Mutual Inductance and Selfinductance in Mesoscopic Systems

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The role played by the magnetostatic interaction in mesoscopic multi-channel systems is discussed. We show that the interaction of currents from different channels, when taken in the selfconsistent mean field approximation, leads to selfinductance terms in the Hamiltonian producing an internal magnetic flux. Such multichannel systems can exhibit spontaneous flux or flux expulsion. The dependence of these phenomena on the parameters of the system is discussed.

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1. Introduction

In this paper we want to discuss and elucidate the role played by the mutual inductance and selfinductance in creating persistent selfsustaining currents in mesoscopic metallic or semiconducting systems. We perform some model calculations of persistent currents [1] in mesoscopic rings and cylinders induced by the static magnetic field in the presence of the magnetostatic (current–current) interaction. We assume spinless electrons, the inclusion of spin changes the picture mainly quantitatively. In the presented model calculations we also neglect the influence of disorder on persistent selfsustaining currents as it has been the subject of some earlier papers [2, 3]. We also argue with some earlier results found in the literature [4].

2. A system of two coupled rings

The calculation of persistent selfsustaining currents for a system of two electrically isolated rings can be performed exactly [4]. The model of two interacting (in $T = 0$ K limit) mesoscopic and quasi one-dimensional rings proposed by [4] can be extended. The current–flux characteristics $I(\phi)$ of such a ring is a piecewise-linear function

$$I(\phi) = -2I_0\left[\frac{\phi}{\phi_0} - q\left(\frac{\phi}{\phi_0}\right)\right],\tag{1}$$

where $I_0 = \frac{e\hbar N}{4\pi m R^2}$, N is the number of conducting electrons, $\phi_0 = \frac{h}{e}$ is the flux unit and

$$q(x) = \begin{cases} \|x\| & \text{if } N \text{ is odd,} \\ \|x\| + \frac{1}{2} & \text{if } N \text{ is even,} \end{cases}\tag{2}$$

where $\|x\|$ indicates the integral part of x.

Let us consider the system of two coaxially placed rings of selfinductance L_i (where $i = 1, 2$ is the number of the ring) coupled by the mutual inductance $\mathcal{M}$. The magnetic fluxes ϕ_1 and ϕ_2 in the rings depend on the appropriate currents I_1 and I_2 according to

$$\phi_1, 2 = L_{1,2}I_{1,2} + \mathcal{M}I_{2,1}.\tag{3}$$

Equation (1) applied to both rings together with (3) form a system of four selfconsistent equations. For simplicity we shall assume the same geometry of both rings but possibly different number of electrons N_1 and N_2. Further it is assumed that N_1 and N_2 differ by no more than a few electrons so we may neglect the difference of I_0 for the rings. The solutions of the system (1) and (3) depend on the parity of N_1 and N_2 and yield spontaneous fluxes ϕ_{s_1}, and ϕ_{s_2} on both rings

$$\frac{\phi_{s_{1,2}}}{\phi_0} = \frac{2I_0[\phi_0(\mathcal{M}q_{2,1} + Lq_{1,2}) + 2I_0q_{1,2}(L^2 - \mathcal{M}^2)]}{\phi_0^2 + 4I_0\phi_0L + 4I_0^2(L^2 - \mathcal{M}^2)}.\tag{4}$$

The parameters q_i, $i = 1, 2$ indicate different solutions and take, in principle, arbitrary integer (if N_i is odd) or half integer (if N_i is even) values. The number of possible spontaneous solutions is however limited by the values of L and $\mathcal{M}$. After some algebra one finds the following conditions for the possible solutions of (4):

$$\frac{\pm 2mq_2 \mp 2lq_1 \mp 2q_1 + 1 + 2l + l^2 - m^2}{1 + 2l + l^2 - m^2} > 0,$$
$$\frac{\pm 2mq_1 \mp 2lq_2 \mp 2q_2 + 1 + 2l + l^2 - m^2}{1 + 2l + l^2 - m^2} > 0,\tag{5}$$

where l and m are real numbers defined by $L = l\frac{\phi_0}{2I_0}$ and $\mathcal{M} = m\frac{\phi_0}{2I_0}$.

