
Wormhole Solution and Energy in Teleparallel Theory of Gravity

G.G.L. NASHED*

Mathematics Department, Faculty of Science, Ain Shams University
Cairo, Egypt

(Received December 10, 2006)

An exact solution is obtained in the tetrad theory of gravitation. This solution is characterized by two parameters k_1 , k_2 of spherically symmetric static Lorentzian wormhole which is obtained as a solution of the equation $\rho = \rho_t = 0$ with $\rho = T_{i,j}u^i u^j$, $\rho_t = (T_{ij} - \frac{1}{2}Tg_{ij})u^i u^j$, where $u^i u_i = -1$. From this solution which contains an arbitrary function we can generate the other two solutions obtained before. The associated metric of this space-time is a static Lorentzian wormhole and it includes the Schwarzschild black hole, a family of naked singularity and a disjoint family of Lorentzian wormholes. Calculating the energy content of this tetrad field and using the gravitational energy momentum given by Møller in the teleparallel space-time we find that the resulting form depends on the arbitrary function and does not depend on the two parameters k_1 and k_2 characterizing the wormhole. Using the regularized expression of the gravitational energy momentum we get the value of energy which does not depend on the arbitrary function.

PACS numbers: 04.20.Cv, 04.50.+h, 04.20.-q

1. Introduction

At present, teleparallel theory seems to be popular again, and there is a trend of analyzing the basic solutions of general relativity with teleparallel theory and comparing the results. It is considered as an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory [1–7] or metric-affine gravity [8] as well as a possible physical relevant geometry by itself-teleparallel description of gravity [9, 10]. Teleparallel approach is used for positive-gravitational-energy proof [11]. A relation between spinor Lagrangian and teleparallel theory is established [12]. In [13] it is shown that the teleparallel equivalent of general relativity (TEGR) is not consistent in the presence of minimally coupled spinning matter. The consistency of the coupling of the Dirac fields to the TEGR

*e-mail: nashed@asunet.shams.edu.eg

is demonstrated [14]. However, it is shown that this demonstration is not correct [15, 16].

For a satisfactory description of the total energy of an isolated system it is necessary that the energy density of the gravitational field is given in terms of first- and/or second-order derivatives of the gravitational field variables. It is well known that there exists no covariant, nontrivial expression constructed out of the metric tensor. However, covariant expressions that contain a quadratic form of first-order derivatives of the tetrad field are feasible. Thus it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy momentum are related to the geometrical description of the gravitational field rather than are an intrinsic drawback of the theory [17, 18]. Møller has shown that the problem of energy-momentum complex has no solution in the framework of gravitational field theories based on Riemannian space-time [19]. In a series of papers [19–22] he was able to obtain a general expression for a satisfactory energy-momentum complex in the teleparallel space-time.

It was recognized by Flamm [23] in 1916 that our universe may not be simply connected, there may exist handles or tunnels now called wormholes, in the space-time topology linking widely separated regions of our universe or even connecting us with different universes altogether. That such wormholes may be traversable by humanoid travellers was first conjectured by Morris and Thorne [24], thereby suggesting that interstellar travel and even time travel may some day be possible [25, 26].

Morris and Thorne (MT) wormholes are static and spherically symmetric and connect asymptotically flat space-times. The metric of this wormhole is given by

$$ds^2 = -\exp(2\Phi(r))dt^2 + \frac{dr^2}{1-b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $\Phi(r)$ is the redshift function and $b(r)$ is the shape function. The shape function describes the spatial shape of the wormhole when viewed. The metric (1) is spherically symmetric and static. The geometric significance of the radial coordinate r is that the circumference of a circle centered on the throat of the wormhole is given by $2\pi r$. The coordinate r is nonmonotonic in that it decreases from $+\infty$ to a minimum value b_0 , representing the location of the throat of the wormhole, and then it increases from b_0 to $+\infty$. This behavior of the radial coordinate reflects the fact that the wormhole connects two separate external universes. At the throat $r = b = b_0$, there is a coordinate singularity where the metric coefficient g_{rr} becomes divergent but the radial proper distance

$$l(r) = \pm \int_{b_0}^r \frac{dr}{\sqrt{1-b(r)/r}}, \quad (2)$$

must be required to be finite everywhere [27]. At the throat, $l(r) = 0$, while $l(r) < 0$ on the left side of the throat and $l(r) > 0$ on the right side. For a worm-

hole to be traversable it must have no horizon which implies that g_{tt} must never be allowed to vanish, i.e., $\Phi(r)$ must be finite everywhere.

Traversable Lorentzian has been in vogue ever since Morris et al. [28] came up with the exciting possibility of constructing time machine models with these exotic objects. MT paper demonstrated that the matter required to support such space-times necessarily violates the null energy condition. Semiclassical calculations based on techniques of quantum fields in curved space-time, as well as an old theorem of Epstein et al. [29], raised hopes about generation of such space-times through quantum stresses.

There have been innumerable attempts at solving the “exotic matter problem” in wormhole physics in the last few years [25, 30]. Alternative theories of gravity [31] evolving wormhole space-times [32–35] with varying definitions of the throat have been tried out as possible avenues of resolution.

