

The Ground State of Ising Model $\pm J$ on the Tube

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A method of linear equations is proposed allowing a reduction of a physical problem of determination of energy, entropy, or magnetisation for the systems with frozen disorder to a mathematical problem of solving a set of linear equations. Apart from an exact method for determination of energy and entropy, a type of the mean field approach is presented, which permits a summation over series representing entropy. Moreover, it was established that magnetisation can appear only when the antiferromagnetic integrals occur at the zero probability.

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1. Introduction

Results of this paper have already been presented at the conference “Physics of Magnetism’02” [1].

From among the spin glass models only those of infinite range have proved useful for analytical studies. The information we have on spin glass has been mostly obtained from computer simulations allowing a determination of physical properties of increasingly less idealised models with increasing accuracy. The first simulations were of the Monte Carlo type, at present they are of the type called exact calculations. In these simulations performed for small systems with randomly chosen exchange integrals the following thermodynamical quantities are calculated: energy, entropy, and magnetisation. In the next step new signs of the exchange integrals are randomly chosen and the calculations are repeated. As a result of the sampling the mean values and errors of their determination are obtained, see Hartmann [2].

The main reason why the analytical calculations have been replaced by computer simulations is the lack of calculation methods for the systems with frozen disorder. Although physics has developed a wide spectrum of methods for determination of the average values of some quantities over temperature, they cannot be applied for a frozen system of exchange integrals. The reason is that for such a system there is no quantity which would reach an extreme value at the equilibrium state and would correspond to free energy or entropy for the mean taken over spin states.

The paper is directly related to the works of Derida et al. [3] and Dress et al. [4], in which the thermodynamical quantities in the ground state were determined from the recurrent relations. The description of the model (Sec. 2) is followed by presentation of the method reducing the problem of determination of energy to the task of solving a set of linear equations (Sec. 3). When compared with the hitherto available methods it is definitely the most effective. Section 4 presents entropy expressed in the form of a series, apart from a direct summation of the series, a type of the mean field approximation has been applied to obtain a compact formula for entropy. Section 5 is devoted to analysis of the possibility of appearance of magnetisation as a function of the concentration of antiferromagnetic integrals. The conclusions are given in Sec. 6.

2. Description of the model

Let us consider the Ising model described by the Hamiltonian

$$\mathcal{H} = - \sum_{i,j} J_{ij} S_i S_j, \quad (1)$$

where (ij) numbers pairs of neighbouring spins, $J_{ij} = J$ with a probability $1 - p$ and $J_{ij} = -J$ with a probability p . We will consider a square lattice wrapped around tubes of infinite lengths. If the signs of the exchange integrals are distributed at random, there are two types of squares: unfrustrated ones of an even number of antiferromagnetic bonds and frustrated ones with an odd number of antiferromagnetic bonds. When the width of the tube is equal to a single square, the vertical (orthogonal to the tube) ferromagnetic bonds can be assigned to the $|+\rangle$ state, while the vertical antiferromagnetic bonds can be assigned to the $|-\rangle$ state. The periodicity of the structure implies that the horizontal bonds in the square are equal. If we assume that the probabilities of the vertical bonds are included in the probabilities of the squares occurring on the left of such a bond, then the probability that a square is unfrustrated and all its bonds are ferromagnetic is $(1 - p)^2$. The change sign of the exchange integrals of horizontal bonds leads to the probability of $p(1 - p)$. The other unfrustrated squares correspond to the case of vertical antiferromagnetic bonds and occur with the probabilities $(1 - p)p$ and p^2 . Finally the matrix for the unfrustrated square takes the form

$$\mathbf{u} = \begin{bmatrix} (1-p)^2 + p(1-p) & 0 \\ 0 & p(1-p) + p^2 \end{bmatrix} \quad (2)$$

and for the frustrated square can be written as

$$\mathbf{f} = \begin{bmatrix} 0 & p(1-p) + p^2 \\ (1-p)^2 + p(1-p) & 0 \end{bmatrix}. \quad (3)$$

By $\mathbf{pr1}$ we will denote the probability of occurrence of the vertical bond and $\mathbf{pr1} = [(1-p) \ p]$. When we consider a square, which can be preceded on the right by any frustrated and unfrustrated square, then the state of the square on the right is described by a vector $\mathbf{Je1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The above quantities permit writing the probabilities of the selected square being unfrustrated as $\mathbf{pr1} \cdot \mathbf{u} \cdot \mathbf{Je1}$ and being frustrated as $\mathbf{pr1} \cdot \mathbf{f} \cdot \mathbf{Je1}$.

In order to describe the ground state, let us resort to the concept of the frustrating string introduced by Toulouse [5] and Kirkpatrick [6]. The string joins the centres of frustrated squares in pairs cutting the unsatisfied bonds of elevated energy. The search for the ground state energy is equivalent to the search for a frustrating string of the minimum length whose number of possible positions equals the degeneracy. The tube of a one-square width has some artificial properties because different squares can be joined by vertical bonds only and only such vertical bonds can be unsatisfied. Therefore, we shall consider the tube of a width of two squares.

3. The energy

For a tube of a two-bond width there are four matrices of probabilities of finding a pair of squares. Let us denote by \mathbf{uu} , \mathbf{uf} , \mathbf{fu} , and \mathbf{ff} the matrices of probabilities corresponding to the situations of two unfrustrated squares, the upper one unfrustrated and the lower frustrated, the upper one frustrated and the lower unfrustrated, and both frustrated. The base of the matrices is determined by the vertical exchange integrals and can be written as $[ff, fa, af, aa]$, where f and a stand for the ferromagnetic and antiferromagnetic exchange integrals. As before, the probability of the bond on the left of the square has been included in the probability of the preceding square. The probability of the occurrence of these bonds is described by the row vector \mathbf{pr} of the form

$$\mathbf{pr} = [(1-p)^2 \ p(1-p) \ p(1-p) \ p^2]. \quad (4)$$

The energy of the pair of squares depends on whether they are frustrated or not and on the state of the frustrating string reaching them let us say from the right. The state $(--)$ corresponds to the situation without the string, the states $(+-)$ and $(-+)$ correspond to the situation when the string leaves the upper and lower

square, respectively. We neglect the state $(++)$ corresponding to the occurrence of two unsatisfied bonds across the tube, but this case can always, without a change in energy, be replaced by the state whose unsatisfied bond separates the upper and lower square. The probability normalisation condition leads to

$$\mathbf{P}(--)+\mathbf{P}(+-)+\mathbf{P}(-+)=\mathbf{J}e, \quad (5)$$

where $\mathbf{J}e$ is the four-dimensional column vector of unitary terms.

When we do not know anything about the squares preceding the pair of squares considered, so when the state is described by the vector $\mathbf{J}e$, then $\mathbf{pr} \cdot \mathbf{uu} \cdot \mathbf{J}e$, $\mathbf{pr} \cdot \mathbf{uf} \cdot \mathbf{J}e$, $\mathbf{pr} \cdot \mathbf{fu} \cdot \mathbf{J}e$, and $\mathbf{pr} \cdot \mathbf{ff} \cdot \mathbf{J}e$ are the probabilities of occurrence of the following pairs of squares: both unfrustrated, upper unfrustrated and lower frustrated, upper frustrated and lower unfrustrated, and both frustrated.