We are going to analyze some special cases. The existence of the solution for parallel ($q_1 = q_2 = \frac{1}{2}$ or equivalently $q_1 = q_2 = -\frac{1}{2}$), antiparallel ($q_1 = -q_2 = \frac{1}{2}$ or

$q_1 = -q_2 = -\frac{1}{2}$) coupling of two rings with even number of electrons as well as the trivial ($q_1 = q_2 = 0$, $\phi_{s_i} = 0$, $i = 1, 2$) solution for rings hosting an odd number of electrons is, in principle, proven since for any positive l and m (in a typical system $l > m$) conditions (5) are satisfied. The same takes place for two coupled rings, one of which hosts an even while the other an odd number of conducting electrons if one considers $q_1 = \frac{1}{2}$ (or $q_1 = -\frac{1}{2}$) and $q_2 = 0$ or vice versa.

Finally we analyze a parallel ($q_1 = q_2 = q$) and antiparallel ($q_1 = -q_2 = q$) cases for $|q|$ exceeding $\frac{1}{2}$. For the first of above mentioned cases conditions (5) are equivalent to

$$l + m > 2q - 1,$$

while for the other

$$l - m > 2q - 1.$$

The "excited" solutions, with $|q_i|$ exceeding $\frac{1}{2}$, are very unlikely to obtain due to large necessary values of l and m. For a typical ring with a radius 500 Å, thickness 5 Å and $N = 10^4$ conducting electrons the value of l is of the order of 10^{-3}, 10^{-4}, i.e. it is too small to obtain the spontaneous flux bigger than $|q| < \frac{1}{2}$. One can also see that for positive values of m to obtain "excited" spontaneous solutions the values of selfinductance and the mutual inductance add constructively for the parallel alignment while in the antiparallel case the mutual inductance weakens the magnetic coupling and acts destructively.

Concluding this case we see that at zero temperature selfinductance and mutual inductance are capable to maintain spontaneous currents only at the lowest possible level provided at least one of the rings carries an even number of electrons. The case of two coupled rings is important mainly for didactic purposes as the calculations can be performed exactly and it helps to understand the nature of magnetostatic interaction leading to spontaneous solutions. However in reality to support the current in the absence of an external flux one needs a large number of rings (channels) interacting via the magnetostatic interaction. Such a situation will be considered in the following chapters.

3. The system of M coupled rings in a mean field approximation

Let us assume that we have M mesoscopic rings carrying currents in the presence of the external flux ϕ_e. The calculation of persistent selfsustaining currents cannot be performed exactly for large number of rings. One needs to apply some approximations.

The Hamiltonian of such system with the inclusion of the mutual interaction of ring currents is of the form [2]:

$$H = \frac{1}{2m} \sum_{\lambda=1}^{M} \sum_{i=1}^{N_\lambda} \left(p_{i\lambda}^0 - \frac{e\phi_e}{2\pi R} \right)^2 - \sum_{\lambda=1}^{M} \sum_{\lambda'=1, \lambda \neq \lambda'}^{M} \mathcal{M}_{\lambda\lambda'} I_\lambda I_{\lambda'}, \tag{6}$$

where R is the ring radius, $p_{i\lambda}^0 - \frac{e\phi_e}{2\pi R}$ is the canonical momentum of the electron of the i-th electron in the λ-th ring, I_λ is the current in the λ-th ring, $\mathcal{M}_{\lambda\lambda'}$ is the mutual inductance coefficient and N_λ is the number of electrons in a λ-th ring. The interaction is long range and depends strongly on the sample geometry.

The mutual inductance for two coaxial circular rings is given by [5]

$$\mathcal{M}_{\lambda\lambda'} = \mu_0 \sqrt{R_\lambda R_{\lambda'}} \left[\left(\frac{2}{k} - k \right) K(k) - \frac{2}{k} E(k) \right], \tag{7}$$

where $k^2 = \frac{4R_\lambda R_{\lambda'}}{(R_\lambda + R_{\lambda'})^2 + z_{\lambda\lambda'}^2}$ R_λ is the radius of the λ-th ring and $z_{\lambda\lambda'}$ is the distance between centers of the rings labeled by λ and λ'. K and E are complete elliptic integrals.