It is the aim of the present work to derive a wormhole in Møller’s tetrad theory of gravitation. To do so we first begin with a tetrad having spherical symmetry with three unknown functions of the radial coordinate [36]. Applying this tetrad to the field equations of Møller’s theory we obtain a set of nonlinear partial differential equations. It is our aim to find a general solution to these differential equations and discuss its physical properties. In Sect. 2 a brief survey of Møller’s tetrad theory of gravitation is presented. The exact solution of the set of nonlinear partial differential equations is given in Sect. 3. In Sect. 4 the energy content of the tetrad field is calculated and the form of energy depending on the arbitrary function and not depending on the two parameters k_1 and k_2 that characterize the wormhole is obtained. In Sect. 5 the energy is recalculated using the regularized expression of the gravitational energy momentum. Discussion and conclusion of the obtained results are given in Sect. 6.

2. Møller’s tetrad theory of gravitation

In a space-time with absolute parallelism the parallel vector fields e_i^μ define the nonsymmetric affine connection

$$\Gamma_{\mu\nu}^\lambda \stackrel{\text{def.}}{=} e_i^\lambda e_{\mu,\nu}^i, \quad (3)$$

where $e_{\mu,\nu}^i = \partial_\nu e_i^\mu$. The curvature tensor defined by $\Gamma_{\mu\nu}^\lambda$ is identically vanishing, however.

Møller constructed a gravitational theory based on this space-time. In this theory the field variables are the 16 tetrad components e_i^μ , from which the metric tensor is derived by

$$g^{\mu\nu} \stackrel{\text{def.}}{=} \eta^{ij} e_i^\mu e_j^\nu, \quad (4)$$

where η^{ij} is the Minkowski metric $\eta_{ij} = \text{diag}(+1, -1, -1, -1)$.

We note that, associated with any tetrad field e_i^μ , there is a metric field defined uniquely by (4), while a given metric $g^{\mu\nu}$ does not determine the tetrad field completely; for any local Lorentz transformation of the tetrads b_i^μ leads to a new

set of tetrads which also satisfy (4). The Lagrangian L is an invariant constructed from $\gamma_{\mu\nu\rho}$ and $g^{\mu\nu}$, where $\gamma_{\mu\nu\rho}$ is the contorsion tensor given by

$$\gamma_{\mu\nu\rho} \stackrel{\text{def.}}{=} e_{i\ \mu} e_{i\nu;\ \rho}, \quad (5)$$

where the semicolon denotes covariant differentiation with respect to Christoffel symbols. The most general Lagrangian density invariant under the parity operation is given by the following form [22]:

$$\mathcal{L} \stackrel{\text{def.}}{=} \sqrt{-g} (\alpha_1 \Phi^\mu \Phi_\mu + \alpha_2 \gamma^{\mu\nu\rho} \gamma_{\mu\nu\rho} + \alpha_3 \gamma^{\mu\nu\rho} \gamma_{\rho\nu\mu}), \quad (6)$$

where

$$g \stackrel{\text{def.}}{=} \det(g_{\mu\nu}), \quad (7)$$

and Φ_μ is the basic vector field defined by

$$\Phi_\mu \stackrel{\text{def.}}{=} \gamma^\rho_{\ \mu\rho}. \quad (8)$$

Here α_1, α_2 , and α_3 are constants determined by Møller such that the theory coincides with general relativity in the weak fields

$$\alpha_1 = -\frac{1}{\kappa}, \quad \alpha_2 = \frac{\lambda}{\kappa}, \quad \alpha_3 = \frac{1}{\kappa}(1 - 2\lambda), \quad (9)$$

where κ is the Einstein constant and λ is a free dimensionless parameter[†]. The same choice of the parameters was also obtained by Hayashi and Nakano [37].

Møller applied the action principle to the Lagrangian density (6) and obtained the field equation in the following form:

$$G_{\mu\nu} + H_{\mu\nu} = -\kappa T_{\mu\nu}, \quad F_{\mu\nu} = 0, \quad (10)$$

where $G_{\mu\nu}$ is the Einstein tensor, $H_{\mu\nu}$ and $F_{\mu\nu}$ are given by

$$\begin{aligned} H_{\mu\nu} \stackrel{\text{def.}}{=} & \lambda [\gamma_{\rho\sigma\mu} \gamma^{\rho\sigma}_{\ \nu} + \gamma_{\rho\sigma\mu} \gamma_{\nu}^{\rho\sigma} + \gamma_{\rho\sigma\nu} \gamma_{\mu}^{\rho\sigma} \\ & + g_{\mu\nu} (\gamma_{\rho\sigma\tau} \gamma^{\tau\sigma\rho} - \frac{1}{2} \gamma_{\rho\sigma\tau} \gamma^{\rho\sigma\tau})], \end{aligned} \quad (11)$$

and

$$F_{\mu\nu} \stackrel{\text{def.}}{=} \lambda [\Phi_{\mu,\nu} - \Phi_{\nu,\mu} - \Phi_\rho (\gamma^\rho_{\ \mu\nu} - \gamma^\rho_{\ \nu\mu}) + \gamma_{\mu\nu}{}^\rho{}_{;\rho}], \quad (12)$$

and they are symmetric and skew symmetric tensors, respectively.

Møller assumed that the energy-momentum tensor of matter fields is symmetric. In the Hayashi–Nakano theory, however, the energy-momentum tensor of spin-1/2 fundamental particles has a non-vanishing antisymmetric part arising from the effects due to intrinsic spin, and the right-hand side of antisymmetric field Eq. (10) does not vanish when we take into account the possible effects of intrinsic spin.