The choice of the states permits an introduction of the Markov chain. The state $(--)$ acting on a pair of unfrustrated squares gives $(--)$, which we write $\mathbf{uu} \cdot \mathbf{P}(--)\rightarrow\mathbf{P}(--)$. From here the matrix multiplication is denoted by a dot, provided that the matrices are not given in the explicit form. The same state acting on a pair of frustrated squares gives: $\mathbf{ff} \cdot \mathbf{P}(--)\rightarrow\mathbf{P}(--)$. When one of the squares is frustrated we have $\mathbf{fu} \cdot \mathbf{P}(--)\rightarrow\mathbf{P}(+-)$ and $\mathbf{uf} \cdot \mathbf{P}(--)\rightarrow\mathbf{P}(-+)$. These relations are illustrated in Fig. 1. Writing the analogous relations for $\mathbf{P}(+-)$ and $\mathbf{P}(-+)$ we arrive at the set of equations

$$\begin{aligned} \mathbf{P}(--)&= (\mathbf{ff} + \mathbf{uu}) \cdot \mathbf{P}(--)+(\mathbf{fu} + \mathbf{uf}) \cdot \mathbf{P}(+-)+(\mathbf{uf} + \mathbf{fu}) \cdot \mathbf{P}(-+), \\ \mathbf{P}(+-)&= \mathbf{fu} \cdot \mathbf{P}(--)+\mathbf{uu} \cdot \mathbf{P}(+-)+\mathbf{ff} \cdot \mathbf{P}(-+), \\ \mathbf{P}(-+)&= \mathbf{uf} \cdot \mathbf{P}(--)+\mathbf{ff} \cdot \mathbf{P}(+-)+\mathbf{uu} \cdot \mathbf{P}(-+). \end{aligned} \quad (6)$$

The periodic conditions and the system symmetries lead to the identity of the coordinates of these vectors

$$\begin{aligned} \mathbf{P}(--)(2) &= \mathbf{P}(--)(3) = 0, & \mathbf{P}(+-)(1) &= \mathbf{P}(+-)(4) = 0, \\ \mathbf{P}(-+)(1) &= \mathbf{P}(-+)(4) = 0, & \mathbf{P}(+-)(2) &= \mathbf{P}(-+)(3), \\ \mathbf{P}(+-)(3) &= \mathbf{P}(-+)(2). \end{aligned} \quad (7)$$

Taking advantage of the normalisation condition (5), the vectors of probabilities can be written as

$$\mathbf{P}(--)=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{P}(+-)=\begin{bmatrix} 0 \\ x \\ 1-x \\ 0 \end{bmatrix}, \quad \mathbf{P}(-+)=\begin{bmatrix} 0 \\ 1-x \\ x \\ 0 \end{bmatrix}. \quad (8)$$

Putting these vectors into one of Eq. (6), we get

$$x = \frac{2p - p^2 - 4p^3 + 4p^4}{1 - 2p + 10p^2 - 16p^3 + 8p^4}. \quad (9)$$

The energy per spin can be written as

$$\begin{aligned} \frac{E}{J} = \frac{1}{2}pr \cdot \{ & [-4uu - 2uf - 2fu - 2ff] \cdot \mathbf{P}(-) \\ & + [-2uu - 2uf - 4fu - 2ff] \cdot \mathbf{P}(+) \\ & + [-2uu - 4uf - 2fu - 2ff] \cdot \mathbf{P}(-+)\}. \end{aligned} \quad (10)$$

The coefficients preceding the probability matrices are the energies of the pair of squares assuming that the energy of the vertical bonds is included to that of the squares on their right. This convention becomes necessary when we consider a complete set of the states leading to degeneration. Equation (10) can be rewritten as

$$\frac{E}{J} = \frac{-4(1 - 4p + 15p^2 - 26p^3 + 23p^4 - 12p^5 + 4p^6)}{2 - 4p + 20p^2 - 32p^3 + 16p^4} \quad (11)$$

and introducing $z = 1 - 2p$, we get Eq. (17) from the paper of Derrida et al. [3].

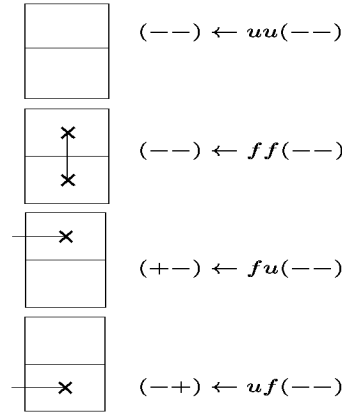


Fig. 1. The states obtained from the state $(--)$ for four types of square pairs. The frustrated squares marked by crosses are the sources of frustrated string intersecting the unsatisfied bonds.

Analogous calculations were performed for tubes of the width of 3 and 4 bonds. The number of states that must be considered increases to 8 and 9, and the number of variables increases to 6 and 13, respectively. The results are in the form of fractions, whose numerator and denominator are polynomials of the 22nd and 19th degree, and 38th and 34th degree for the tube width of 3 and 4 squares, respectively. The form of the results in detail is given in Appendix A. The expression for the energy of a tube of three bonds in width has been given by Derrida and Vanniemenus and by putting $p = (1 - z)/2$, the equation for ϵ_3/J from Appendix A is transformed into Eq. (12) from Ref. [7]. Figure 1 in [1] presents the energy per site as a function of the concentration of antiferromagnetic integrals for three different widths of the tube. For a tube of 3 squares in width (the broken line) the energy ceases to be invariant with the replacement $p \leftrightarrow (1 - p)$, as the vertical bonds form triangles breaking this symmetry.

4. Entropy

When considering entropy we cannot restrict our analysis to the states $(--)$, $(+-)$, $(-+)$, but we have to take into account all possible locations of the frustrating string. Thus, we have to consider the following four probabilities: $\mathbf{P}_1(--)$, $\mathbf{P}_2(+--)(-+, +\mathbf{2})$, $\mathbf{P}_3(-+)(+-, +\mathbf{2})$, $\mathbf{P}_4(--)(++, +\mathbf{2})$. Apart from $\mathbf{P}_1(--)$, these are the probabilities of alternative states. $\mathbf{P}_2(+--)(-+, +\mathbf{2})$ is the probability of the state with the frustrating string leaving the upper square or the lower square, but the second possibility corresponds to the energy by $2J$ higher than the first one. For $\mathbf{P}_3(-+)(+-, +\mathbf{2})$ the sequence of the possibilities because of the string location has been changed. $\mathbf{P}_4(--)(++, +\mathbf{2})$ corresponds to the state when the frustrating string does not leave the squares or when there are two strings leaving the upper and the lower square, the probability of the latter case is by $2J$ greater. Similarly as above, we write a set of equations determining the probabilities of these four states

$$\begin{aligned}
 \mathbf{P}_1 &= uu \cdot \mathbf{P}_1 + fu \cdot \mathbf{P}_2 + uf \cdot \mathbf{P}_3 + uu \cdot \mathbf{P}_4, \\
 \mathbf{P}_2 &= fu \cdot \mathbf{P}_1 + uu \cdot \mathbf{P}_2 + ff \cdot \mathbf{P}_3 + fu \cdot \mathbf{P}_4, \\
 \mathbf{P}_3 &= uf \cdot \mathbf{P}_1 + ff \cdot \mathbf{P}_2 + uu \cdot \mathbf{P}_3 + uf \cdot \mathbf{P}_4, \\
 \mathbf{P}_4 &= ff \cdot \mathbf{P}_1 + uf \cdot \mathbf{P}_2 + fu \cdot \mathbf{P}_3 + ff \cdot \mathbf{P}_4.
 \end{aligned} \tag{12}$$

As follows from the set of equations, the states $\mathbf{P}_1(--)$, $\mathbf{P}_2(+--)(-+, +\mathbf{2})$, $\mathbf{P}_3(-+)(+-, +\mathbf{2})$ and $\mathbf{P}_4(--)(++, +\mathbf{2})$ can be also defined as the states obtained from the state without the frustrating string after the transformation of the following pairs of squares: uu , fu , uf , and ff . The set of Eqs. (12) is equivalent to the matrix of probabilities of transitions between the pair of states, denoted by \mathbf{x} :

$$\mathbf{x} = \begin{bmatrix} uu & fu & uf & uu \\ fu & uu & ff & fu \\ uf & ff & uu & uf \\ ff & uf & fu & ff \end{bmatrix}. \tag{13}$$