The dependence of $\mathcal{M}_{\lambda\lambda'}$ on the distance between the rings centers for the coaxial situation is presented in Fig. 1.
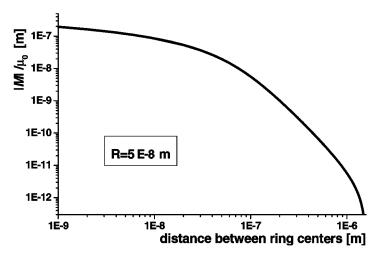

Fig. 1.   Mutual inductance in the case of coaxial (Eq. (7)) alignment of two rings.

A calculation for $M \gg 1$ interacting mesoscopic rings cannot be done exactly but we can perform, following [2], a selfconsistent mean field approximation (SMFA) for the current–current interaction in Eq. (6).

Let us take a mesoscopic cylinder of length l made of a set of stacked rings of radius R. The effective Hamiltonian obtained in SMFA [2] is of the form

$$H_{\mathrm{eff}} = \frac{1}{2m} \sum_{\lambda=1}^{M} \sum_{i=1}^{N_\lambda} \left[p_{i\lambda}^0 - \frac{e}{2\pi R}(\phi_e + \phi_I) \right]^2 + \frac{1}{2}\tilde{L}I^2, \tag{8}$$

where $\tilde{L} = \frac{\mu_0 \pi R^2}{l}$ is the selfinductance of a cylinder, I is the total current in the cylinder, $\phi_I = \tilde{L}I$ and

$$I = \sum_{\lambda=1}^{M} \langle I_\lambda \rangle. \tag{9}$$

It follows from Eq. (8) that each electron feels an effective magnetic flux which is a sum of an external magnetic flux and the flux ϕ_I coming from the currents itself

$$\phi = \phi_e + \phi_I. \tag{10}$$

We see that the mutual interaction of currents from different rings results in a selfinductance of cylinder.

Till now we considered only the magnetic interaction of currents from different rings. However a ring of finite thickness has many transverse channels and we now take into account both the interaction of currents from different channels in each ring and the interaction of currents from M stacked rings. The effective Hamiltonian in the SMFA reads

$$H_{\text{eff}} = \frac{1}{2m} \sum_{\lambda=1}^{M} \sum_{\kappa=1}^{M_\lambda} \sum_{i=1}^{N_{\kappa\lambda}} \left[p_{i\kappa\lambda}^0 - \frac{e}{2\pi R}(\phi_e + \phi_{I\lambda}) \right]^2$$
$$+\frac{1}{2}\tilde{L}I^2 + \frac{1}{2} \sum_{\lambda=1}^{M} L_\lambda \langle I_\lambda \rangle^2, \tag{11}$$

where

$$\phi_{I\lambda} = \tilde{L}I + L_\lambda \langle I_\lambda \rangle$$

and

$$L_\lambda = \mu_0 R \left(\ln \frac{16R}{d_\lambda} - \frac{7}{4} \right).$$

L_λ is a selfinductance of a λ-th ring. Here again the mutual interaction of currents from different channels in a single ring results in a selfinductance of a ring. M_λ is the number of channels in the λ-th ring, $M_\lambda = \frac{\pi(d_\lambda/2)^2}{a^2}$ where d_λ is the thickness of the λ-th ring ($d_\lambda \ll R$) and a is the lattice constant. $N_{\kappa\lambda}$ is the number of electrons in the κ-th channel of the λ-th ring.

In the following, for simplicity, we assume that the rings are identical, the extension to different rings is straightforward, but the precise calculations have to be done then numerically.