It can be shown [9] that the tensors, $H_{\mu\nu}$ and $F_{\mu\nu}$, consist of only those terms which are linear or quadratic in the axial-vector part of the torsion tensor, a_μ , defined by

$$a_\mu \stackrel{\text{def.}}{=} \frac{1}{3} \epsilon_{\mu\nu\rho\sigma} \gamma^{\nu\rho\sigma}, \quad \text{where} \quad \epsilon_{\mu\nu\rho\sigma} \stackrel{\text{def.}}{=} \sqrt{-g} \delta_{\mu\nu\rho\sigma}, \quad (13)$$

[†]Throughout this paper we use the relativistic units, $c = G = 1$ and $\kappa = 8\pi$.

where $\delta_{\mu\nu\rho\sigma}$ is completely antisymmetric and normalized as $\delta_{0123} = -1$. Therefore, both $H_{\mu\nu}$ and $F_{\mu\nu}$ vanish if the a_μ is vanishing. In other words, when the a_μ is found to vanish from the antisymmetric part of the field equations (10), the symmetric part will coincide with the Einstein field equation in teleparallel equivalent of general relativity.

3. Spherically symmetric solutions

Let us begin with the tetrad [36]:

$$(e_l^\mu) = \begin{pmatrix} A & Dr & 0 & 0 \\ 0 & B \sin \theta \cos \phi & \frac{B}{r} \cos \theta \cos \phi & -\frac{B \sin \phi}{r \sin \theta} \\ 0 & B \sin \theta \sin \phi & \frac{B}{r} \cos \theta \sin \phi & \frac{B \cos \phi}{r \sin \theta} \\ 0 & B \cos \theta & -\frac{B}{r} \sin \theta & 0 \end{pmatrix}, \quad (14)$$

where A , D , B are functions of the radial coordinate r . The associated metric of the tetrad (14) has the form

$$ds^2 = -\frac{B^2 - D^2 r^2}{A^2 B^2} dt^2 - 2\frac{Dr}{AB^2} dr dt + \frac{1}{B^2} dr^2 + \frac{r^2}{B^2} (d\theta^2 + \sin^2 \theta d\phi^2). \quad (15)$$

As it is clear from (15) that there is a cross term which can be eliminated by performing the coordinate transformation

$$dT = dt + \frac{ADr}{B^2 - D^2 r^2} dr, \quad (16)$$

using the transformation (16) in the tetrad (14) we obtain

$$(e_l^\mu) = \begin{pmatrix} \frac{A}{1 - \mathcal{D}^2 R^2} & R\mathcal{D} - R^2 \mathcal{D}\mathcal{B}' & 0 & 0 \\ \frac{A\mathcal{D}R \sin \theta \cos \phi}{1 - \mathcal{D}^2 R^2} & (1 - R\mathcal{B}') \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{R} & -\frac{\sin \phi}{R \sin \theta} \\ \frac{A\mathcal{D}R \sin \theta \sin \phi}{1 - \mathcal{D}^2 R^2} & (1 - R\mathcal{B}') \sin \theta \sin \phi & \frac{\cos \theta \sin \phi}{R} & \frac{\cos \phi}{R \sin \theta} \\ \frac{A\mathcal{D}R \cos \theta}{1 - \mathcal{D}^2 R^2} & (1 - R\mathcal{B}') \cos \theta & -\frac{\sin \theta}{R} & 0 \end{pmatrix}, \quad (17)$$

where \mathcal{A} , \mathcal{D} , and \mathcal{B} are now unknown functions of the new radial coordinates R which is defined by

$$R = \frac{r}{B}, \quad \mathcal{B}' = \frac{d\mathcal{B}(R)}{dR}. \quad (18)$$

Applying the tetrad fields (17) to the field equations (10) we obtain the following nonlinear partial differential equations:

$$\begin{aligned} \rho(R) &= \frac{1}{\kappa} [2(R^3 \mathcal{D}^2 \mathcal{B}' - R^2 \mathcal{D}^2 - R\mathcal{B}' + 1)\mathcal{B}'' + (2R^3 \mathcal{D}\mathcal{D}' + 5R^2 \mathcal{D}^2 - 3)\mathcal{B}'^2 \\ &\quad - 2(2R^2 \mathcal{D}\mathcal{D}' + 4R\mathcal{D}^2 - \frac{2}{R})\mathcal{B}' + (2R\mathcal{D}' + 3\mathcal{D})\mathcal{D}], \\ \tau(R) &= \frac{1}{\kappa R \mathcal{A}} [(2R^3 \mathcal{A}\mathcal{D}\mathcal{D}' - 2R^3 \mathcal{D}^2 \mathcal{A}' + 3R^2 \mathcal{A}\mathcal{D}^2 + 2R\mathcal{A}' - \mathcal{A})R\mathcal{B}'^2 \\ &\quad - 2(2R^3 \mathcal{A}\mathcal{D}\mathcal{D}' - 2R^3 \mathcal{D}^2 \mathcal{A}' + 3R^2 \mathcal{D}^2 \mathcal{A} + 2R\mathcal{A}' - \mathcal{A})\mathcal{B}' + (2R^2 \mathcal{A}\mathcal{D}\mathcal{D}' \\ &\quad - 2R^2 \mathcal{D}^2 \mathcal{A}' + 3R\mathcal{A}\mathcal{D}^2 + 2\mathcal{A}'), \end{aligned}$$