Now, let us define the degeneracy related to the transitions between two states joined by a pair of squares. As the degeneracy depends on which of the alternative states are realised, only when the initial and final state is the state 1, the degeneracy is a number, and in the other cases it will be expressed as a matrix. For example

$$\begin{aligned}
 \langle \mathbf{P}_1(--)|\mathbf{y}|\mathbf{P}_1(--)\rangle &= \begin{bmatrix} 1 \end{bmatrix}, \\
 \langle \mathbf{P}_3(-+)(+-, +\mathbf{2})|\mathbf{y}|\mathbf{P}_1(--)\rangle &= \begin{bmatrix} 1 \\ 2 \end{bmatrix},
 \end{aligned}$$

$$\langle \mathbf{P}_2(+ -)(- +, + \mathbf{2}) | \mathbf{y} | \mathbf{P}_2(+ -)(- +, + \mathbf{2}) \rangle = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

The locations of the frustrating string corresponding to these three cases are shown in Fig. 2. In the same way we obtain the matrix of degeneracy denoted by \mathbf{y} :

$$\mathbf{y} = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}. \quad (14)$$

The system is divided into blocks, defined as the smallest elements bringing additive contributions to entropy. Thus, a block starts and ends with the state 1. The entropy per spin is obtained by multiplying the block entropy by the probability of its occurrence, and summing up the result over all blocks. Thus, we can write

$$\begin{aligned} \frac{S}{k} &= \frac{1}{2} \sum_{n=1}^{\infty} \sum_{i_1=2}^4 \dots \sum_{i_n=2}^4 \mathbf{p} \mathbf{r} \cdot \langle 1 | \mathbf{x} | i_1 \rangle \cdot \langle i_1 | \mathbf{x} | i_2 \rangle \dots \\ &\quad \langle i_n | \mathbf{x} | 1 \rangle \cdot \mathbf{P}_1 \ln[\langle 1 | \mathbf{y} | i_1 \rangle \cdot \langle i_1 | \mathbf{y} | i_2 \rangle \dots \langle i_n | \mathbf{y} | 1 \rangle]. \end{aligned} \quad (15)$$

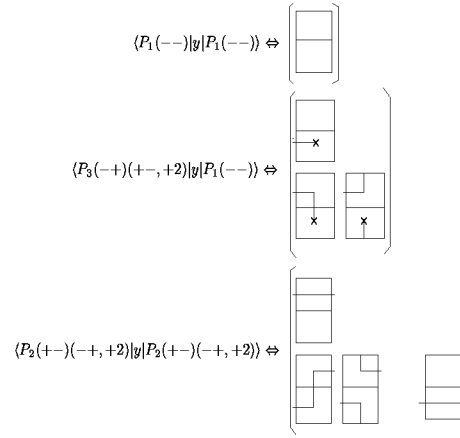


Fig. 2. Possible positions of the frustrating string for the cases described in the text.

Figure 3 presents the entropy per spin versus the concentration of antiferromagnetic integrals p , taking into account the blocks of the lengths from 10 to 18 bonds. For $p = 0.5$, for blocks up to 18 bonds long we get the entropy per site $S/k = 0.1427$. Because the blocks longer than 18 bonds still bring significant contributions to entropy, it was found that these contributions can be approximated by the quadratic equation in the logarithmic coordinates. The validity of this approximation is illustrated in Fig. 4. The contributions to the entropy as a function of the block length are marked by circles and the corresponding approximate values are marked by squares.

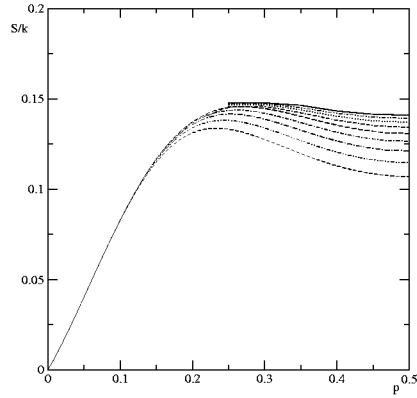


Fig. 3

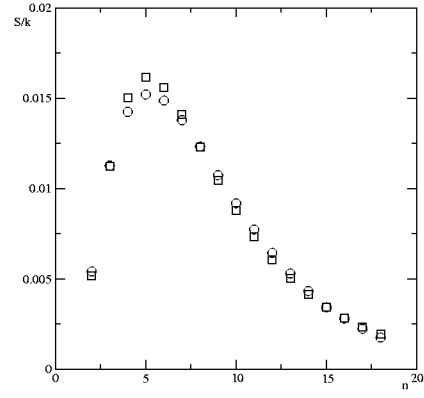


Fig. 4

Fig. 3. The entropy per spin versus the concentration of the antiferromagnetic integrals p for the blocks up to 10 bonds long — broken line, for the blocks up to 18 bonds long — solid line.

Fig. 4. The contributions coming from blocks to entropy per spin as a function of the blocks length — circles, and the corresponding approximate values — squares.

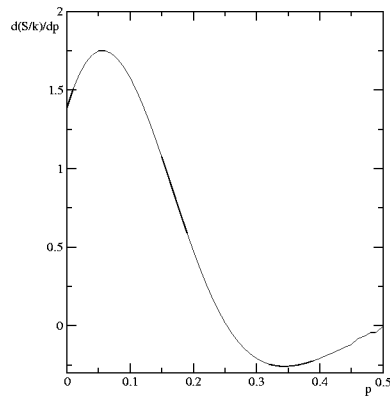


Fig. 5. The derivative of entropy per spin as a function of the concentration of the antiferromagnetic integrals p .

are marked by squares. On the basis of this approximation, the contributions from 19th to 50th and to 500th bonds were summed up, getting the entropy per spin of $0.1513k$ and $0.1515k$. Figure 5 presents the concentration p dependence of the derivative of entropy obtained on the basis of the symbolic expression derived for the 11-bond long blocks. It reaches a maximum for the concentration $p \approx 0.06$. For the concentrations higher than $p \approx 0.26$ the derivative of the entropy is negative.

Knowing that the series (15) is slowly convergent for the majority of concentrations, it would be interesting to find a probability of each of the alternative states and determine the entropy from them. Although such a procedure leads to exact expressions for any measurable quantity (e.g. energy or magnetisation), it gives only approximate expression for entropy. The measurable quantities are expressed through one- and two-spin correlation functions and their derivatives. These quantities are linear with respect to both the probability of the states of the system at a fixed distribution of admixed bonds and the probability of exchange integrals. Because of these properties, determination of the mean values of such quantities can be performed according to the procedure of determination of the mean internal energy. For a certain distribution of the exchange integrals we find the energy of a pair of squares and multiply it by the probability of occurrence of this pair of squares. The procedure is repeated for different distributions of exchange integrals and the obtained sum of the corresponding energies multiplied by the appropriate probabilities is the searched mean value. As the entropy is not a measurable quantity, the smallest element of the system for which it can be defined is not a pair of squares but a block, which in general is built of more squares and its length is not limited. We can follow another procedure, in which the entropy of a pair of spins is determined, if the pair of spins directly proceeding the above-considered pair, occurs in a certain fixed state. In this way we get the entropy value for a pair of squares for the assumed exchange integrals. This method is related to an error following from the impossibility of separation of the spin probability from the probability of exchange integrals which lead to the spin probability. Although a change in the sequence of taking the average and taking the logarithm is mathematically justified only for small logarithm arguments, this approximation is eliminated for measurable quantities, so if a measurable quantity is expressed by entropy in this approximation, it must be exact.