Thus

$$\phi_{I\lambda} \equiv \phi_I = \tilde{L}(1+\delta)I = \left(\tilde{L} + \frac{L}{M} \right) I, \tag{12}$$

where

$$\delta = \frac{L/M}{\tilde{L}}, \quad L_\lambda \equiv L = \mu_0 R \left(\ln \frac{16R}{d} - \frac{7}{4} \right), \tag{13}$$

and the last two terms in Eq. (11) take the form

$$\frac{1}{2}\tilde{L}I^2 + \frac{1}{2} \sum_{\lambda=1}^{M} L_\lambda \langle I_\lambda \rangle^2 = \frac{1}{2}\tilde{L}(1+\delta)I^2.$$

The flux ϕ_I contains now the contribution coming from the selfinductance L of each ring and that from the selfinductance $\tilde{L}$ of the cylinder, the parameter δ reflects the ratio of the two contributions.

4. Spontaneous flux and flux expulsion

It is well known [6, 7] that persistent currents in multichannel system depend on the correlation of currents from different channels i.e. on the shape of the Fermi surface (FS). Assuming that each ring is made of a material of density ρ with flat FS, the total current in the cylinder is given by the formula (9) with

$$\langle I_\lambda \rangle = M_{\mathrm{r}} \frac{e\hbar}{2\pi m R^2} \sum_{n=-\infty}^{\infty} \left(n - \frac{\phi}{\phi_0} \right) f_{\mathrm{FD}}(T), \tag{14}$$

where n is the orbital quantum number for the electron going along the circumference of the ring, $f_{\mathrm{FD}}(T)$ is the Fermi–Dirac distribution, $\phi_0 = \frac{h}{e}$, $M_{\mathrm{r}} = \pi \frac{d^2}{4a^2}$ is the number of transverse channels of a single ring, a is the lattice constant.

In the following we discuss the $T = 0$ case. The temperature dependence of the persistent and selfsustaining currents has been given elsewhere (see e.g. [6, 7]).

Let us consider at first a stack of rings with an even number of electrons in each channel. Such rings react with the paramagnetic reaction to ϕ_{e}. Equations (9), (10), (12) form the selfconsistent equations for the flux and one can look for the spontaneous flux solutions in the absence of an external flux. After some algebra one obtains the formula for spontaneous flux in a cylinder

$$\phi_{\mathrm{s}} = \frac{1}{2 + \frac{\phi_0}{\tilde{L}(1+\delta)M I_0}} \phi_0, \tag{15}$$

where $I_0 = \frac{e\hbar N}{4\pi m R^2}$, N is the number of electrons in a single ring.

Assuming that the distance between the ring centers in the cylinder is b one gets from Eq. (13)

$$\delta = \frac{b(\ln \frac{16R}{d} - \frac{7}{4})}{\pi R} \tag{16}$$

and the formula for ϕ_{s} takes the form

$$\phi_{\mathrm{s}} = \frac{1}{2 + \frac{16\lambda^2 b}{\pi d^2 R(1+\delta)}} \phi_0, \tag{17}$$

where

$$\lambda^2 = \frac{m}{\mu_0 \rho e^2}. \tag{18}$$

Assuming the electron density ρ as for copper $\rho = 4 \times 10^{-2}$ Å^{-3} we get $\lambda = 2.7 \times 10^2$ Å.

We are now ready to discuss the solutions of Eq. (15) (or equivalently of Eq. (17)) in different cases. For the set of densely packed rings, so that $b \ll R$,

the parameter δ is much smaller than one ($\delta \ll 1$) and ϕ_s is determined by the inductance $\tilde{L}$ of the cylinder. For thin rings e.g. $d \approx 5$ Å, $R \approx 10^4$ Å, $b = 10^2$ Å:

$$\phi_s \approx 7 \times 10^{-3} \phi_0,$$

i.e. it is negligible. However, ϕ_s increases with increasing R and d and for e.g. $d \approx 30$ Å, $R \approx 5 \times 10^4$ Å, $b = 10^2$ Å one gets a substantial ϕ_s,

$$\phi_s \approx 0.33 \phi_0.$$

One should stress that the considerations performed here and in [4] neglect the r dependence of the flux (r is the coordinate measured along the ring thickness) and are thus valid for $d \ll R$.