$$\begin{aligned}
p(R) = \kappa T^3_3 = \frac{1}{\kappa R \mathcal{A}^2} [& (\mathcal{A}R - \mathcal{A}D^2R^5\mathcal{B}'^2 - \mathcal{A}D^2R^3 - 2\mathcal{A}R^2\mathcal{B}' + 2\mathcal{A}D^2R^4\mathcal{B}' \\
& + \mathcal{A}R^3\mathcal{B}'^2)\mathcal{A}'' + (\mathcal{A}^2R - \mathcal{A}^2R^4\mathcal{D}\mathcal{D}' + \mathcal{A}D^2R^4\mathcal{A}' - \mathcal{A}D^2R^5\mathcal{A}'\mathcal{B}' \\
& + \mathcal{A}^2\mathcal{D}R^5\mathcal{B}'\mathcal{D}' + \mathcal{A}R^3\mathcal{A}'\mathcal{B}' + 2\mathcal{A}^2\mathcal{D}^2R^4\mathcal{B}' - \mathcal{A}^2R^2\mathcal{B}' - 2\mathcal{A}^2\mathcal{D}^2R^3 \\
& - \mathcal{A}R^2\mathcal{A}')\mathcal{B}'' + (\mathcal{A}^2\mathcal{D}R^5\mathcal{B}'^2 + \mathcal{A}^2\mathcal{D}R^3 - 2\mathcal{A}^2\mathcal{D}R^4\mathcal{B}')\mathcal{D}'' + (2\mathcal{D}^2R^3 - 2R \\
& - 4\mathcal{D}^2R^4\mathcal{B}' + 4R^2\mathcal{B}')\mathcal{A}'^2 + (\mathcal{A}^2R^5\mathcal{D}'^2 + 2\mathcal{D}^2R^5\mathcal{A}'^2 - \mathcal{A}^2R + 2\mathcal{A}R^2\mathcal{A}' \\
& + 7\mathcal{A}^2\mathcal{D}R^4\mathcal{D}' - 3\mathcal{A}DR^5\mathcal{A}'\mathcal{D}' - 5\mathcal{A}D^2R^4\mathcal{A}' + 5\mathcal{A}^2\mathcal{D}^2R^3 - 2R^3\mathcal{A}'^2)\mathcal{B}'^2 \\
& + (\mathcal{A}^2R^3 - 2\mathcal{A}^2R^4\mathcal{B}')\mathcal{D}'^2 + (\mathcal{A} - 4\mathcal{A}D^2R^2)\mathcal{A}' + (9\mathcal{A}D^2R^3\mathcal{A}' - 3\mathcal{A}R\mathcal{A}' \\
& - 13\mathcal{A}^2\mathcal{D}R^3\mathcal{D}' + 6\mathcal{A}DR^4\mathcal{A}'\mathcal{D}' + \mathcal{A}^2 - 8\mathcal{A}^2\mathcal{D}^2R^2)\mathcal{B}' \\
& + (6\mathcal{A}^2\mathcal{D}R^2 - 3\mathcal{A}DR^3\mathcal{A}')\mathcal{D}' + 3\mathcal{A}^2\mathcal{D}^2R], \tag{19}
\end{aligned}$$

where $\rho(R) = T^0_0$, $\tau(R) = T^1_1$, $p(R) = T^2_2 = T^3_3$, with $\rho(R)$ being the *energy density*, $\tau(R)$ is the *radial pressure* and $p(R)$ is the *tangential pressure*. (Let us note that $\tau(R)$ as defined above is simply the radial pressure p_r , and differs by a minus sign from the conventions in [24, 25].)

Now let us try to solve the above differential Eqs. (19).

The general solution. It is our purpose to find a general solution to the differential Eqs. (19) when the stress-energy momentum tensor is not vanishing. From the first equation of (19) when $\rho(R) = 0$, we can get the unknown function $\mathcal{D}(\mathcal{R})$ in terms of the unknown functions $\mathcal{B}(\mathcal{R})$ to have the form

$$\mathcal{D}(\mathcal{R}) = \frac{1}{1 - R\mathcal{B}'} \sqrt{\frac{2m}{R^3} + \frac{\mathcal{B}'}{R}(R\mathcal{B}' - 2)}, \tag{20}$$

substituting (20) into (19) we can obtain the unknown function $\mathcal{A}(\mathcal{R})$ in terms of the unknown function $\mathcal{B}(\mathcal{R})$ to have the form

$$\mathcal{A}(\mathcal{R}) = \frac{1}{(1 - R\mathcal{B}') \left(k_2 + \frac{k_1}{\sqrt{1 - 2m/R}} \right)}. \tag{21}$$

As it is clear from (20) and (21) that the solution depends on the arbitrary function \mathcal{B} , i.e., we can generate the previous solutions obtained before by Nashed [38] by choosing the arbitrary function \mathcal{B} to have the form

$$\begin{aligned}
\mathcal{B}(\mathcal{R}) = \ln \left[R \left(R - m + R\sqrt{1 - \frac{2m}{R}} \right) \right] - 2\sqrt{1 - \frac{2m}{R}}, \\
\text{and } \mathcal{B}(\mathcal{R}) = 1. \tag{22}
\end{aligned}$$

Using (20) and (21) in (10) we can get that the components of the energy-momentum tensor turn out to have the form

$$\rho(R) = 0, \quad \tau(R) = -\frac{1}{\kappa} \left[\frac{2mk_1}{R^3 \left(k_1 + k_2\sqrt{1 - 2m/R} \right)} \right],$$

$$p(R) = \frac{1}{\kappa} \left[\frac{mk_1}{R^3 (k_1 + k_2 \sqrt{1 - 2m/R})} \right]. \quad (23)$$

The weak

$$\rho \geq 0, \quad \rho + \tau \geq 0, \quad \rho + p \geq 0, \quad (24)$$

and null energy conditions

$$\rho + \tau \geq 0, \quad \rho + p \geq 0 \quad (25)$$

are both violated as it is clear from (23). The violation of the energy condition stems from the violation of the inequality $\rho + \tau \geq 0$.

