Exact Periodic Wave Solutions of Two Types of Modified Boussinesq Equations*

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Exact periodic wave solutions to two types of modified Boussinesq equations are obtained by the use of the Jacobi elliptic function method in a unified form. Some new, general solitary wave solutions are presented.

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1. Introduction

The travelling wave solution, meaning a solution of constant form moving with a fixed velocity, is one of the fundamental objects in the study of equations modelling wave phenomena. Of particular interest are three types of travelling waves: the solitary waves, which are localized travelling waves, asymptotically zero at large distances, the periodic waves, and the kink waves, which rise or descend from one asymptotic state to another. Recently, the periodic wave solutions to nonlinear evolution equations has attracted considerable interest [1–6]. Porubov et al. [1–3] have obtained some exact periodic wave solutions to some nonlinear wave equations in terms of the Weierstrass elliptic function. However, their method involves complicated deducing. Liu et al. [4, 5] proposed the Jacobi elliptic function method for finding periodic wave solutions to nonlinear evolution equations, but this method involves tedious calculation. In this paper, we use the Jacobi elliptic function method in a unified form to find exact periodic wave solutions to two types of modified Boussinesq equations. Some new solitary wave solutions are presented.

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2. Periodic wave solutions to the first type of modified Boussinesq equations

The first type of modified Boussinesq equations [7] is

$$H_t + (Hu)_x + u_{xxx} = 0, \quad u_t + H_x + uu_x = 0, \tag{1}$$

where the subscripts denote partial derivatives. We seek for its travelling wave solution of the form

$$H(x,t) = H(\xi), \quad u(x,t) = u(\xi), \quad \xi = kx - \omega t. \tag{2}$$

Without loss of generality, we define $k > 0$. Substituting Eq. (2) into Eq. (1) and integrating once, we obtain

$$-\omega H + k(Hu)' + k^3 u'' = C_1, \quad -\omega u + kH + \tfrac{1}{2}ku^2 = C_2, \tag{3}$$

where the prime denotes the derivative with respect to ξ (throughout the paper), C_1 and C_2 — integral constants. We assume that Eq. (3) has solution in the form

$$H = A_0 + A_1 f + A_2 f^2, \quad u = B_0 + B_1 f, \tag{4}$$

with f satisfying the following equation:

$$f'' = pf + qf^3, \quad f'^2 = pf^2 + \tfrac{1}{2}qf^4 + r, \tag{5}$$

where p, q, and r are constants to be determined.

Remark. Generally, H, u should be expanded into $\sum_{i=0}^{l} A_i f^i$ and $\sum_{i=0}^{n} B_i f^i$, respectively, where l and n are determined by balancing the highest order of derivative term and nonlinear term in Eq. (1), that is $l = 2$, and $n = 1$. Thus we obtain Eq. (4). The use of Eq. (5) is the key of our method. If we use the Jacobi elliptic function method [4, 5], in Eq. (4) f should be replaced by $\mathrm{sn}\xi$, $\mathrm{cn}\xi$, $\mathrm{dn}\xi$, and $\mathrm{cs}\xi$, respectively.

The substitution of Eq. (4) into Eq. (3) and use of Eq. (5) yields (equate coefficients of like powers of f to zero)

$$A_2 B_1 + qk^2 B_1 = 0, \quad -\omega A_2 + k(A_1 B_1 + A_2 B_0) = 0,$$

$$-\omega A_1 + k(A_0 B_1 + A_1 B_0) + pk^3 B_1 = 0, \quad -\omega A_0 + kA_0 B_0 = C_1,$$

$$kA_2 + \tfrac{1}{2}kB_1^2 = 0, \quad -\omega B_1 + kA_1 + kB_0 B_1 = 0,$$

$$-\omega B_0 + kA_0 + \tfrac{1}{2}kB_0^2 = C_2, \tag{6}$$

from which we have

$$A_0 = -pk^2, \quad A_1 = 0, \quad A_2 = -qk^2, \quad B_0 = \frac{\omega}{k}, \quad B_1 = \pm\sqrt{2q}k, \tag{7}$$

which demands $q \geq 0$.