Therefore, contrary to conclusions in [4], we will never obtain the maximal value of ϕ_s, $\phi_s^{max} = \frac{\phi_0}{2}$ in a cylinder of small thickness and R in the mesoscopic regime. The calculations for $d \geq R$ are in progress and results will be published in a forthcoming paper.

The above calculations were performed for material with flat FS. There is then the perfect correlation of currents from different channels and the current is the strongest.

If the rings forming a cylinder are made of a material with a spherical FS there is almost no correlation of currents from different channels [6, 7] and the formula for the spontaneous flux takes then the form

$$\phi_s^{sph} = \frac{1}{2 + \frac{8\lambda^2 b}{\sqrt{\pi} dRa}} \phi_0. \tag{19}$$

In this situation we get much smaller spontaneous flux $\phi_s^{sph} \approx 0.1 \phi_0$ for the set of parameters as in the previous example.

Till now we discussed the solutions of Eq. (17) in the case of densely packed rings ($b \ll R$) where the effect of ring selfinductance was negligible. The influence of L increases with increase in the distance b of the ring centers and for $b \approx R$ both $\tilde{L}$ and L determine the magnitude of spontaneous flux. With further increase in b, for $b \gg R$, $\delta \gg 1$ and ϕ_s is determined mainly by L. If one considers at first rings with flat FS and even number of electrons in each channel, one obtains from Eq. (17) in the limit $b \gg R$ the following equation for ϕ_s:

$$\phi_s = \frac{1}{2 + \frac{16\lambda^2}{d^2(\ln\frac{16R}{d} - \frac{7}{4})}} \phi_0. \tag{20}$$

We see that ϕ_s increases with increasing d and for e.g. $R = 10^4$ Å, $d = 3 \times 10^2$ Å we obtain a relatively large spontaneous flux

$$\phi_s = 0.21 \phi_0.$$

However, with reducing the thickness d of the ring, ϕ_s decreases and for quasi 1D ring we get $\phi_s \approx 10^{-4} \phi_0$. Thus the statement in [4] that ϕ_s approaches its maximal value $\phi_s^{max} = \frac{\phi_0}{2}$ as the thickness of the wire is vanishingly small is incorrect.

If it were true then each atom, molecule or quasi 1D ring made e.g. by means of litography could expel the flux or produce the large spontaneous flux which is not the case. Although in principle the selfinductance L diverges as $d \to 0$ it is only the limit in a mathematical sense and for the smallest realistic values of d and R in the mesoscopic regime L is finite and small. The product LI_0 which matters is proportional to $d^2(\ln\frac{16R}{d} - \frac{7}{4})$ and gives substantial values of ϕ_s only for relatively thick rings. Thus the selfinductance of electrons from a single channel (i.e. in a quasi 1D ring) is unable to support the flux. Only the collective action of electrons from many different channels can lead to selfsustaining flux.

If the rings are made of a material with spherical FS one obtains in the limit $b \gg R$ the following formula for the spontaneous flux ϕ_s^{sph}:

$$\phi_s^{\mathrm{sph}} = \frac{1}{2 + \frac{8\sqrt{\pi}\lambda^2}{da(\ln\frac{16R}{d} - \frac{7}{4})}}\phi_0. \tag{21}$$

Comparing Eq. (20) with Eq. (21) we see again that

$$\phi_s^{\mathrm{sph}} < \phi_s,$$

where ϕ_s was calculated with flat FS. For FS intermediate between spherical and flat spontaneous flux will be intermediate between ϕ_s^{sph} and ϕ_s.

Till now we discussed rings and cylinders exhibiting a paramagnetic reaction to the external magnetic field. Let us investigate now a mesoscopic ring made of a material with flat FS when the number of electrons in each channel is odd [8]. Such ring reacts with a diamagnetic reaction to ϕ_e ($\phi_e < \frac{\phi_0}{2}$) and making use of Eq. (9) and (14) one obtains the following formula for $\phi_I = LI$:

$$\phi_I = -\alpha\phi, \tag{22}$$

where $\alpha = (\frac{d}{\lambda})^2\frac{\ln\frac{16R}{d} - \frac{7}{4}}{8}$.