The complete line element of the above solution (20) and (21) is

$$ds^2 = -\eta_1(R)dT^2 + \frac{dR^2}{\eta_2(R)} + R^2 d\Omega^2, \quad (26)$$

$$\text{where } \eta_1(R) = \left(k_1 + k_2 \sqrt{1 - \frac{2m}{R}} \right)^2, \quad \eta_2(R) = 1 - \frac{2m}{R},$$

with $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. If one replaces k_2 by $-k_2$ in the above solution (20) and (21), the resulting form will also be a solution to the nonlinear partial differential Eqs. (19). All the above solutions have a common property that their scalar Ricci tensor vanishes identically, i.e.,

$$R(\{\}) = 0.$$

The metric (26) makes sense only for $R \geq 2m$ so to really make the wormhole explicit one needs two conditions patches

$$R_1 \in (2m, \infty), \quad R_2 \in (2m, \infty),$$

which we then have to sew together at $R = 2m$. More discussion for such wormholes can be found in [34]. We are interested in the evaluation of energy since it is the most important test for any gravitational energy expression, local or quasi-local, since the geometrical setting corresponds to an intricate configuration of the gravitational field [18].

4. Energy content

The superpotential is given by

$$\mathcal{U}_\mu{}^{\nu\lambda} = \frac{(-g)^{1/2}}{2\kappa} P_{\chi\rho\sigma}{}^{\tau\nu\lambda} [\phi^\rho g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu} \gamma^{\chi\rho\sigma} - (1 - 2\lambda) g_{\tau\mu} \gamma^{\sigma\rho\chi}], \quad (27)$$

where $P_{\chi\rho\sigma}{}^{\tau\nu\lambda}$ is

$$P_{\chi\rho\sigma}{}^{\tau\nu\lambda} \stackrel{\text{def.}}{=} \delta_\chi{}^\tau g_{\rho\sigma}{}^{\nu\lambda} + \delta_\rho{}^\tau g_{\sigma\chi}{}^{\nu\lambda} - \delta_\sigma{}^\tau g_{\chi\rho}{}^{\nu\lambda} \quad (28)$$

with $g_{\rho\sigma}{}^{\nu\lambda}$ being a tensor defined by

$$g_{\rho\sigma}{}^{\nu\lambda} \stackrel{\text{def.}}{=} \delta_\rho{}^\nu \delta_\sigma{}^\lambda - \delta_\sigma{}^\nu \delta_\rho{}^\lambda. \quad (29)$$

The energy is expressed by the surface integral [39–41]

$$E = \lim_{r \rightarrow \infty} \int_{r=\text{const}} \mathcal{U}_0^{0\alpha} n_\alpha dS, \quad (30)$$

where n_α is the unit 3-vector normal to the surface element dS .

Now we are in a position to calculate the energy associated with the solution (20) and (21) using the superpotential (27). As it is clear from (30), the only component which contributes to the energy is $\mathcal{U}_0^{0\alpha}$. Thus substituting the solution (20) and (21) into (27) we obtain the following non-vanishing value:

$$\mathcal{U}_0^{0\alpha} = \frac{2x^\alpha}{\kappa r^3} (2m - R^2 \mathcal{B}' + R^3 \mathcal{B}'^2). \quad (31)$$

Substituting (31) into (30) we get

$$E(R) = 2m - R^2 \mathcal{B}' + R^3 \mathcal{B}'^2. \quad (32)$$

We accept the formula of the energy to depend on the physical quantities but we do not accept the formula to depend on an arbitrary function [18]. Now we are going to follow a procedure similar to that followed by Brown and York [42].

5. Regularized expression for the gravitational energy momentum

An important property of the tetrad fields that satisfy the condition

$$e_{i\mu} \cong \eta_{i\mu} + \frac{1}{2} h_{i\mu} \left(\frac{1}{r} \right), \quad (33)$$

is that in the flat space-time limit $e^i{}_\mu(t, x, y, z) = \delta^i{}_\mu$, and therefore the torsion tensor defined by

$$T^\lambda{}_{\mu\nu} \stackrel{\text{def.}}{=} e_a{}^\lambda T^a{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}, \quad (34)$$

is vanishing, i.e., $T^\lambda{}_{\mu\nu} = 0$. Hence for the flat space-time it is normally to consider a set of tetrad fields such that $T^\lambda{}_{\mu\nu} = 0$ in any coordinate system. However, in general an arbitrary set of tetrad fields that yields the metric tensor for the asymptotically flat space-time does not satisfy the asymptotic condition given by (33). Moreover for such tetrad fields the torsion $T^\lambda{}_{\mu\nu} \neq 0$ for the flat space-time [18, 43, 44]. It might be argued, therefore, that the expression for the gravitational energy momentum (30) is restricted to particular class of tetrad fields, namely, to the class of frames such that $T^\lambda{}_{\mu\nu} = 0$ if $e^i{}_\mu$ represents the flat space-time tetrad field [43]. To explain this, let us calculate the flat space-time tetrad field of (14) with (20) and (21) which is given by

$$(E_i{}^\mu) = \begin{pmatrix} 1 - R\mathcal{B}' & \sqrt{R^2\mathcal{B}'^2 - 2R\mathcal{B}'} & 0 & 0 \\ \sqrt{R^2\mathcal{B}'^2 - 2R\mathcal{B}'} \sin \theta \cos \phi & (1 - R\mathcal{B}') \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{R} & \frac{-\sin \phi}{R \sin \theta} \\ \sqrt{R^2\mathcal{B}'^2 - 2R\mathcal{B}'} \sin \theta \sin \phi & (1 - R\mathcal{B}') \sin \theta \sin \phi & \frac{\cos \theta \sin \phi}{R} & \frac{\cos \phi}{R \sin \theta} \\ \sqrt{R^2\mathcal{B}'^2 - 2R\mathcal{B}'} \cos \theta & (1 - R\mathcal{B}') \cos \theta & \frac{\sin \theta}{R} & 0 \end{pmatrix}. \quad (35)$$