Therefore the exact solution of Eq. (1) reads

$$H = -pk^2 - qk^2 f^2, \quad u = \frac{\omega}{k} \pm \sqrt{2q}kf, \tag{8}$$

with f satisfying Eq. (5).

As examples, we discuss the following two cases.

Case 1. $p = -(1+m^2),\ q = 2m^2,\ r = 1.$

In this case, Eq. (5) has solution $f = \mathrm{sn}\xi$, so we obtain periodic wave solution to Eq. (1)

$$H = (1+m^2)k^2 - 2m^2k^2\mathrm{sn}^2(kx-\omega t), \quad u = \frac{\omega}{k} \pm 2mk\mathrm{sn}(kx-\omega t). \tag{9}$$

As $m \to 1$, $\mathrm{sn}\xi \to \tanh\xi$, and we get solitary wave solution of Eq. (1)

$$H = 2k^2\mathrm{sech}^2(kx-\omega t), \quad u = \frac{\omega}{k} \pm 2k\tanh(kx-\omega t). \tag{10}$$

Case 2. $p = 2-m^2,\ q = 2,\ r = m'^2 \equiv 1-m^2.$

Now the solution of Eq. (5) reads $f = \mathrm{cs}\xi \equiv \mathrm{cn}\xi/\mathrm{sn}\xi$. Thus another periodic wave solution to Eq. (1) is

$$H = -(2-m^2)k^2 - 2k^2\mathrm{cs}^2(kx-\omega t), \quad u = \frac{\omega}{k} \pm 2k\mathrm{cs}(kx-\omega t). \tag{11}$$

For $m \to 1$, $\mathrm{cs}\xi \to \mathrm{csch}\xi$, Eq. (11) degenerates as

$$H = -k^2 - 2k^2\mathrm{csch}^2(kx-\omega t), \quad u = \frac{\omega}{k} \pm 2k\mathrm{csch}(kx-\omega t), \tag{12}$$

which is a new solitary wave solution (singular) of Eq. (1). Because $q \geq 0$ in Eq. (8), Eq. (1) does not admit cn- and dn-function wave solutions of polynomial form. Let us notice that $\mathrm{sn}\xi$, $\mathrm{cn}\xi$, and $\mathrm{dn}\xi$ are the Jacobi elliptic sine function, cosine function, and the third kind of the Jacobi elliptic function, respectively, and m is the modulus of the Jacobi elliptic functions, about which detailed discussion can be found in [8–10].

3. Periodic wave solutions to the second type of modified Boussinesq equations

The second type of modified Boussinesq equation [11] is

$$h_t + u_x + (hu)_x - \alpha u_{xxx} = 0, \quad u_t + uu_x + h_x - 3\alpha u_{xxt} = 0. \tag{13}$$

We suppose that Eq. (13) has solution of the form

$$\begin{aligned} h(x,t) &\equiv h(\xi) = A_0 + A_1 f + A_2 f^2, \\ u(x,t) &\equiv u(\xi) = B_0 + B_1 f + B_2 f^2, \end{aligned} \tag{14}$$

with f satisfying Eq. (5). Substituting Eq. (14) into Eq. (13) and using Eq. (5), with the same procedure as in Sec. 2, we obtain

$$A_0 = -1 + \frac{k^2}{36\omega^2} + 2p\alpha k^2, \quad A_1 = 0, \quad A_2 = 3q\alpha k^2,$$
$$B_0 = \frac{\omega}{k} + \frac{k}{6\omega} - 12p\alpha\omega k, \quad B_1 = 0, \quad B_2 = -18q\alpha\omega k. \tag{15}$$

Therefore the exact solution of Eq. (13) is

$$h = -1 + \frac{k^2}{36\omega^2} + 2p\alpha k^2 + 3q\alpha k^2 f^2,$$
$$u = \frac{\omega}{k} + \frac{k}{6\omega} - 12p\alpha\omega k - 18q\alpha\omega k f^2, \tag{16}$$

where f satisfies Eq. (5).

In what follows several cases are discussed.

Case 1. $p = -(1 + m^2),\ q = 2m^2,\ r = 1$.