Now, we shall describe the way of determining the probabilities of alternative states. Let us note that the degeneration matrices introduced determine the possible degenerations, which realise after a block is defined. For instance

$$\begin{aligned} & \langle 1|y|2 \rangle \cdot \langle 2|y|2 \rangle \cdot \langle 2|y|2 \rangle \cdot \langle 2|y|1 \rangle \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1, \end{aligned} \quad (16)$$

so the state $\mathbf{P}_2(+)(-+,\mathbf{+2})$ can be always replaced by the state $(+-)$, and the excited state $(-+,\mathbf{+2})$ is never realised; thus the above product could be

replaced by

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (17)$$

After this replacement, each matrix presents the actual degeneration of a pair of sites. The situation is similar for the other alternative states. They can occur in the normal form (alternative states) or in the reduced form — when only the state of a lower energy can be realised. Consequently, instead of the four states originally considered we have seven. In order to find out which of the alternative states is realised in the system, each of the alternative states is replaced by two new ones, and thus the total number of the states increases to ten. Finally, because of the need to distinguish the spin states this number is doubled, as each of the states is replaced by two states: with the spin up and down. Further, because of the need to determine the state of the upper spin, each of the matrices of probabilities is replaced by a pair of matrices: one for the ferromagnetic upper bond and one for the antiferromagnetic upper bond. Assuming that the upper bond is satisfied, they correspond to the case of no change or a change in the upper spin respectively. For example, for a pair of unfrustrated squares we have *uun* and *uuc*. Because of the above changes the matrix of degeneration $\langle 2|\mathbf{y}|2 \rangle$ is replaced by the matrices of the probabilities of transitions between the states

$$\langle 2|\mathbf{y}_n|2 \rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \langle 2|\mathbf{y}_c|2 \rangle = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (18)$$

when the states are reduced and

$$\langle 2|\mathbf{y}_n|2 \rangle = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1 \end{bmatrix}, \quad \langle 2|\mathbf{y}_c|2 \rangle = \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1 \\ 1/3 & 1/3 & 1 & 0 \end{bmatrix}, \quad (19)$$

when the states are normal. The bases of the matrices are the states: $(+-)(\uparrow)$, $(+-)(\downarrow)$, $(-+, +2)(\uparrow)$, $(-+, +2)(\downarrow)$. Figure 6 presents the position of the frustrating string and spins of the bond distinguished for the non-zero elements of the matrices. We should define now when the states are reduced and when normal. If any of the states 2, 3, or 4 is followed by the state 4 and not by 1 occurring closer to this state on the left, then this state is normal; if state 1 is closer to the state considered then it is reduced. In order to write down the equations for the probability of states, we shall introduce the following abbreviations for these probabilities. For the normal states

$$\begin{aligned}
 P_2(+)(\uparrow) &= P21, & P_2(+)(\downarrow) &= P22, \\
 P_2(-, +2)(\uparrow) &= P23, & P_2(-, +2)(\downarrow) &= P24, \\
 P_3(-)(\uparrow) &= P31, & P_3(-)(\downarrow) &= P32, \\
 P_3(+, +2)(\uparrow) &= P33, & P_3(+, +2)(\downarrow) &= P34, \\
 P_4(-)(\uparrow) &= P41, & P_4(-)(\downarrow) &= P42, \\
 P_4(+, +2)(\uparrow) &= P43, & P_4(+, +2)(\downarrow) &= P44,
 \end{aligned}$$

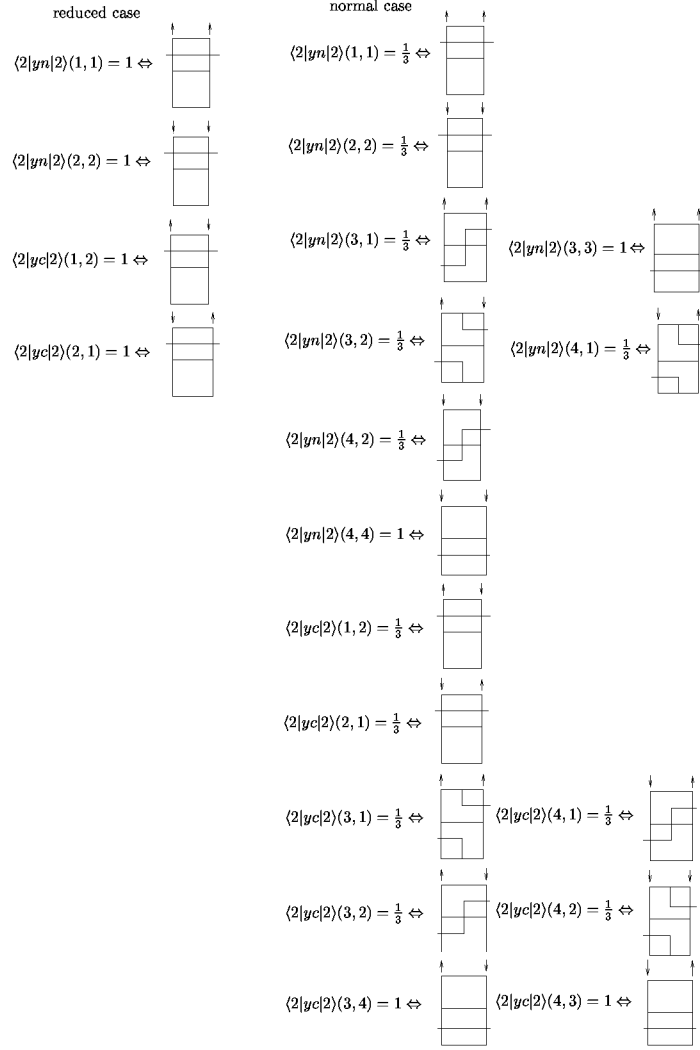


Fig. 6. The positions of the frustrating string and spins of the upper bonds for the non-zero matrix elements $\langle 2|y_n|2\rangle$, $\langle 2|y_c|2\rangle$, for the reduced and normal states.

for the reduced states

$$P_2(+)(\uparrow) = P25, \quad P_2(+)(\downarrow) = P26, \quad P_3(-)(\uparrow) = P35, \\ P_3(-)(\downarrow) = P36, \quad P_4(-)(\uparrow) = P45, \quad P_4(-)(\downarrow) = P46,$$

and for state 1, which can be either reduced or normal

$$P_1(-)(\uparrow) = P11 \quad \text{and} \quad P_1(-)(\downarrow) = P12.$$

Ten of the twenty equations (only for the spin down) for the probabilities of states are given below.

$$P11 = uun \cdot P11 + uuc \cdot P12 + fun \cdot P25 + fuc \cdot P26 + ufn \cdot P35 \\ + ufc \cdot P36 + uun \cdot P45 + uuc \cdot P46,$$

$$P21 = \frac{1}{3}fun \cdot P11 + \frac{1}{3}fuc \cdot P12 + \frac{1}{3}uun \cdot P21 + \frac{1}{3}uuc \cdot P22 + ffn \cdot P31 \\ + ffc \cdot P32 + \frac{1}{3}fun \cdot P41 + \frac{1}{3}fuc \cdot P42,$$

$$P23 = \frac{1}{3}fun \cdot P11 + \frac{1}{3}fuc \cdot P11 + \frac{1}{3}fun \cdot P12 + \frac{1}{3}fuc \cdot P12 + \frac{1}{3}uun \cdot P21 \\ + \frac{1}{3}uuc \cdot P21 + \frac{1}{3}uun \cdot P22 + \frac{1}{3}uuc \cdot P22 + uun \cdot P23 + uuc \cdot P24 \\ + ffn \cdot P33 + ffc \cdot P34 + \frac{1}{3}fun \cdot P41 + \frac{1}{3}fuc \cdot P41 + \frac{1}{3}fun \cdot P42 \\ + \frac{1}{3}fuc \cdot P42 + fun \cdot P43 + fuc \cdot P44,$$

$$P25 = fun \cdot P11 + fuc \cdot P12 + uun \cdot P25 + uuc \cdot P26 + ffn \cdot P35 \\ + ffc \cdot P36 + fun \cdot P45 + fuc \cdot P46,$$

