag{31}$$

We denoted by $(\mathrm{d}\boldsymbol{x}_+/\mathrm{d}t) = \boldsymbol{\nu}_+$ the "forward" speed and by $(\mathrm{d}\boldsymbol{x}_-/\mathrm{d}t) = \nu_-$ the "backward" speed.

We have no way to favor $\boldsymbol{\nu}_+$ rather than $\boldsymbol{\nu}_-$. Both choices are equally qualified for the description of the physics laws. The only solution to this problem

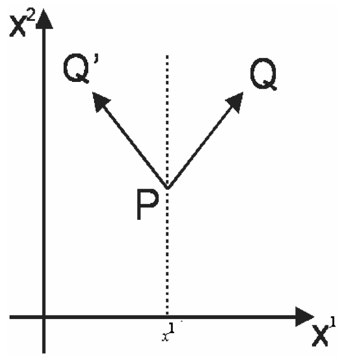

Fig. 1. The continuous curves which are not fractals but have points where they are not differentiable.

is to consider both the forward ($\mathrm{d}t > 0$) and backward ($\mathrm{d}t < 0$) processes together. Therefore, we can introduce the complex speed [10, 12–14]:

$$\boldsymbol{V} = \frac{\boldsymbol{\nu}_+ + \boldsymbol{\nu}_-}{2} - \mathrm{i}\frac{\boldsymbol{\nu}_+ - \boldsymbol{\nu}_-}{2} = \frac{\mathrm{d}\boldsymbol{x}_+ + \mathrm{d}\boldsymbol{x}_-}{2\mathrm{d}t} - \mathrm{i}\frac{\mathrm{d}\boldsymbol{x}_+ - \mathrm{d}\boldsymbol{x}_-}{2\mathrm{d}t}, \tag{32}$$

where $(\boldsymbol{\nu}_+ + \boldsymbol{\nu}_-)/2$ may be considered as classical speed, the difference between them, i.e. $(\boldsymbol{\nu}_+ - \boldsymbol{\nu}_-)/2$ is the fractal speed.

From (32) the operator results

$$\delta = \frac{\mathrm{d}_+ + \mathrm{d}_-}{2\mathrm{d}t} - \mathrm{i}\frac{\mathrm{d}_+ - \mathrm{d}_-}{2\mathrm{d}t}. \tag{33}$$

Let us suppose that the motion of microphysical objects takes places on continuous but non-differentiable curves immersed in a 3-dimensional space. Let us also consider a function $f(X, t)$ and the following Taylor series expansion up to the first order

$$\mathrm{d}f = f(X^i + \mathrm{d}X^i, t + \mathrm{d}t) - f(X^i, \mathrm{d}t) = \left(\frac{\partial}{\partial X^i}\mathrm{d}X^i + \frac{\partial}{\partial t}\mathrm{d}t\right) f(X^i, t) \tag{34}$$

with X^i ($i = \overline{1,3}$) the components of the position vector $\boldsymbol{X}$ of a point on the curve. Using the notations, $\mathrm{d}X^i_\pm = \mathrm{d}_\pm X^i$, the forward and backward average values of this relation take the form

$$\langle \mathrm{d}_\pm f\rangle = \left\langle \frac{\partial f}{\partial t}\mathrm{d}t\right\rangle + \langle \nabla f \cdot \mathrm{d}_\pm \boldsymbol{X}\rangle. \tag{35}$$

We make the following stipulations: the mean values of the function f and its derivatives coincide with themselves, and the differentials $\mathrm{d}_\pm X^i$ and $\mathrm{d}t$ are independent, therefore the averages of their products coincide with the product of average (for other details see [11–14]). Then (35) becomes

$$\mathrm{d}_\pm f = \frac{\partial f}{\partial t}\mathrm{d}t + \nabla f \langle \mathrm{d}_\pm \boldsymbol{X}\rangle \tag{36}$$

so that, using (29) in the form (28),

$$\mathrm{d}_\pm f = \frac{\partial f}{\partial t}\mathrm{d}t + \nabla f \langle \mathrm{d}_\pm \boldsymbol{x}\rangle. \tag{37}$$

In the previous works [10–14] a supplementary term in the Taylor expansion was considered. Then, the non-zero expressions, $\langle \mathrm{d}\xi_\pm^i \mathrm{d}\xi_\pm^l \rangle$ had appeared, and fractal dimension D_F (for details see [16]) must be considered. In relation (36), the zero term $\mathrm{d}\xi_\pm^i$ appears and fractal dimension D_F is not present.