Inserting Eq. (22) into (10) we obtain the formula for the total flux ϕ in the ring

$$\phi = \frac{\phi_e}{1 + \alpha}. \tag{23}$$

We see that we get a substantial screening of the external flux ϕ_e if $\alpha \geq 1$, i.e. if the ring has many transverse channels.

For e.g. $R = 10^4$ Å, $d = 2 \times 10^2$ Å, $\alpha = 0.34$ and

$$\phi = 0.75\phi_e.$$

Increasing d to e.g. $d = 3.2 \times 10^2$ Å we get $\alpha = 0.78$ and $\phi = 0.56\phi_e$ so we obtain a relatively large flux expulsion. However, for very thin, let us say quasi 1D ring, $\alpha \ll 1$, $\phi \approx \phi_e$ and we can neglect the selfinductance effect. We can easily check that in the case of a ring with spherical FS exhibiting diamagnetic reaction to ϕ_e the flux expulsion is much smaller than in the former case.

The calculations of flux expulsion can be extended to the case of mesoscopic cylinder made of a set of rings exhibiting a diamagnetic reaction to ϕ_e.

Making use of Eqs. (10), (12) and (9) one obtains for total flux in the system

$$\phi = \frac{\phi_e}{1 + \alpha_c}, \tag{24}$$

where

$$\alpha_c = \frac{\pi d^2 R}{8\lambda^2 b} \left[1 + \frac{b(\ln \frac{16R}{d} - \frac{7}{4})}{\pi R} \right].$$

The flux expulsion increases with increase in the thickness d of the rings and with decrease in the distance b of the ring centers. Assuming e.g. $d = 35$ Å, $b = 10^2$ Å, $R = 10^4$ Å one gets

$$\phi = 0.59\phi_e.$$

5. Conclusions

In macroscopic systems magnetostatic interaction is known to be small [9], however in mesoscopic systems this interaction, giving the internal magnetic flux, can be important because magnetic flux has a very dramatic effect on such systems leading e.g. to persistent currents.

The system of two magnetically coupled quasi one-dimensional rings can, at zero temperature, exhibit different spontaneous flux solutions in the first period of the current–flux characteristics provided at least one of the rings has an even number of electrons. The existence of higher realizations is very unlikely and depends on the values of coupling constants.

To obtain the spontaneous flux solutions in real systems one needs many interacting entities. We have discussed the role played by the mutual interaction of currents in the multichannel system and have shown that when taken in the SMFA it leads to selfinductance terms in the Hamiltonian producing an internal magnetic flux. Let us notice the minus sign in Eq. (6) which is pertinent for the interaction of persistent currents in microscopic and mesoscopic systems. The sign plus in Eq. (6) would be used for the interaction of dissipative currents where the energy should be supplied from the outside to maintain the current [5].

In the case of a cylinder made of set of stacked rings we should take into account both the selfinductance of each single ring (L) and that of a cylinder ($\tilde{L}$) and the general formula for ϕ_s contains both terms. However, when the rings are densely packed ($b < R$), the formula for ϕ_s is determined mainly by $\tilde{L}$. It justifies the approximation used by us in the earlier papers [2, 7] in which we neglected the ring selfinductance. If we increase the distance b between the rings forming a cylinder, the ring selfinductance becomes important and for $b \gg R$, L determines the magnitude of ϕ_s. The rings are then so far apart that $\tilde{L}$ coming from the mutual interaction of different rings is negligible.

We have shown that the magnitude of spontaneous flux depends on the thickness d of the ring and on the shape of the FS. For very thin ring ϕ_s is negligible contrary to some statements in the literature [4] but it increases with increase in d and with increase in the curvature of the FS.

We have also shown that the rings and cylinders exhibiting the diamagnetic reaction to ϕ_e give a substantial flux expulsion if they have many transverse channels.

The calculations presented above were performed under the assumption $d \ll R$, but it will be shown in a forthcoming paper that for thick rings and cylinders one can obtain a full flux expulsion and quantized flux trapped in such nonsuperconducting structures.

Acknowledgment

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