$$P31 = \frac{1}{3}ufn \cdot P11 + \frac{1}{3}ufc \cdot P12 + ffn \cdot P21 + ffc \cdot P22 + \frac{1}{3}uun \cdot P31 \\ + \frac{1}{3}uuc \cdot P32 + \frac{1}{3}ufn \cdot P41 + \frac{1}{3}ufc \cdot P42,$$

$$P33 = \frac{1}{3}ufn \cdot P11 + \frac{1}{3}ufc \cdot P11 + \frac{1}{3}ufn \cdot P12 + \frac{1}{3}ufc \cdot P12 + ffn \cdot P23 \\ + ffc \cdot P24 + \frac{1}{3}uun \cdot P31 + \frac{1}{3}uuc \cdot P31 + \frac{1}{3}uun \cdot P32 + \frac{1}{3}uuc \cdot P32 \\ + uun \cdot P33 + uuc \cdot P34 + \frac{1}{3}ufn \cdot P41 + \frac{1}{3}ufc \cdot P41 + \frac{1}{3}ufn \cdot P42 \\ + \frac{1}{3}ufc \cdot P42 + ufn \cdot P43 + ufc \cdot P44,$$

$$P35 = ufn \cdot P11 + ufc \cdot P12 + ffn \cdot P25 + ffc \cdot P26 + uun \cdot P35 \\ + uuc \cdot P36 + ufn \cdot P45 + ufc \cdot P46,$$

$$P41 = \frac{1}{3}ffn \cdot P11 + \frac{1}{3}ffc \cdot P11 + \frac{1}{3}ffn \cdot P12 + \frac{1}{3}ffc \cdot P12 + \frac{1}{3}ufn \cdot P21 \\ + \frac{1}{3}ufc \cdot P21 + \frac{1}{3}ufn \cdot P22 + \frac{1}{3}ufc \cdot P22 + ufn \cdot P23 + ufc \cdot P24 \\ + \frac{1}{3}fun \cdot P31 + \frac{1}{3}fuc \cdot P31 + \frac{1}{3}fun \cdot P32 + \frac{1}{3}fuc \cdot P32 + fun \cdot P33 \\ + fuc \cdot P34 + \frac{1}{3}ffn \cdot P41 + \frac{1}{3}ffc \cdot P41 + \frac{1}{3}ffn \cdot P42 + \frac{1}{3}ffc \cdot P42 \\ + ffn \cdot P43 + ffc \cdot P44,$$

$$\begin{aligned}
P43 &= \frac{1}{3}ffn \cdot P11 + \frac{1}{3}ffc \cdot P12 + \frac{1}{3}ufn \cdot P21 + \frac{1}{3}ufc \cdot P22 + \frac{1}{3}fun \cdot P31 \\
&\quad + \frac{1}{3}fuc \cdot P32 + \frac{1}{3}ffn \cdot P41 + \frac{1}{3}ffc \cdot P42, \\
P45 &= \frac{1}{2}ffn \cdot P11 + \frac{1}{2}ffc \cdot P11 + \frac{1}{2}ffn \cdot P12 + \frac{1}{2}ffc \cdot P12 + \frac{1}{2}ufn \cdot P21 \\
&\quad + \frac{1}{2}ufc \cdot P21 + \frac{1}{2}ufn \cdot P22 + \frac{1}{2}ufc \cdot P22 + ufn \cdot P23 + ufc \cdot P24 \\
&\quad + \frac{1}{2}fun \cdot P31 + \frac{1}{2}fuc \cdot P31 + \frac{1}{2}fun \cdot P32 + \frac{1}{2}fuc \cdot P32 + fun \cdot P33 \\
&\quad + fuc \cdot P34 + \frac{1}{2}ffn \cdot P41 + \frac{1}{2}ffc \cdot P41 + \frac{1}{2}ffn \cdot P42 + \frac{1}{2}ffc \cdot P42 \\
&\quad + ffn \cdot P43 + ffc \cdot P44. \tag{20}
\end{aligned}$$

As a state can be normal or reduced, we have the following two normalisation conditions:

$$\begin{aligned}
P11 + P12 + P21 + P22 + P23 + P24 + P31 + P32 \\
+ P33 + P34 + P41 + P42 + P43 + P44 &= Je, \\
P11 + P12 + P25 + P26 + P35 + P36 + P45 + P46 &= Je. \tag{21}
\end{aligned}$$

In these conditions and due to the symmetries, the set (20) is reduced to a scalar system with 8 variables. The solutions to these equations are given in Appendix B, which also gives the probabilities of the states. To solve the set (20) two normalisation conditions are needed.

We would like to express this set in such a way to give the information whether the state is normal or reduced and to be able to give only one normalisation condition for all states. In order to do this we shall introduce the matrices **a41**, **a42**, **a43**, **a44** of the probabilities of transitions from the first, second, third, and fourth state to the fourth state

$$\begin{aligned}
a41 &= \langle 4|x|1 \rangle + r \cdot s \cdot t1, \\
a42 &= r \cdot s \cdot \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad a43 = r \cdot s \cdot \begin{bmatrix} 0 \\ I \end{bmatrix}, \\
a44 &= \langle 4|x|4 \rangle + r \cdot s \cdot t4, \tag{22}
\end{aligned}$$

where

$$\begin{aligned}
r &= [\langle 4|x|2 \rangle, \langle 4|x|3 \rangle] = [uf, fu], \\
s &= \sum_{k=0}^{\infty} \begin{bmatrix} \langle 2|x|2 \rangle & \langle 2|x|3 \rangle \\ \langle 3|x|2 \rangle & \langle 3|x|3 \rangle \end{bmatrix}^k = \sum_{k=0}^{\infty} \begin{bmatrix} uu & ff \\ ff & uu \end{bmatrix}^k, \\
t1 &= \begin{bmatrix} \langle 2|x|1 \rangle \\ \langle 3|x|1 \rangle \end{bmatrix} = \begin{bmatrix} fu \\ uf \end{bmatrix}, \quad t4 = \begin{bmatrix} \langle 2|x|4 \rangle \\ \langle 3|x|4 \rangle \end{bmatrix} = \begin{bmatrix} fu \\ uf \end{bmatrix}.
\end{aligned}$$

Matrices **I** and **0** have their usual meaning of the identity and zero matrix, respectively. Replacing the final state 4 by 1 we get the matrices **a11**, **a12**, **a13**, **a14**. In a similar way we define the matrices of conditional probabilities. For

example $a41ff$ is the matrix of transition probabilities from the state 1 to 4, if the state 1 is followed by the matrix ff , so $a41ff = I$. Due to so defined probabilities the set (20) can be rewritten without the need to distinguish the normal and reduced parts. For example, the eighth equation from the set (20) takes the form

$$\begin{aligned}
a44 \cdot P41 &= a44 \cdot (\frac{1}{3} a41ff \cdot ffn \cdot P11 + \frac{1}{3} a41ff \cdot ffc \cdot P11 \\
&+ \frac{1}{3} a41ff \cdot ffn \cdot P12 + \frac{1}{3} a41ff \cdot ffc \cdot P12 + \frac{1}{3} a42uf \cdot ufn \cdot P21 \\
&+ \frac{1}{3} a42uf \cdot ufc \cdot P21 + \frac{1}{3} a42uf \cdot ufn \cdot P22 + \frac{1}{3} a42uf \cdot ufc \cdot P22 \\
&+ a42uf \cdot ufn \cdot P23 + a42uf \cdot ufc \cdot P24 + \frac{1}{3} a43fu \cdot fun \cdot P31 \\
&+ \frac{1}{3} a43fu \cdot fuc \cdot P31 + \frac{1}{3} a43fu \cdot fun \cdot P32 + \frac{1}{3} a43fu \cdot fuc \cdot P32 \\
&+ a43fu \cdot fun \cdot P33 + a43fu \cdot fuc \cdot P34 + \frac{1}{3} a44ff \cdot ffn \cdot P41 \\
&+ \frac{1}{3} a44ff \cdot ffc \cdot P41 + \frac{1}{3} a44ff \cdot ffn \cdot P42 + \frac{1}{3} a44ff \cdot ffc \cdot P42 \\
&+ a44ff \cdot ffn \cdot P43 + a44ff \cdot fuc \cdot P44). \tag{23}
\end{aligned}$$

The sum of the left hand side of the equations of the modified set gives the normalisation condition of the state probabilities.

$$\begin{aligned}
pr \cdot [P11 + P12 + a42 \cdot (P21 + P22 + P23 + P24) + a12 \cdot (P25 + P26) \\
+ a43 \cdot (P31 + P32 + P33 + P34) + a13 \cdot (P35 + P36) \\
+ a44 \cdot (P41 + P42 + P43 + P44) + a14 \cdot (P45 + P46)] = 1. \tag{24}
\end{aligned}$$

The equations of this modified set permit expressing the entropy in the unperturbed but approximate form. The approximation is a consequence of a change in the sequence of taking average and taking logarithm, and as above, it is eliminated for observables.