If we divide by $\mathrm{d}t$, relation (36) is reduced to

$$\frac{\mathrm{d}_\pm f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \boldsymbol{\nu}_\pm \nabla f_\pm. \tag{38}$$

Now, let us calculate $(\delta f/\mathrm{d}t)$. Taking into account (33) and (38), we have

$$\begin{aligned}\frac{\delta f}{\mathrm{d}t} &= \frac{1}{2}\left[\frac{\mathrm{d}_+ f}{\mathrm{d}t} + \frac{\mathrm{d}_- f}{\mathrm{d}t} - \mathrm{i}\left(\frac{\mathrm{d}_+ f}{\mathrm{d}t} - \frac{\mathrm{d}_- f}{\mathrm{d}t}\right)\right] = \frac{1}{2}\left(\frac{\partial f}{\partial t} + \boldsymbol{\nu}_+ \nabla f\right)\\ &+\frac{1}{2}\left(\frac{\partial f}{\partial t} + \boldsymbol{\nu}_- \nabla f\right) - \frac{\mathrm{i}}{2}\left[\left(\frac{\partial f}{\partial t} + \boldsymbol{\nu}_+ \nabla f\right) - \left(\frac{\partial f}{\partial t} + \boldsymbol{\nu}_- \nabla f\right)\right]\\ &= \frac{\partial f}{\partial t} + \left(\frac{\boldsymbol{\nu}_+ + \boldsymbol{\nu}_-}{2} - \mathrm{i}\frac{\boldsymbol{\nu}_+ - \boldsymbol{\nu}_-}{2}\right)\nabla f\end{aligned} \tag{39}$$

or using (32)

$$\frac{\delta f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \boldsymbol{V}\cdot\nabla f. \tag{40}$$

This relation also allows us to give the definition of the non-differentiable operator:

$$\frac{\delta}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla. \tag{41}$$

We are applying now the principle of scale covariance, and we are postulating that the passage from classical (differentiable) mechanics to the non-differentiable mechanics can be introduced by replacing the standard time derivative $\mathrm{d}/\mathrm{d}t$ by the complex operator $\delta/\mathrm{d}t$ (for details on the method see Refs. [10–14]). As a consequence, we are able to write the Newton equation in its covariant form

$$\frac{\delta \boldsymbol{V}}{\mathrm{d}t} = \frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\nabla\boldsymbol{V} = -\left(\frac{U}{m_0}\right). \tag{42}$$

This means that the non-differentiability manifests only through the complex speed field $\boldsymbol{V}$. The fractal dimension does not appear explicitly, but implicitly only by means of the $\boldsymbol{V}$ field.

From here and using the operational relation, $\boldsymbol{V}\cdot\nabla\boldsymbol{V} = \nabla(\boldsymbol{V}^2/2) - \boldsymbol{V}\times(\nabla\times\boldsymbol{V})$, we obtain the Euler type equation in the non-differentiable space-time

$$\frac{\delta \boldsymbol{V}}{\mathrm{d}t} = \frac{\partial \boldsymbol{V}}{\partial t} + \nabla\left(\frac{\boldsymbol{V}^2}{2}\right) - \boldsymbol{V}\times(\nabla\times\boldsymbol{V}) = -\left(\frac{U}{m_0}\right). \tag{43}$$

If the microparticle motion is irrotational, i.e. $\boldsymbol{\Omega} = \nabla\times\boldsymbol{V} = 0$, we can choose $\boldsymbol{V}$ of the form

$$\boldsymbol{V} = \nabla\phi = -\mathrm{i}\frac{\hbar}{m_0}\nabla(\ln\psi_0), \tag{44}$$

— see also the relation (26), with ϕ the complex speed potential. By substituting (44) in (43) with U given by (25a,b), and integrating it, up to an arbitrary phase factor which may be set to zero by a suitable choice of the phase of ψ_0, a WD type equation in the fractal space-time is obtained

$$\begin{aligned} \mathrm{i}\hbar\partial_t\psi_0 &= -\frac{\hbar^2}{2m_0}\Delta\psi_0 + \frac{\Lambda_0\hbar^2}{6m_0}|\psi_0|^2\psi_0 \\ &+ \frac{1}{2m_0}\left[\frac{R_0\hbar^2}{6} - m_0^2c^2 + \frac{\hbar^2}{4m_0^2}\Delta(\ln\psi_0)\right]\psi_0. \end{aligned} \tag{45}$$

ψ_0 becomes simultaneously wave-function and speed potential and $\hbar/2m_0$ defines the differential–non-differential transition, i.e. the transition from the explicit scale dependence to scale independence.

4. Conclusions

The main conclusions of the paper are the following. (i) Hydrodynamic models of the WD theory in the relativistic and non-relativistic approaches are established. (ii) Any particle is in a permanent interaction with a "subquantum level" through a quantum potential. (iii) The "subquantum level" is identified with a quantum fluid described by the laws of conservation of the current density and the momentum, respectively. (iv) The quantum potential depends in the non-relativistic case only by the imaginary part of the complex speed. (iv) A connection between the WD theory and SRT through the WD hydrodynamic model in the non-relativistic approach was found. In such conjecture, considering that the motions of the microparticles take place on continuous but non-differentiable curves, i.e. on fractals, a WD type equation is obtained.

All these results reflect the fractal structure of the space-time, a concept in agreement with the new ideas on the space-time [30, 31].

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