Expression (35) yields the following non-vanishing torsion components:

$$T_{001} = \mathcal{B}', \quad T_{112} = -r \cos(\theta) \cos \phi \mathcal{B}', \quad T_{113} = \sin(\theta) \sin \phi \mathcal{B}',$$

$$\begin{aligned}
T_{114} &= -\frac{\sin(\theta) \cos \phi \sqrt{\mathcal{B}'(1 - \mathcal{B}')}}{\sqrt{R^2 \mathcal{B}' - 2R}}, & T_{124} &= \cos \theta \cos \phi \sqrt{R^2 \mathcal{B}'^2 - 2R\mathcal{B}'}, \\
T_{134} &= -\sin(\theta) \sin \phi \sqrt{R^2 \mathcal{B}'^2 - 2R\mathcal{B}'}, & T_{212} &= -R \cos(\theta) \sin \phi \mathcal{B}', \\
T_{213} &= -R \sin(\theta) \cos \phi \mathcal{B}', & T_{214} &= -\frac{\sin \theta \sin \phi \sqrt{\mathcal{B}'(1 - \mathcal{B}')}}{\sqrt{R^2 \mathcal{B}' - 2R}}, \\
T_{224} &= \cos(\theta) \sin \phi \sqrt{R^2 \mathcal{B}'^2 - 2R\mathcal{B}'}, & T_{234} &= \sin(\theta) \cos \phi \sqrt{R^2 \mathcal{B}'^2 - 2R\mathcal{B}'}, \\
T_{312} &= R \sin(\theta) \mathcal{B}', & T_{314} &= -\frac{\cos(\theta) \sqrt{\mathcal{B}'(1 - \mathcal{B}')}}{\sqrt{R^2 \mathcal{B}' - 2R}}, \\
T_{324} &= -\sin(\theta) \sqrt{R^2 \mathcal{B}'^2 - 2R\mathcal{B}'}. & & (36)
\end{aligned}$$

Maluf et al. [18, 43, 44] discussed the above problem in the framework of TEGR and constructed a regularized expression for the gravitational energy momentum in this frame. They checked this expression for a tetrad field that suffers from the above problems and obtained very satisfactory results [43]. In this section we will follow the same procedure to derive a regularized expression for the gravitational energy momentum defined by Eq. (30). It can be shown that one can define the gravitational energy momentum contained within an arbitrary volume V of the three-dimensional space-like hypersurface in the form [30, 40]:

$$P_\mu = \int_V d^3x \partial_\alpha \mathcal{U}_\mu^{0\alpha}, \quad (37)$$

where $\mathcal{U}_\mu^{\nu\lambda}$ is given by Eq. (27). Expression (37) bears no relationship to the Arnowitt–Deser–Misner (ADM) energy momentum [44]. P_μ transforms as a vector under the global $SO(3,1)$ group.

Our assumption is that the space-time is asymptotically flat. In this case the total gravitational energy momentum is given by

$$P_\mu = \oint_{S \rightarrow \infty} dS_\alpha \mathcal{U}_\mu^{0\alpha}. \quad (38)$$

The field quantities are evaluated on a surface S in the limit $r \rightarrow \infty$.

In Eqs. (37) and (38) it is implicitly assumed that the reference space is determined by a set of tetrad fields $e^i{}_\mu$ for flat space-time such that the condition $T^\lambda{}_{\mu\nu} = 0$ is satisfied. However, in general there exist flat space-time tetrad fields for which $T^a{}_{\mu\nu} \neq 0$. In this case Eq. (37) may be generalized [43, 44] by adding a suitable reference space subtraction term, exactly like in the Brown–York formalism [42].

We will denote $T^a{}_{\mu\nu}(E) = \partial_\mu E^a{}_\nu - \partial_\nu E^a{}_\mu$ and $\mathcal{U}_\mu^{0\alpha}(E)$ as the expression of $\mathcal{U}_\mu^{0\alpha}$ constructed out of the flat tetrad $E^i{}_\mu$. The regularized form of the gravitational energy momentum P_μ is defined by

$$P_\mu = \int_V d^3x \partial_\alpha [\mathcal{U}_\mu^{0\alpha}(e) - \mathcal{U}_\mu^{0\alpha}(E)]. \quad (39)$$

This condition guarantees that the energy momentum of the flat space-time always

vanishes. The reference space-time is determined by tetrad fields $E^i{}_\mu$, obtained from $e^i{}_\mu$ by requiring the vanishing of the physical parameters like mass, angular momentum, etc. Assuming that the space-time is asymptotically flat, then Eq. (39) can have the form

$$P_\mu = \oint_{S \rightarrow \infty} dS_\alpha [\mathcal{U}_\mu{}^{0\alpha}(e) - \mathcal{U}_\mu{}^{0\alpha}(E)], \quad (40)$$

where the surface S is established at space-like infinity. Equation (40) transforms as a vector under the global SO(3,1) group [22]. Now we are in a position to prove that the tetrad field (14) with (20) and (21) yields a satisfactory value for the total gravitational energy momentum.