A normal state with two ferromagnetic bonds and with the upper and bottom spin directed upwards can be either the first coordinate of the vector $P11$ and the first coordinate of the vector $P41$. The state is denoted $\uparrow\uparrow(1,4)$ where the first coordinate is the number of the state vector coordinate and the second one determines the state of the bond on the left hand side and when the latter is 4, the state is normal, while when it is 1, the state is reduced. As we wish to determine the entropy of the state after the transformation, thus, the probability that after the appearance of ffn squares the system is in the state with ferromagnetic bonds and spins directed upwards or antiferromagnetic bonds with the spins up and down, is denoted as $P(\uparrow\uparrow, 0, 0, \uparrow\downarrow, ffn \uparrow\uparrow(1,4))$. The first four coordinates of this probability are the states of the pair of spins when the exchange integrals are ff , fa , af , and aa . The fifth variable describes the type of the pair of squares and the state of the pair of spins transformed. The set of equations analogous to Eq. (23) permits writing

$$P(\uparrow\uparrow, 0, 0, \uparrow\downarrow, ffn \uparrow\uparrow (1, 4)) = pr$$

$$\cdot \left(\frac{1}{3} a_{44} \cdot a_{41} ff \cdot ffn \begin{bmatrix} \mathbf{P11}(1) \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3} a_{44} \cdot a_{41} ff \cdot ffn \begin{bmatrix} \mathbf{P41}(1) \\ 0 \\ 0 \\ 0 \end{bmatrix} \right). \quad (25)$$

Expressions for $P(\downarrow\downarrow, 0, 0, \uparrow\downarrow, ffn \uparrow\uparrow (1, 4))$ and $P(\uparrow\downarrow, 0, 0, \uparrow\downarrow, ffn \uparrow\uparrow (1, 4))$ are identical. These expressions represent all the possible transitions from the state with both spins upwards, if the ffn squares occur. Thus, the total probability is

$$P(ffn \uparrow\uparrow (1, 4)) = P(\uparrow\uparrow, 0, 0, \uparrow\downarrow, ffn \uparrow\uparrow (1, 4)) \\ + P(\downarrow\downarrow, 0, 0, \uparrow\downarrow, ffn \uparrow\uparrow (1, 4)) + P(\uparrow\downarrow, 0, 0, \uparrow\downarrow, ffn \uparrow\uparrow (1, 4)). \quad (26)$$

Now, we are able to find the conditional probability of occurrence of any of these states, e.g. that with the spins up for the ferromagnetic bonds, if any of the three states has occurred. For this exemplary state $P(\uparrow\uparrow, 0, 0, \uparrow\downarrow, ffn \uparrow\uparrow (1, 4))/P(ffn \uparrow\uparrow (1, 4))$. The contribution to entropy coming from this state can be written as

$$P(\uparrow\uparrow, 0, 0, \uparrow\downarrow, ffn \uparrow\uparrow (1, 4)) \ln[P(\uparrow\uparrow, 0, 0, \uparrow\downarrow, ffn \uparrow\uparrow (1, 4))/P(ffn \uparrow\uparrow (1, 4))]$$

and the other two contributions are found analogously. Then we also sum up the contributions for the situation when the state after the transformation is reduced. Similar calculations are performed for the other vector coordinates (signs of the exchange integrals), other directions of spins and possible matrices of probabilities of pairs of squares. Because of the approximation used for determination of entropy, only its linear term is correct, although the results are close to those obtained for blocks up to 18 bonds long: $S(0.5)/k = 0.142104$, the maximum entropy of 0.147344 occurs at $p = 0.312$, and the maximum of the entropy derivative is 0.8437 and occurs at $p = 0.0525$. The approximation is removed if we forbid summation of the probabilities of the states of the same spin state but preceded by different sequences of frustrated and unfrustrated states in the exemplary sequences made of **P11** and **P44**. However, this leads to a necessity of considering a greater number of squares. If the ban on the summation of the probabilities is realised so that the only probabilities considered are **P11** and **P12**, and so the possible blocks start and end with these states, we obtain the series described above. As follows from the above considerations, apart from the definition of blocks as the smallest elements of the system bringing additive contributions to the system entropy, they can be also defined as the smallest elements of the system with the fixed spin state at the ends and the fixed probability of the state occurrence. A comparison of the values of entropy per site obtained in this approximation and by tabulation of series (15) for blocks of up to 18 squares in length, is given in Fig. 2 in [1].

5. Magnetisation

5.1. Strictly one-dimensional case

In this section we shall review the possible methods of determination of magnetisation and give the definitions of magnetisation. In the absence of a magnetic field, the magnetisation is determined from the relation

$$\langle s \rangle^2 = \lim_{n \rightarrow \infty} \langle s_0 s_n \rangle \quad (27)$$

and because

$$\langle s_0 s_n \rangle = \text{th}(\beta J_{0,1}) \text{th}(\beta J_{1,2}) \dots \text{th}(\beta J_{n-1,n}) \quad (28)$$

and in the ground state $\text{th}(\beta J) = 1$ with the probability $1 - p$ and $\text{th}(\beta J) = -1$ with the probability p , the mean value of the spin correlations taken over the distribution of the exchange integrals $\langle s_0 s_n \rangle = (1 - 2p)^n$, which implies $\langle s \rangle = 0$ for $p \neq 0$. If the exchange integrals (except the ferromagnetic and antiferromagnetic ones) can take zero with a non-zero probability, then irrespective of the concentration of ferromagnetic bonds we get $\langle s \rangle = 0$. In the presence of a magnetic field B , when we calculate $\langle s \rangle$ at first taking the limit for $T \rightarrow 0$ and then for $B \rightarrow 0$, we get $\langle s \rangle \neq 0$, see Doman [8]. Probably for infinite systems these two definitions are equivalent.