As case 1 in Sec. 2, we have periodic solution to Eq. (13)

$$h = -1 + \frac{k^2}{36\omega^2} - 2(1 + m^2)\alpha k^2 + 6m^2\alpha k^2 \mathrm{sn}^2(kx - \omega t),$$
$$u = \frac{\omega}{k} + \frac{k}{6\omega} + 12(1 + m^2)\alpha\omega k - 36m^2\alpha\omega k \mathrm{sn}^2(kx - \omega t). \tag{17}$$

Its corresponding general solitary wave solution is

$$h = -1 + \frac{k^2}{36\omega^2} - 4\alpha k^2 + 6\alpha k^2 \tanh^2(kx - \omega t),$$
$$u = \frac{\omega}{k} + \frac{k}{6\omega} + 24\alpha\omega k - 36\alpha\omega k \tanh^2(kx - \omega t). \tag{18}$$

Case 2. $p = 2m^2 - 1,\ q = -2m^2,\ r = m'^2 \equiv 1 - m^2$.

From Eq. (5) we have $f = \mathrm{cn}\xi$, and Eq. (13) has periodic wave solution

$$h = -1 + \frac{k^2}{36\omega^2} + 2(2m^2 - 1)\alpha k^2 - 6m^2\alpha k^2 \mathrm{cn}^2(kx - \omega t),$$
$$u = \frac{\omega}{k} + \frac{k}{6\omega} - 12(2m^2 - 1)\alpha\omega k + 36m^2\alpha\omega k \mathrm{cn}^2(kx - \omega t). \tag{19}$$

For $m \to 1$, $\mathrm{cn}\xi \to \mathrm{sech}\xi$, and Eq. (19) degenerates as Eq. (18).

Case 3. $p = 2 - m^2,\ q = -2,\ r = -m'^2$.

The solution of Eq. (5) reads $f = \mathrm{dn}\xi$. Therefore we get another periodic wave solution of Eq. (13)

$$h = -1 + \frac{k^2}{36\omega^2} + 2(2 - m^2)\alpha k^2 - 6\alpha k^2 \mathrm{dn}^2(kx - \omega t),$$
$$u = \frac{\omega}{k} + \frac{k}{6\omega} - 12(2 - m^2)\alpha\omega k + 36\alpha\omega k \mathrm{dn}^2(kx - \omega t). \tag{20}$$

As $m \to 1$, $\mathrm{dn}\xi \to \mathrm{sech}\xi$, and we obtain the solution to Eq. (18) again.

Case 4. $p = 2 - m^2,\ q = 2,\ r = m'^2$.

As case 2 in Sec. 2, we get periodic wave solution to Eq. (13)

$$\begin{aligned} h &= -1 + \frac{k^2}{36\omega^2} + 2(2-m^2)\alpha k^2 + 6\alpha k^2 \mathrm{cs}^2(kx - \omega t), \\ u &= \frac{\omega}{k} + \frac{k}{6\omega} - 12(2-m^2)\alpha\omega k - 36\alpha\omega k \mathrm{cs}^2(kx - \omega t). \end{aligned} \tag{21}$$

As $m \to 1$, we obtain a new solitary wave solution of Eq. (13)

$$\begin{aligned} h &= -1 + \frac{k^2}{36\omega^2} + 2\alpha k^2 + 6\alpha k^2 \mathrm{csch}^2(kx - \omega t), \\ u &= \frac{\omega}{k} + \frac{k}{6\omega} - 12\alpha\omega k - 36\alpha\omega k \mathrm{csch}^2(kx - \omega t). \end{aligned} \tag{22}$$

4. Conclusion

We have obtained exact periodic wave solutions of two types of modified Boussinesq equations by means of the Jacobi elliptic function method in a unified form. In contrast to the Jacobi elliptic function method, some merits are obviously available for our method, which is applicable to a large variety of nonlinear partial differential equations, as long as the odd- and even-order derivative terms do not coexist in the equation under consideration. It is shown that this method is more general than the hyperbolic tangent function method. In fact we may obtain more Jacobi elliptic wave solutions through Eq. (5). Due to the limitation of space, we do not discuss it.

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