We will integrate Eq. (40) over a surface of constant radius $x^1 = r$ and require $r \rightarrow \infty$. Therefore, the index α in (40) takes the value $\alpha = 1$. We need to calculate the quantity $\mathcal{U}_0{}^{01}$ and we find

$$\mathcal{U}_0{}^{01}(e) \cong -\frac{1}{4\pi} R \sin(\theta) \left(\frac{2m}{R} - R\mathcal{B}' + R^2\mathcal{B}'^2 \right), \quad (41)$$

and the expression of $\mathcal{U}_0{}^{01}(E)$ is obtained by just making $m = 0$ in Eq. (41). It is given by

$$\mathcal{U}_0{}^{01}(E) \cong -\frac{1}{4\pi} R \sin(\theta) (R^2\mathcal{B}'^2 - R\mathcal{B}'). \quad (42)$$

Thus the gravitational energy contained within a surface S of constant radius r is given by

$$\begin{aligned} P_0 \cong \int_{R \rightarrow \infty} d\theta d\phi \frac{1}{4\pi} \sin(\theta) \left[-R \left(\frac{2m}{R} - R\mathcal{B}' + R^2\mathcal{B}'^2 \right) \right. \\ \left. + (R^3\mathcal{B}'^2 - R^2\mathcal{B}') \right] = 2m, \end{aligned} \quad (43)$$

this value of $2m$ is the value obtained by several different approaches [40, 41].

6. Discussion and conclusion

In this paper we have applied the tetrad having spherical symmetry with three unknown functions of the radial coordinate [37] to the field equations of Møller's tetrad theory of gravitation [22]. From the resulting partial differential equation we have obtained an exact non vacuum solution. This solution is characterized by an arbitrary function $\mathcal{B}(\mathcal{R})$ and from it one can generate the other two solutions. The solutions in general are characterized by three parameters m , k_1 , and k_2 . If the two parameters $k_1 = 0$ and $k_2 = 1$, then one can obtain the previous solutions [45]. The energy-momentum tensor has the property that $\rho = 0$. The line element associated with these solutions has the form (26).

To make the picture more clear we discuss the geometry of each solution. The line element of this solution in the isotropic form is given by

$$ds^2 = -\eta_1(R)dT^2 + \frac{dR^2}{\eta_2(R)} + R^2 d\Omega^2, \quad (44)$$

where $\eta_1(R) = \left(k_1 + k_2 \sqrt{1 - \frac{2m}{R}}\right)^2$, $\eta_2(R) = 1 - \frac{2m}{R}$. If $g_{tt} = 0$, one obtains a real naked singularity region. Outside these regions naked singularity does not appear and one obtains a traverse wormhole. The throat of this wormhole $g_{tt}(R = 2m)$ gives the conditions that $g_{tt} = -k_1^2 \Rightarrow (k_1 \neq 0$ is required to ensure the traversability). The properties of this wormhole are discussed by Dadhich et al. [34].

We calculate the energy content of the solution (20) and (21) using the energy-momentum complex given by [39, 40]. We find that the energy does not depend on the two parameters k_1 and k_2 characterizing the wormhole (32). On the contrary, it depends on the arbitrary function $\mathcal{B}(\mathcal{R})$. This is in fact not acceptable that we accept the energy to depend on the physical quantities like mass m and the charge q , etc.

Maluf et al. [18, 43, 44] have derived a simple expression for the energy-momentum flux of the gravitational field. This expression is obtained on the assumption that Eq. (37) represents the energy momentum of the gravitational field on a volume V of the three-dimensional space-like hypersurface. They [43, 44] gave this definition for the gravitational energy momentum in the framework of TEGR, which requires $T^\lambda{}_{\mu\nu}(E) = 0$ for the flat space-time. They extended this definition to the case where the flat space-time tetrad fields $E^a{}_\mu$ yield $T^\lambda{}_{\mu\nu}(E) \neq 0$. They show that [44] in the context of the regularized gravitational energy-momentum definition it is not strictly necessary to stipulate asymptotic boundary conditions for tetrad fields that describe asymptotically flat space-times.

Using the definition of the torsion tensor given by Eq. (34) and applying it to the tetrad field (35) we show that the flat space-time associated with this tetrad field has a non-vanishing torsion components Eq. (36). However, using the regularized expression of the gravitational energy-momentum Eq. (40) and calculating all the necessary components we finally get Eq. (43) which shows that the total energy of the tetrad field (14) with (20) and (21) does not depend on the arbitrary function.

As a punchline we obtain a traversable wormhole in tetrad theory of gravitation given by Møller [22] using a spherical symmetric tetrad given by Robertson [37] without using the line element given by Eq. (1) [24]. Lemos et al. [46] has studied Morris–Thorne wormholes with a cosmological constant using the tetrad form of the line element (1) in the diagonal form. Now one can do the same procedure with the non-diagonal tetrads given by the solutions (21) and (22).

Acknowledgment

The author would like to thank Professor J.G. Pereira, Universidade Estadual Paulista, Brazil.