Let us denote by $P(n, \uparrow)$ the probability of getting the total spin of the system as n and the last spin directed upwards. Then, we shall make a Markov chain by addition of another bond. This Markov chain can be written as

$$\begin{cases} P(n+1, \uparrow) \leftarrow (1-p)P(n, \uparrow) + pP(n, \downarrow), \\ P(n-1, \downarrow) \leftarrow pP(n, \uparrow) + (1-p)P(n, \downarrow), \end{cases} \quad (29)$$

where the sign of substitution \leftarrow becomes $=$, when it is the only way of getting a probability on the left hand side. For $n = 0$, Eq. (29) gives

$$\begin{cases} P(1, \uparrow) \leftarrow (1-p)P(0, \uparrow) + pP(0, \downarrow), \\ P(1, \downarrow) \leftarrow pP(0, \uparrow) + (1-p)P(0, \downarrow), \end{cases} \quad (30)$$

where the last substitution was made with the inversion of the system in order to preserve non-negative magnetisation. Equation (30) implies

$$P(1, \uparrow) = P(0, \uparrow) + P(0, \downarrow), \quad (31)$$

but $P(0, \uparrow) = 0$ as otherwise the probability with negative magnetisation must have been non-zero, so

$$P(1, \uparrow) = P(0, \downarrow). \quad (32)$$

Let us introduce the notation

$$U = \sum_{n=0}^{\infty} P(n, \uparrow), \quad V = \sum_{n=0}^{\infty} P(n, \downarrow), \quad (33)$$

then

$$U + V = 1, \quad U - V = \langle s \rangle. \quad (34)$$

Summing up (29) for $n = 1, 2, \dots$, we get

$$\begin{cases} U - P(1, \uparrow) = (1-p)U + p[V - P(0, \downarrow)], \\ V = pU + (1-p)[V - P(0, \downarrow)], \end{cases} \quad (35)$$

but since $P(1, \uparrow) = P(0, \downarrow)$, each of the equations is equivalent to

$$p\langle s \rangle = (1-p)P(0, \downarrow). \quad (36)$$

Since $P(0, \downarrow) = pP(1, \uparrow) + (1-p)P(1, \downarrow)$, thus when $p \neq 1$ we have $P(1, \uparrow) = P(1, \downarrow)$, and similarly $P(2, \uparrow) = (1-p)P(1, \uparrow) + pP(1, \downarrow)$ gives $P(2, \uparrow) = P(1, \uparrow)$.

In this way for $i \geq 1$, $p \neq 1$ we get

$$P(i, \uparrow) = P(i, \downarrow), \quad P(i+1, \uparrow) = P(i, \uparrow). \quad (37)$$

These expressions with the normalisation condition give $P(0, \downarrow) = 0$ for $p \neq 1$, so always

$$p\langle s \rangle = 0. \quad (38)$$

The same equation can be obtained as a result of averaging over the last but one spin

$$\begin{aligned} \langle s \rangle &= (1-p)U + pU - (1-p)[V - P(0, \downarrow)] + (1-p)P(0, \downarrow) - pV \\ &= U - V + 2(1-p)P(0, \downarrow) = \langle s \rangle + 2p\langle s \rangle. \end{aligned} \quad (39)$$

Investigation of a total spin of a system in such a way to guarantee that it is positive is very difficult and therefore it is more convenient to determine two-spin correlations. In order to do this we shall denote by $P_i(\uparrow)$ and $P_i(\downarrow)$ the probabilities that the spin of the i -th node is directed upwards and downwards, respectively. These probabilities are determined as

$$\begin{cases} P_{i+1}(\uparrow) = (1-p)P_i(\uparrow) + pP_i(\downarrow), \\ P_{i+1}(\downarrow) = pP_i(\uparrow) + (1-p)P_i(\downarrow). \end{cases} \quad (40)$$

The total spin of the system can be positive or negative, so despite the condition $P_i(\uparrow) + P_i(\downarrow) = 1$, the expression $P_i(\uparrow) - P_i(\downarrow)$ is not necessarily equal to $\langle s_i \rangle$. The relations (40) can be rewritten in the matrix form

$$\begin{bmatrix} P_{i+1}(\uparrow) \\ P_{i+1}(\downarrow) \end{bmatrix} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \begin{bmatrix} P_i(\uparrow) \\ P_i(\downarrow) \end{bmatrix}. \quad (41)$$

We can determine the probabilities over n bonds

$$\begin{bmatrix} P_n(\uparrow) \\ P_n(\downarrow) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + (1-2p)^n & 1 - (1-2p)^n \\ 1 - (1-2p)^n & 1 + (1-2p)^n \end{bmatrix} \begin{bmatrix} P_0(\uparrow) \\ P_0(\downarrow) \end{bmatrix}, \quad (42)$$

thus

$$\begin{aligned}
\langle s_0 s_n \rangle &= P_0(\uparrow)P_n(\uparrow) - P_0(\uparrow)P_n(\downarrow) - P_0(\downarrow)P_n(\uparrow) + P_0(\downarrow)P_n(\downarrow) \\
&= (1 - 2p)^n [P_0(\uparrow)P_0(\uparrow) - P_0(\uparrow)P_0(\downarrow) - P_0(\downarrow)P_0(\uparrow) + P_0(\downarrow)P_0(\downarrow)] \\
&= (1 - 2p)^n [P_0(\uparrow) - P_0(\downarrow)]^2.
\end{aligned} \tag{43}$$

In order for the magnetisation to be non-zero, apart from the symmetry breaking, it is necessary that the non-zero limit of the expression $(1 - 2p)^n$ existed for $n \rightarrow \infty$, which happens when $p = 0$. Another way of finding p at which spontaneous magnetisation can appear is to look for the stationary solutions of (42). Both solutions are then identical and we get

$$\frac{1}{2}[1 - (1 - 2p)^n]P(\uparrow) = \frac{1}{2}[1 - (1 - 2p)^n]P(\downarrow). \tag{44}$$

The equation is fulfilled when $P(\uparrow) = P(\downarrow)$ or when $\frac{1}{2}[1 - (1 - 2p)^n] = 0$, which happens for any n for $p = 0$. This example suggests that the operation of bond addition must not be many times repeated and the spontaneous symmetry breaking takes place when the set of Eqs. (40) has, apart from the symmetric solutions $P(\uparrow) = P(\downarrow)$, also unsymmetrical solutions $P(\uparrow) - P(\downarrow) = z \neq 0$, provided that the stationary condition is fulfilled. In the assumed notation each of Eqs. (40) is equivalent to $pz = 0$.

5.2. The tube

From among the above presented methods of determination of the concentration p at which the spontaneous magnetisation can appear, the last is the most effective and has been applied to a two-bond wide tube. In order to find the probabilities **P11**, ..., **P46**, we introduced the variables $x[i]$, $i = 1, \dots, 14$, which by the normalisation conditions were reduced to 8 independent variables. Because we have to distinguish between the cases with the spin upwards and downwards, the number of the variables is doubled. Therefore, $x[1]$ is replaced by $x[1u]$ or $x[1d]$. Similarly as in the one-dimensional case we introduce $z[1]$ satisfying the relation $x[1u] = x[1] + z[1]$, getting the variables $z[i]$, $i = 1, \dots, 14$. Expressing the probability vectors **P11**, ..., **P46** in the variables $z[i]$ and substituting them into Eq. (20) we get a uniform and scalar set of equations with 14 unknowns $z[i]$. In order for this set to have non-zero solutions (apart from zero ones) the matrix made of its coefficients should be singular. In this way we arrive at

$$\begin{aligned}
&p(1 - 2p + 6p^2 - 4p^3)^2(162 - 702p + 2934p^2 - 6873p^3 + 14010p^4 \\
&- 20258p^5 + 23520p^6 - 25071p^7 + 25596p^8 - 32424p^9 \\
&+ 35328p^{10} - 15260p^{11} - 15312p^{12} + 29040p^{13} - 21888p^{14} \\
&+ 9408p^{15} - 2304p^{16} + 256p^{17}) = 0.
\end{aligned} \tag{45}$$

In the range $p \in \langle 0, 1 \rangle$ it has only one real solution $p = 0$, therefore, only for $p = 0$, so when the antiferromagnetic integrals occur with the zero probability, the magnetisation can be non-zero. When the calculations are performed not for (20) but for its modified form (23), then the determinant of Eq. (45) is enlarged by the factor

$$\frac{p^{12}(1-p)^{12}(1-2p)^{12}}{(1-4p+16p^2-40p^3+60p^4-48p^5+16p^6)^{12}} \quad (46)$$

and the occurrence of non-zero magnetisation will be also possible for $p = 1/2$ and $p = 1$.

6. Conclusions

The states defining the position of the frustrating string were defined. The probabilities of these states together with the probabilities of the frustrated and unfrustrated squares permit determination of the ground state energy. In order to determine entropy and magnetisation, the above states were split into the normal and reduced ones, with the distinguished spin directed either up or down. The probabilities of the two types of states (corresponding to energy or entropy) can be found from a set of linear equations. On the basis of the solutions of this set, the expressions for energy and entropy can be given. The entropy can be expressed in the form of a series of blocks, defined as the smallest elements of the system bringing additive contributions to entropy. The need to express entropy in the form of a series is a consequence of its nonlinearity. Assuming the approximation in which entropy is treated as a measurable quantity, it was possible to sum up the series. Apart from determination of the energy and entropy, the probabilities of the states permitted identification of the concentration range in which magnetisation must be zero. Despite the fact that numerous computer simulations have estimated the critical value of concentration p for two-dimensional lattice as $p_c = 0.1$, our results indicate that at a finite width of the tube spontaneous magnetisation would occur only for $p = 0$. Analysis of the behaviour of the entropy derivative provides the information on the concentration of antiferromagnetic integrals at which a phase transition would occur on a two-dimensional square lattice. We get a similar result from a search for the conditions in which the relation describing the spontaneous magnetisation becomes singular, it happens in the complex plane for such concentrations p that $\text{Re} p \in (0.06, 0.17)$. As a continuation, it would be interesting to apply this method to systems with non-zero magnetic field and non-zero temperature.

Appendix A

The energy per a site for tubes three bonds e_3 or four bonds e_4 wide can be expressed as

$$\frac{e3}{J} = \frac{N(3)}{D(3)},$$

$$\begin{aligned} N(3) &= 2(3 - 18p + 129p^2 - 522p^3 + 1899p^4 - 5325p^5 + 12074p^6 \\ &\quad - 22682p^7 + 35807p^8 - 47991p^9 + 58407p^{10} - 72747p^{11} + 101616p^{12} \\ &\quad - 153434p^{13} + 221980p^{14} - 277440p^{15} + 284832p^{16} - 237376p^{17} \\ &\quad + 160512p^{18} - 86272p^{19} + 34304p^{20} - 8704p^{21} + 1024p^{22}), \\ D(3) &= 3(-1 + 4p - 34p^2 + 120p^3 - 435p^4 + 1176p^5 - 2514p^6 \\ &\quad + 4354p^7 - 6221p^8 + 7278p^9 - 7929p^{10} + 10304p^{11} - 16344p^{12} \\ &\quad + 25772p^{13} - 35296p^{14} + 38480p^{15} - 30848p^{16} + 16704p^{17} \\ &\quad - 5376p^{18} + 768p^{19}), \end{aligned}$$

$$\frac{e4}{J} = \frac{N(4)}{D(4)},$$

$$\begin{aligned} N(4) &= -2850816 + 2138112z^2 - 1250560z^4 - 1360736z^6 \\ &\quad - 750976z^8 + 45400z^{10} + 387976z^{12} + 257517z^{14} \\ &\quad - 1110466z^{16} + 193636z^{18} + 368386z^{20} - 202591z^{22} \\ &\quad - 143924z^{24} + 109416z^{26} - 7252z^{28} - 27265z^{30} \\ &\quad + 12118z^{32} - 2468z^{34} + 234z^{36} - 45z^{38}, \\ D(4) &= 64(32768 - 24576z^2 + 4864z^4 + 13216z^6 + 8704z^8 \\ &\quad - 1672z^{10} - 5080z^{12} - 2291z^{14} + 9594z^{16} - 1337z^{18} \\ &\quad - 3452z^{20} + 1438z^{22} + 1352z^{24} - 910z^{26} - 12z^{28} \\ &\quad + 245z^{30} - 98z^{32} + 15z^{34}), \\ z &= 1 - 2p. \end{aligned}$$

Appendix B

The probabilities of the states occurrence can be expressed as

$$p[11] = p[12] = \begin{bmatrix} x[1] \\ 0 \\ 0 \\ x[2] \end{bmatrix}, \quad p[21] = p[22] = \begin{bmatrix} 0 \\ x[3] \\ x[4] \\ 0 \end{bmatrix},$$

$$p[23] = p[24] = \begin{bmatrix} 0 \\ x[5] \\ x[6] \\ 0 \end{bmatrix}, \quad p[25] = p[26] = \begin{bmatrix} 0 \\ x[11] \\ x[12] \\ 0 \end{bmatrix},$$

$$p[31] = p[32] = \begin{bmatrix} 0 \\ x[4] \\ x[3] \\ 0 \end{bmatrix}, \quad p[33] = p[34] = \begin{bmatrix} 0 \\ x[6] \\ x[5] \\ 0 \end{bmatrix},$$

$$p[35] = p[36] = \begin{bmatrix} 0 \\ x[12] \\ x[11] \\ 0 \end{bmatrix}, \quad p[41] = p[42] = \begin{bmatrix} x[7] \\ 0 \\ 0 \\ x[8] \end{bmatrix},$$

$$p[43] = p[44] = \begin{bmatrix} x[9] \\ 0 \\ 0 \\ x[10] \end{bmatrix}, \quad p[45] = p[46] = \begin{bmatrix} x[13] \\ 0 \\ 0 \\ x[14] \end{bmatrix},$$

with the following normalisation conditions for the non-reduced part:

$$x[1] + x[7] + x[9] = \frac{1}{2}, \quad x[2] + x[8] + x[10] = \frac{1}{2},$$

$$x[3] + x[4] + x[5] + x[6] = \frac{1}{2},$$

while for the reduced part:

$$x[1] + x[13] = \frac{1}{2}, \quad x[2] + x[14] = \frac{1}{2}, \quad x[11] + x[12] = \frac{1}{2}.$$

Solving Eq. (20) with respect to $x[1], x[2], x[3], x[4], x[5], x[7], x[8], x[11]$ we get

$$x[1] = \frac{1}{D1}(1 - 4p + 11p^2 - 12p^3 + 4p^4),$$

$$x[2] = x[11] = \frac{p}{D2}(2 - p - 4p^2 + 4p^3),$$

$$x[3] = \frac{3p}{D3}(-6 + 8p + 4p^2 - 5p^3 - 26p^4 + 46p^5 - 32p^6 + 8p^7),$$

$$x[4] = \frac{3}{D4}(-3 + 12p - 28p^2 + 28p^3 - 5p^4 - 26p^5 + 46p^6 - 32p^7 + 8p^8),$$

$$\begin{aligned}
x[5] &= \frac{p}{D5}(-36 + 3p + 60p^2 - 278p^3 + 1364p^4 - 2827p^5 + 2808p^6 \\
&\quad - 1036p^7 - 1224p^8 + 2588p^9 - 2336p^{10} + 1216p^{11} - 384p^{12} + 64p^{13}), \\
x[7] &= -\frac{p}{D7}(36 + 21p - 232p^2 + 1074p^3 - 4042p^4 + 9337p^5 - 14104p^6 \\
&\quad + 14924p^7 - 11120p^8 + 6012p^9 - 2912p^{10} + 1472p^{11} - 512p^{12} + 64p^{13}), \\
x[8] &= \frac{1-p}{D8}(-18 + 48p - 144p^2 - 72p^3 + 905p^4 - 3987p^5 + 10708p^6 \\
&\quad - 16964p^7 + 16244p^8 - 9028p^9 + 2208p^{10} + 320p^{11} - 320p^{12} + 64p^{13}), \\
D1 = D2 &= 2 - 4p + 20p^2 - 32p^3 + 16p^4, \\
D3 = D4 &= -54 + 36p - 246p^2 + 268p^3 + 478p^4 - 1416p^5 + 1704p^6 \\
&\quad - 1248p^7 + 552p^8 - 160p^9 + 32p^{10}, \\
D5 = D7 = D8 &= 2(1 - 2p + 10p^2 - 16p^3 + 8p^4)(-27 + 18p - 123p^2 + 134p^3 \\
&\quad + 239p^4 - 708p^5 + 852p^6 - 624p^7 + 276p^8 - 80p^9 + 16p^{10}).
\end{aligned}$$

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