References

- [1] Y. Itin, *Class. Quant. Grav.* **19**, 173 (2002).
- [2] Y. Itin, *J. Math. Phys.* **46**, 012501 (2005).
- [3] F.W. Hehl, Y. Neeman, J. Nitsch, P. Von der Heyde, *Phys. Lett. B* **78**, 102 (1978).
- [4] F.W. Hehl, in: *Spin, Torsion and Supergravity*, Eds. P.G. Bergmann, V. De Sabbata, Plenum, New York 1980, p. 5.
- [5] T. Kawai, *Phys. Rev. D* **49**, 2862 (1994); *Phys. Rev. D* **62**, 104014 (2000).
- [6] M. Blagojević, M. Vasilović, *Class. Quant. Grav.* **17**, 3785 (2000).
- [7] M. Blagojević, I.A. Nikolić, *Phys. Rev. D* **62**, 024021 (2000).
- [8] F.W. Hehl, J.D. MacCrea, E.W. Mielke, Y. Ne'eman, *Phys. Rep.* **258**, 1 (1995).
- [9] K. Hayashi, T. Shirafuji, *Phys. Rev. D* **19**, 3524 (1979).
- [10] J. Nitsch, F.W. Hehl, *Phys. Lett. B* **90**, 98 (1980).
- [11] E.W. Mielke, *Phys. Rev. D* **42**, 3388 (1990).
- [12] R.S. Tung, J.M. Nester, *Phys. Rev. D* **60**, 021501 (1999).
- [13] M. Leclerc, *Phys. Rev. D* **71**, 027503 (2005).
- [14] E.W. Mielke, *Phys. Rev. D* **69**, 128501 (2004).
- [15] Yu.N. Obukhov, J.G. Pereira, *Phys. Rev. D* **69**, 128502 (2004).
- [16] Yu.N. Obukhov, J.G. Pereira, *Phys. Rev. D* **67**, 044016 (2003).
- [17] J.W. Maluf, *J. Math. Phys.* **35**, 335 (1994).
- [18] J.W. Maluf, J.F. da Rocha-neto, T.M.L. Toribio, K.H. Castello-Branco, *Phys. Rev. D* **65**, 124001 (2002).
- [19] C. Møller, *Ann. Phys.* **12**, 118 (1961).
- [20] C. Møller, *Mat. Fys. Medd. Dan. Vid. Selsk.* **1**, 10 (1961).
- [21] C. Møller, *Nucl. Phys.* **57**, 330 (1964).
- [22] C. Møller, *Mat. Fys. Medd. Dan. Vid. Selsk.* **39**, 13 (1978).
- [23] L. Flamm, *Phys. Z.* **17**, 48 (1916).
- [24] M.S. Morris, K.S. Thorne, *Am. J. Phys.* **56**, 395 (1988).
- [25] M. Visser, *Lorentzian Wormholes — From Einstein to Hawking*, American Institute of Physics, New York 1996.
- [26] P.K.F. Kuhfittig, *Phys. Rev. D* **67**, 064015 (2003).
- [27] T.A. Roman, *Phys. Rev. D* **47**, 1370 (1993).
- [28] M.S. Morris, K.S. Thorne, U. Yurtsever, *Phys. Rev. Lett.* **61**, 1446 (1988).
- [29] H. Epstein, E. Glaser, A. Yaffe, *Nuovo Cimento* **36**, 2296 (1965).
- [30] M. Visser, D. Hochberg, gr-qc/9710001.
- [31] D. Hochberg, *Phys. Lett. B* **251**, 349 (1990); B. Bhawal, S. Kar, *Phys. Rev. D* **46**, 2464 (1992).
- [32] S. Kar, *Phys. Rev. D* **49**, 862 (1994); S. Kar, D. Sahdev, *Phys. Rev. D* **53**, 722 (1996).

- [33] D. Hochberg, M. Visser, *Phys. Lett.* **81**, 764 (1998); *Phys. Rev. D* **58**, 044021 (1998).
- [34] N. Dadhich, S. Kar, S. Mukherjee, M. Visser, *Phys. Rev. D* **65**, 064004 (2002).
- [35] D. Hochberg, M. Visser, gr-qc/9901020.
- [36] H.P. Robertson, *Ann. Math. (Princeton)* **33**, 496 (1932).
- [37] K. Hayashi, T. Nakano, *Prog. Theor. Phys.* **38**, 491 (1967).
- [38] G.G.L. Nashed, submitted to *Nuovo Cimento B*.
- [39] C. Møller, *Ann. Phys.* **4**, 347 (1958); **12**, 118 (1961).
- [40] F.I. Mikhail, M.I. Wanas, A. Hindawi, E.I. Lashin, *Int. J. Theor. Phys.* **32**, 1627 (1993).
- [41] T. Shirafuji, G.G.L. Nashed, K. Hayashi, *Prog. Theor. Phys.* **95**, 665 (1996).
- [42] J.D. Brown, J.W. York, Jr., in: *Proc. Joint Summer Research Conf. on Mathematical Aspects of Classical Field Theory*, in series: *Contemporary Mathematics*, Vol. 132, Eds. M.J. Gotay, J.E. Marsden, V. Moncrief, American Mathematical Society, Providence (USA) 1992, p. 129; *Phys. Rev. D* **47**, 1407 (1993).
- [43] J.W. Maluf, M.V.O. Veiga, J.F. da Rocha-neto, gr-qc/0507122.
- [44] J.W. Maluf, J.F. da Rocha-neto, *Phys. Rev. D* **64**, 084014 (2001).
- [45] G.G.L. Nashed, *Gen. Rel. Grav.* **34**, 1047 (2002).
- [46] J.P.S. Lemos, F.S.N. Lobo, S.Q. de Oliveira, *Phys. Rev. D* **68**, 064004 (2003).